R.1.- *Let* f *be an o-regular function from a normal topological space* S *to a finite directed graph* G *and* Y *a closed subset of* S. *Then, if for* $a \in G$ *we have* $A^f \cap Y = \phi$ and $\overline{A^f} \cap Y \neq \phi$ *there exists an o-regular function* $g : S \to G$, *which is o-homotopic to* f *and such that* $A^g \cap Y = \phi$ (See Lemma 6).

R.2 - *In the construction of* R.1, *if there exist* n *vertices* $p_1, \dots, p_n \in G$, *such that* $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} = \phi$ *then also it follows* $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} = \phi$. (See Corollary 7).

R.3.- *Let* f *be an o-regular function from* S, S' *to* G, G', *where* S *is a normal topological space,* S' *a closed subspace of* S, Y *a closed subset of* S', G *a finite directed graph and* G' *a subgraph of* G. *Then, if for* $a \in G$ *we have* $A^f \cap Y = \phi$ *and* $\overline{A^f} \cap Y = \phi$, *there exists an o-regular function* $g : S, S' \to G, G'$, *which is o-homotopic to* f *and such that* $\overline{A^g} \cap Y = \phi$. (See Lemma 11).

R.4.- *In the construction of* R. 3, *if there exist* n *vertices* $p_1, \dots, p_n \in G$ *and* m *vertices* $q_1, \dots, q_m \in G'$, such that $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap \overline{Q_1^{f'}} \cap \dots \cap \overline{Q_m^{f'}} = \phi$, *then also it follows that* $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap \overline{Q_1^{g'}} \cap \dots \cap \overline{Q_m^{g'}} = \phi$. *While, from* $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap \dots = \phi$, *it results* $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap S' = \phi$. (See Corollary 12).

By Duality Principle, the results dual to the previous ones are also true for o^*-regular functions.

1) *Headed and totally headed subsets of a graph*

DEFINITION 1.- *Let* G *be a directed graph and* X *a non-empty subset of* G. *A vertex of* X *is called a* head *(resp. a* tail) *of* X *in* G, if it is a *predecessor (resp. a successor) of all the other vertices of* X. *We denote by* $H_G(X)$ *(resp.* $T_G(X)$*) or, more simply, by* $H(X)$ *(resp.* $T(X)$*) the set of the heads (resp. tails) of* X *in* G. *Then* X *is called* headed (*resp.* tailed) *if* $H(X) \neq \phi$ *(resp.* $T(X) \neq \phi$*), otherwise,* X *is called* non-headed (*resp.* non-tailed).

Finally X *is called* totally headed (*resp.* totally tailed), *if all the non-empty subsets of* X *are headed (resp. tailed).*

REMARK 1. - If X is a singleton, we agree to say that $H(X) = T(X) = X$, then X is totally headed and also totally tailed. If X is a pair, X headed $\Longleftrightarrow$ X totally headed $\Longleftrightarrow$ X tailed $\Leftrightarrow$ X totally tailed.

REMARK 2. - This definition and the following ones can be extended to undirected graphs. (See Proposition 6).

REMARK 3. - The concepts of head and tail (headed and tailed subset, etc.) are dual to each other.

DEFINITION 2. - *A non-headed (resp. non-tailed) subset* X *is called* minimal *if all its non-empty proper subsets are headed (resp. tailed).*

DEFINITION 3. - *A finite directed graph* G *is called* almost complete *if the set of its vertices is totally headed.*

REMARK. - A complete finite undirected graph is also almost complete.

PROPOSITION 1. - *A finite directed graph* G *is almost complete iff the diagram* (*) *of the relation* ($\rightarrow$) *includes the diagram of a totally ordered relation* ($<$) *in* G.

(*) We use the term *diagram* rather than *graph* because *graph* is already used in another sense.

Proof. - i) Since G is almost complete, we can choose a vertex $v_1 \in G$, which is a predecessor of all the other vertices of G, as the first one; then a vertex $v_2 \in G - \{v_1\}$, predecessor of all the other vertices of $G - \{v_1\}$, as the second one; and so on.

ii) Since the diagram of the relation (→) includes the diagram of a totally ordered relation (<) in G, we can totally order the vertices of G. Then every vertex of G is a predecessor of the vertices subsequent in the order relation.
Hence G is almost complete.

REMARK. - By ordering the vertices of G as in b) of Proposition 1, we say that the order relation (<) of G is compatible with the relation (→) of G.

PROPOSITION 2. - *Let* G *be an almost complete graph. Then the dually directed graph* G^* *is also almost complete.*

Proof. - Let (<) be a totally order relation, compatible with the relation (→) of G.Then the dual order relation (>) is compatible with the relation (←) of the dually directed graph G^*.

DEFINITION 4. - *Let* G *be a directed graph and* X *a subset of* G. *We call* maximal subgraph induced by X *the subgraph of* G *consisting of those directed edges of* G, *whose vertices are in* X.

PROPOSITION 3. - *A subset* X *of* G *is totally headed iff the maximal subgraph induced by* X *is almost complete.*

PROPOSITION 4. - *A subset* X *of* G *is totally headed iff it is totally tailed.*

Proof. - By Remark 3 to Definition 1 and by Proposition 2,3 we have:

X totally headed in G $\Longleftrightarrow$ the maximal subgraph induced by X is almost complete $\Longleftrightarrow$ the dually directed graph of the maximal subgraph induced by X is almost complete $\Longleftrightarrow$ X is totally headed in G^* $\Longleftrightarrow$ X is totally tailed in G.

PROPOSITION 5. - *A subset* X *of* G *is non-headed minimal iff it is non-tailed minimal.*

Proof. - Since all the subsets of X are totally headed, by Proposition 4, they are also totally tailed. If we assume that X is tailed, then, by Definition 1, it is totally tailed. Hence, by Proposition 4, it is also totally headed. Contradiction.

REMARK. - Then *almost complete graph, totally headed subset, non-headed minimal subset are selfdual concepts*, while it does not follow for headed or tailed subset.

PROPOSITION 6. - *In an undirected graph there does not exist any non-headed minimal* n-*tuple* X *with* n > 2.

Proof.- If all the pairs of vertices of X are headed (i.e. they are vertices of edges), then the maximal subgraph induced by X is complete. Hence it is also totally headed.

EXAMPLES.

1) Let $G=\{a,b,c,d,e\}$ be the graph with the edges $a \to b$, $a \to c$, $a \to d$, $b \to d$, $b \to e$, $c \to d$. Then the subset $\{a,b,e\}$ is non-headed and non-tailed, but it is not minimal non-headed (i.e.minimal non-tailed); $\{a,b,c\}$ is headed and non-tailed; $\{b,c,d\}$ is non-headed and tailed; $\{a,b,c,d\}$ is headed and tailed, but not totally headed (tailed).

2) The graphs $G = \{u,v,w\}$ with the edges $u \to v$, $u \to w$, $v \to w$ and $G' = \{q,r,s,t\}$ with edges $q \to r$, $q \to s$, $q \to t$, $r \to s$, $r \to t$, $s \to t$ are examples of almost complete graphs. Moreover, the sets $\{u,v,w\}$, $\{q,r,s,t\}$ are examples of totally headed (i.e. totally tailed) subsets. Their compatible orders are, respectively, $u < v < w$, $q < r < s < t$.

3) In the graphs $G = \{f,g,h\}$ with the edges $f \to g$, $g \to h$, $h \to f$ and $G' = \{\ell,m,n,p\}$ with the edges $\ell \to m$, $\ell \to n$, $m \to n$, $m \to p$, $n \to \ell$, $n \to p$, $p \to \ell$, $p \to m$ the sets $\{f,g,h\}$ and $\{\ell,m,n,p\}$ are examples of non-headed minimal (i.e. non-tailed minimal) subsets.

2) *Singularities of a regular function.*

PROPOSITION 7. - *Let* S *be a topological space,* G *a finite directed graph,* $f : S \to G$ *an o-regular function from* S *to* G *and* $X = \{v_1, v_2, \ldots, v_n\}$ *a non-headed subset of* G $(n \geq 2)$. *Then it holds:*

$$v_1^f \cap \overline{v_2^f} \cap \ldots \cap \overline{v_n^f} = \phi;$$

$$\overline{v_1^f} \cap v_2^f \cap \ldots \cap \overline{v_n^f} = \phi$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\overline{v_1^f} \cap \ldots \cap \overline{v_{n-1}^f} \cap v_n^f = \phi$$

Proof. - Since X is a non-headed subset, there is no vertex v_i, which is a predecessor of all the other n-1 vertices. Then, for every $i = 1, \ldots, n$ let w_i be a vertex such that $v_i \not\to w_i$. From o-regularity of f it is $v_i \cap \overline{w_i} = \phi$. Since w_i is one of the vertices $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$, it follows $\overline{v_1^f} \cap \ldots \cap \overline{v_{i-1}^f} \cap v_i^f \cap \overline{v_{i+1}^f} \cap \ldots \cap \overline{v_n^f} = \phi$.

DEFINITION 5. - *Let* S *be a topological space,* G *a finite directed*