COMPLEMENTARY DISTRIBUTIONS AND PONTRJAGIN CLASSES

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SUMMARY.- *From a necessary condition for the existence of $k(>2)$ complementary distributions on a manifold we deduce connections between Pontrjagin classes of distributions and of transversal bundles.*

Let M be a riemannian smooth orientable manifold of dimension n (even or odd) and suppose that M has k complementary (smooth) distributions of oriented n_i-planes $(i=1,\dots,k)$; i. e. for every point $p \in M$ the tangent space $T_p(M)$ can be decomposed into the direct sum of the subspaces $T_p^1,\dots,T_p^k$ of $T_p(M)$ where $\dim T_p^i = n_i$ (and hence $n_1+\dots+n_k = n$).

Then one says that M admits an "almost product" or "almost multiproduct" structure.

In a paper of 1969 ([3]) one of the present authors showed that the vanishing of certain Pontrjagin classes is a necessary condition for the existence of k complementary distributions on M. After a review of these results we prove the following

THEOREM A

Let M be a riemannian smooth orientable manifold of dimension n. Furthermore let M have k complementary distributions $E_i \subset TM$ and let the bundle $Q_i = TM/E_i$ have fibre of dimension q_i with

$$q_i = n - n_i \qquad \text{and} \qquad n_1+\dots+n_k = n.$$

Finally let $p_r(E) \in H^{4r}(M;\mathbb{R})$ denote the r-th real Pontrjagin class of the bundle E. Then if

$$P_h(Q_i)P_s(E_i) = 0 \qquad \forall h,s \geq 1 \quad \text{and} \quad h+s = r$$

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one has

$$P_r(Q_i) = 0 \qquad\qquad 2r > \max(n_1,\dots,n_k).$$

Using Bott's "Vanishing Theorem" one can, in a certain sense invert the preceding result obtaining

THEOREM B

Let M be a riemannian smooth orientable manifold of dimension n. Furthermore let M have k complementary distributions $E_i \subset TM$ *and let the bundle* $Q_i = TM/E_i$ *have fibre of dimension* q_i *with*

$$q_i = n - n_i \qquad\qquad \text{and} \qquad n_1+\dots+n_k = n.$$

If Q_i *is integrable, then for* $2r > \max(n_1,\dots,n_k,4q_i)$

$$P_h(Q_i)p_s(E_i) = 0 \qquad\qquad \forall h,s \geq 1 \qquad \text{and} \qquad h+s=r$$

holds.

In order to give a self-contained presentation we recall the necessary preliminaries.

1. Preliminaries

Let M be a smooth (paracompact) manifold and let E be a vector $\mathbb{R}^q$-bundle on M. As is well known, the total (real) Pontrjagin class p(E) of E is defined by

$$p(E) = 1+p_1(E)+\dots+p_{\left[\frac{q}{2}\right]}(E) = \left[\det\left(I-\frac{1}{2\pi}\,\Omega\right)\right]$$

where Ω is the curvature of an arbitrary connection on E and $p_r(E)\in H^{4r}(M;\mathbb{R})$

where $H^*(M;\mathbb{R})$ is the de Rham cohomology ring of M. $P_r(E)$ is called the r-the Pontrjagin class of E.

Clearly $P_r(E) = 0$ for $4r > n$ if $n = \dim M$. Moreover if E is an oriented bundle of even dimension q, then the class $p_{q/2}(E)$, which is locally represented by the closed form $(2\pi)^{-q}\det\Omega$, equals the square of the Euler class; this latter is strictly connected with the Euler-Poincaré characteristic of the manifold under consideration, if $E = TM$.

If E is the tangent bundle TM, then the classes are also called Pontrjagin class of M and are often denoted by $P_r(M)$.

Let Ω is the curvature form of a connection on the principal fibre bundle of orthonormal frames, then the explicit expression of Pontrjagin classes is give by[1]

$$(1.1)\qquad P_r(TM) = \left[\frac{[(2r)!]^2}{(2^r\, r!)^2(2\pi)^{2r}}\sum_{(i)}\Theta^{(2r)}_{i_1\ldots i_{2r}}\wedge\Theta^{(2r)}_{i_1\ldots i_{2r}}\right]$$

where

$$(1.2)\qquad \Theta^{(s)}_{i_1\ldots i_s} = \frac{1}{s!}\sum_{(i)}\delta(i_1,\ldots,i_s;j_1,\ldots j_s)\Omega_{j_1j_2}\wedge\ldots\wedge\Omega_{j_{s-1}j_s}$$

s is an even integer and $\delta(i_1,\ldots,i_s;\, j_1,\ldots,j_s)$ is the generalysed Kronecker symbol.

For n even, the n-form

$$(1.3)\qquad \Omega^{(n)} = \left[2^n\ \pi^{n/2}\ \left(\frac{n}{2}\right)!\right]^{-1}\sum\varepsilon_{i_1\ldots i_n}\wedge\Omega_{i_1i_2}\cdots\wedge\Omega_{i_{n-1}i_n}$$

called Gauss curvature form of M, is a representative of the Euler class of

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1)See J.A. Thorpe [6]

TM and if M is compact and orientable, the Gauss-Bonnet theorem says that

$$\int_M f^* \Omega^{(n)} = \chi(M)$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M and $f:M \to \tilde{G}_n(M)$ is an orientation of M, $\tilde{G}_n(M)$ being the Grassmann bundle of the oriented n-planes tangent to M.

2. A vanishing theorem

We can now prove the following theorem ([3])

(2.1). Theorem

Let M be a riemannian smooth orientable manifold of dimension n which admits k complementary (smooth) distributions of oriented n_i-planes $(i = 1,\dots,k)$. Then the real Pontrjagin classes $P_r(M)$ are null for $2r > \max(n_1,\dots,n_k)$

Proof.

Let $\tilde{E}$ be the principal fibre bundle of the orthonormal frames (associated to the tangent). Its structural group is $\tilde{G} = SO(n)$ (the rotation group). We recall that the Lie algebra $\tilde{\mathfrak{g}}$ of $SO(n)$ can be identified with space of the skew-symmetric matrices of order n.

Let us consider the subbundle E of $\tilde{E}$ formed of the frames "adapted" to the distributions, viz. the orthogonal frames $\{e_i\}(i=1,\dots,n)$ so that the vectors

$$\{e_{\alpha_j}\} \qquad \alpha_j = n_1+\dots+n_{j-1}+1,\dots,n_1+\dots+n_j \qquad (n_\circ = 0)$$

form a basis for T^j. The bundle E can be regarded as having structural group

$$G = SO(n_1) \times SO(n_2) \times \ldots \times SO(n_k).$$

A connection ω on E is represented by a 1-form which takes values in the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{g}$ is the direct product of the Lie algebras of $SO(n_r)$. Hence one obtains

$$\omega_{ii} = 0 \qquad \omega_{ij} + \omega_{ji} = 0 \qquad i,j = 1,\ldots,n$$

$$\omega_{\alpha_i \beta_j} = 0 \quad i \neq j \quad i,j = 1,\ldots,k;\ \alpha_i,\beta_j = n_1+\ldots n_{j-1}+1,\ldots,n_1+\ldots+n_j$$

Analogous relations hold for the components of the curvature form Ω. Therefore, if $2r > \max(n_1,\ldots,n_k)$, then each term in (1.2) will have a factor $\Omega_{\alpha_i \beta_j}$ with $i \neq j$; the assertion $p_r(TM) = P_r(M) = 0$ then follows from (1.1). ⌟

Remarks

(2.2) If $k = n$ and therefore $n_i = 1$ $\forall i$ (i.e. the manifold is parallelizable) then $P_r(M) = 0$ $\forall r$. It follows that the adapted connection vanishes. The manifold is then flat.

(2.3) For $k = 2$ (and obviously for $k = 1$) the theorem is not meaningful. As for $n_1 + n_2 = n$, $\max(n_1,n_2) \geq [n/2]$ and consequently $P_r(M) = 0$ $\forall\ 2r > \max(n_1,n_2)$.

(2.4) It is worth noticing that the existence of a distribution of q-planes implies the existence of a distribution of (n-q)-planes. An argument analogous to the one followed above, using the Gauss curvature form, yields the following

(2.5) Theorem

Let M be a smooth compact orientable manifold of dimension even n and suppose that it has a distribution of oriented q-planes with q odd $(1 \leq q < n)$. *Then the Euler-Poincaré characteristic of M is null.*

3. Proof of theorems A and B.

(3.1) To every distribution there corresponds a smooth subbundle E_i of the tangent bundle with the fibre of dimension n_i. Denote the quotient bundle by $Q_i = TM/E_i$. From the Whitney duality formula $p(TM)=p(Q_i)p(E_i)$ it follows that

$$(3.2) \qquad P_r(TM) = P_r(Q_i)+P_{r-1}(Q_i)P_1(E_i)+\ldots+p_1(Q_i)P_{r-1}(E_i)+p_r(E_i)$$

where the product between classes is the "cup product" in the ring $H^*(M;\mathbb{R})$.

If $2r > n_i$, then $r > n_i/2$ and hence $P_r(E_i) = 0$. If moreover

$$(3.3) \qquad P_h(Q_i)P_s(E_i) = 0 \qquad \forall h,s \geq 1,\ h+s = r$$

then

$$P_r(TM) = P_r(Q_i) \qquad 2r > n_i.$$

Notice that, since the Pontrjagin ring may have divisors of zero, condition (3.3) does not imply that either $P_h(Q_i) = 0$ or $p_s(E_i) = 0$. On account of our assumptions, from theorem (1.1) one can conclude that

$$P_r(Q_i) = 0 \qquad 2r > \max(n_1,\ldots,n_k).$$

This proves theorem A.

(3.4) It is well known that if $E_i \subset TM$ is isomorphic to an integrable subbundle, then by Bott's theorem

$$P_r(Q_i) = 0 \qquad\qquad r > 2q_i$$

where $q_i = n-n_i$. Hence in the assumptions of theorem A, if $m = \max(n_1,\dots,n_k)<4q_i$, then one has for the integers h for which $m < h < 4q_i$

$$P_r(Q_i) = 0 \qquad\qquad 2r > h$$

whithout assuming that Q_i be integrable. This is meaningful if $k > 2$.

(3.5) Conversely let us assume Q_i to be integrable. Then, under our assumptions, one has at the same time

$$P_r(TM) = 0 \qquad P_r(Q_i) = o \qquad 2r > \max(m,4q_i)$$

hence an account of (3.2)

$$P_h(Q_i)\, P_s(E_i) = 0 \qquad\qquad \forall h,s \geq 1 \;,\; h+s = r.$$

This ends the proof of theorem B.

Remark

The results of theorem (2.1) hold in the more general situation of an almost multifoliated riemannian structures on a manifold, i.e.

$$TM = E_1 + \dots + E_k$$

where E_i are not necessarely complementary. Infact by increasing the number of distributions, with a suitable choice of the metric ([7]) it is possible go back to the previous situation.

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