REMARK ON A BROWDER'S FIXED POINT THEOREM(*)

V. CONSERVA - F. FEDELE(**)

ABSTRACT - *In a recent paper, F.E. Browder discussed continuous self-mappings of contractive type in a complete metric space. Browder showed that such mappings have a fixed point and the sequence of iterates of any point, in an invariant subset, converges to the fixed point. In the present paper, the result of Browder is obtained for mappings which are not necessarily continuous.*

1. Introduction. Let (M,d) be a complete metric space, $f : M \to M$ a mapping and let ψ be a contractive gauge function (i.e. ψ is a function from R_+, the non-negative reals, to R_+ non-decreasing and continuous from the right, $\psi(0) = 0$, and $\psi(r) < r$ for all $r > 0$).

In a recent paper [1], in order to get a fixed point theorem of great generality, F.E. Browder proved the following result.

THEOREM 1 - Let M_0 be a subset of M such that f carries M_0 into M_0. For each x in M_0, suppose that there exists a positive integers $n(x)$ and for each $n \geq n(x)$ and for each y in M_0, three subsets $J_1(x,y,n)$, $J_2(x,y,n)$, $J_3(x,y,n)$ of $Z_+ \times Z_+$ (Z_+ is the set of non-negative integers) such that for each $n \geq n(x)$, y in M_0,

$$d(f^n x, f^n y) \leq \psi(\max(\sup_{(i,j)\in J_1} d(f^i x, f^j y), \sup_{(r,s)\in J_2} d(f^r x, f^s x), \sup_{(\ell,t)\in J_3} d(f^\ell y, f^t y))).$$

Then f has a fixed point x_0 in M such that for each x in M_0, $(f^n x)$ converges to x_0 as $n \to \infty$.

(*) Entrato in Redazione il 16/1/1981

(**)Authors address: Istituto di Matematica dell'Università, Via Arnesano 73100 Lecce (Italy).

In the present paper we shall develop this point of view a little further. Therefore we investigate mappings which are not necessarily continuous but satisfy the following weaker condition.

(A) the function $D(x) = d(x,fx)$ has the following property: if $x_n \to x_\circ$ then a subsequence (x_{n_p}) exists such that $D(x_\circ) \leq D(x_{n_p})$.

As a special case of our result we obtain a fixed point theorem of L.Ciric (Theorem 1, [2]) and a fixed point theorem of the first author (Corollary, [3])

2. Let $O(f,x)$ denote the orbit of x under f, i.e. $O(f,x) = \bigcup_{0 \leq n} \{f^n x\}$. We begin this section with the following definition.

DEFINITION 1 - We shall say that x has bounded orbit under f if $\operatorname{diam}(O(f,x)) < +\infty$.

LEMMA 1 - Let ψ be a contractive gauge function, $t_\circ > 0$. Then the sequence $\psi^n(t_\circ) \to 0$ as $n \to \infty$ (where ψ^n is the n - th iterate of ψ).

THEOREM 2 - Suppose that:

(1) for some $x_\circ$ in M, $\operatorname{diam}(O(f,x_\circ)) < +\infty$,

(2) for each x in M there exists a positive integer $n(x)$ and for each $n \geq n(x)$ and for each y in M, three subsets $J_1(x,y,n)$, $J_2(x,y,n)$, $J_3(x,y,n)$ of $Z_+ \times Z_+$ such that for each $n \geq n(x)$, y in M

$$d(f^n x, f^n y) \leq \psi(\max(\sup_{(i,j)\in J_1} d(f^i x, f^j y), \sup_{(r,s)\in J_2} d(f^r x, f^s x), \sup_{(\ell,t)\in J_3} d(f^\ell y, f^t y))).$$

Then f has a unique fixed point u in M and $(f^n x)$ converges to u in M as $n \to \infty$ for each x in M which has bounded orbit under f.