Thus, if $j(k) > 2j(k-1)$ this quantity is unbounded and passing to a subsequence on k will not help. Therefore the condition of Theorem 3 is violated.

CONNECTIONS. -

Now we turn to the main topic of these lectures which is the description of certain connections between the structure theory and inverse function theorems.

Our first remark is an observation that shows how special is the relation (3) which the smoothing operators are assumed to satisfy.

If we have (3), then any $x \in E$, $\theta > 0$ and $k < j < \ell$ we would have,

$$\| x \|_j \leq \| S_\theta x \|_j + \| x - S_\theta x \|_j \leq C(\theta^{j-k} \| x \|_k + \theta^{j-\ell} \| x \|_\ell).$$

We can use calculus to show that the right hand side achieves its minimum value when

$$\theta = \left(\frac{\| x \|_\ell \, (\ell - j)}{\| x \|_k \, (j-k)} \right)^{\frac{1}{\ell - k}},$$

and substituting this value for θ with $k = j-1$, $\ell = j+1$ yields

$$(\| x \|_j)^2 \leq C \, \| x \|_{j-1} \, \| x \|_{j+1} .$$

This immediately implies the condition of Theorem 3 so, with the above discussion, we may conclude that Theorem 2 does not hold for $H(\mathbb{D})$. Actually we have the following much stronger result of D. Vogt.

THEOREM 4. -

A nuclear Fréchet space E has a family of smoothing operators satisfying (3) if and only if E is isomorphic to a coordinate subspace of $C^\infty(T)$.

Thus we see that Theorem 3 as presently formulated is not applicable to a very wide class of Fréchet spaces. It turns out, however, that if we look a little more closely at how the smoothing operators can be defined, the relation (3) can be derived. If we do this for $C^\infty(T)$ or $H(\mathbb{C})$ we get (3), but if we do it for another space, say $H(\mathbb{D})$, we get a different relation which can then be used to prove an inverse function theorem valid for $H(\mathbb{D})$.

There is a method of constructing the smoothing operators. We begin with a nuclear Fréchet space $K(a)$ given in a coordinate representation. If (e_n) is the usual sequence of sequences which are 1 at the n^{th} coordinate and 0 elsewhere, then each $\xi = (\xi_n)$ can be written $\xi = \sum_n \xi_n e_n$, where this series converges in the topology of $K(a)$. Then we define $S_\theta : K(a) \to K(a)$, $\theta > 0$, by

$$S_\theta \xi = \sum_{n \leq \theta} \xi_n e_n$$

and we may calculate for, $k \leq j$,

$$\| S_\theta \xi \|_j = \sup_{n \leq \theta} |\xi_n| \, a_n^j = \sup_{n \leq \theta} |\xi_n| \, a_n^k \, \frac{a_n^j}{a_n^k} \leq \left(\sup_{n \leq \theta} \frac{a_n^j}{a_n^k}\right) \| \xi \|_k \, .$$

A similar calculation for $\xi - S_\theta \xi$ yields

$$\| \xi - S_\theta \xi \|_k \leq \left(\sup_{n > \theta} \frac{a_n^k}{a_n^j}\right) \| \xi \|_j \, .$$

If our space is $C^\infty(T)$ then we can take $a_n^k = n^k$, so we obtain exactly (3).

If the space is $H(\mathbb{C})$ then (3) still holds since $H(\mathbb{C})$ is a coordinate subspace of $C^\infty(T)$. If we use the above derivation for $H(\mathbb{C})$, we get a different inequality which could still be used to prove Theorem 3. Actually, it is not so different. We get the same inequality as in (3) except that θ is replaced by e^θ. Thus if we replace S_θ by $S_{\log\theta}$ we get a family of smoothing operators for $H(\mathbb{C})$ satisfying (3).

For $H(\mathbb{D})$ however, this is impossible as we have seen. The inequalities we get are

$$\| S_\Theta x\|_j \leq C\, e^{(\frac{1}{k} - \frac{1}{j})\Theta} \| x\|_k$$

$$\| x - S_\Theta x\|_k \leq C\, e^{(\frac{1}{j} - \frac{1}{k})\Theta} \| x\|_j \ .$$

Again replacing Θ by $\log\Theta$ we have our condition as follows. For $k \leq j$, $x \in E$, $\Theta > 0$ and an appropriate constant C which depends only on k,j but not on x,Θ there exists a family of smoothing operators S_Θ such that

$$\| S_\Theta x\|_j \leq C\Theta^{(\frac{1}{k} - \frac{1}{j})} \| x\|_k$$

$$\| x - S_\Theta x\|_k \leq C\, \Theta^{(\frac{1}{j} - \frac{1}{k})} \| x\|_j \ . \tag{3'}$$

Thus we see that $H(\mathbb{D})$ does have a family of smoothing operators satisfying (3').

The ultimate goal then would be to find for each $K(a)$ a relation like (3) or (3') and then use it to prove a theorem like Theorem 2. This turns out to be not so easy. So far I have only been able to do this for (3') and, although the proof is not given so we cannot see the actual difficulties as they arise, a clue will be provided by the form of the statement of Theorem 2' and how it differs from Theorem 2.

THEOREM 2'. -

Let E be a Fréchet space which has a family of smoothing operators which satisfy (3') and let $f : U \to E$ be a continuous function on a neighborhood of 0, U in E which has a derivative at each point in U.

Let $f(0) = 0$ and assume that there exist $d_0, d_1 > 0$ and strictly increasing unbounded functions $\alpha, \lambda_0, \lambda_1 : [0,\infty) \to [0,\infty)$ such that, for all $k \geq 0$, we have constants $C_k > 0$ with

$$\| f(x)\|_k \leq C_k \| x\|_{\alpha(k)} \qquad (x \in U)$$

$$\| f'(x)v\|_{d_1} \leq C_o \| v\|_{d_o} \qquad (x\in U, v\in E)$$

$$\| f(x+v)-f(x)-f'(x)v\|_{d_1} \leq C_o \| v\|^2_{d_o} \qquad (x, x+v\in U)$$

Suppose further that $f'(x)$ *has a right inverse* $L(x)$ *for each* $x\in U$ *and*

$$\| L(x)y\|_k \leq C_k(\| x\|_{\lambda_{o(k)}} \| y\|_{d_1} + \| y\|_{\lambda_{1(k)}}) \qquad (x\in U, y\in E).$$

Finally we suppose the following relations hold,

$$\lambda_o(0) \leq d_o,\ \lambda_1(0) \leq d_1,\ d_o < \frac{1}{2},\quad \alpha(d_1) \leq d_o\ .$$

Then $f(U)$ *is a neighborhood of* 0.

There are important differences between this result and Theorem 2 which go beyond the difference between inequalities (3) and (3'). First observe that the restrictions on f' and its approximation by a difference (essentially a condition on f'') need only be made for a single norm here while in Theorem 2 it was on every norm and also the loss could only be from k to $k+d$. This relaxation is also present for the conditions on f and L. In Theorem 2 it could only be a constant linear loss of d while in Theorem 2' the loss (measured by $\alpha, \lambda_o, \lambda_1$) can have any gowth for large k but is mildly restricted for small k. Also in Theorem 2' there is the strange requirement that $d_o < \frac{1}{2}$.

These variations are direct consequences of the proofs. About half of the argument is the same for both theorems and probabily will work for a wide class of space $K(a)$. The other half is quite different and seems to reflect fundamental differences between spaces like $C^\infty(T)$ and $H(\mathbb{C})$ on the one hand and $H(\mathbb{D})$ on the other. This is very similar to other phenomena in the structure theory. It appears that these basic differences will render it unlikely that a unified proof valid for all spaces $K(a)$ can be constructed.

There remains a major consideration in comparing Theorems 2 and 2'. Although,

on the face of it, the hypotheses in Theorem 2' are, at least in some respects, weaker than those of Theorem 2, it is necessary to pin this down with examples. Thus we would want to have a function on $H(\mathbb{D})$ that satisfies the hypotheses of Theorem 2' but not of Theorem 2. Such examples have not yet been discovered and I would consider it to be a major question in this research.

On the other, unsuccessful attempts to find such examples have brought into focus other phenomena which turn out to be important in this and other contexts. I would like to close these notes with a brief explanation.

Perhaps the simplest example of a non-linear function is what we might call a binomial, which is defined as follows. Let $B : E\times E \to E$ be bilinear, symmetric and continuous. Then we define $f : E \to E$ by

$$f(x) = x + B(x,x).$$

We can calculate,

$$f'(x)v = \lim_{t\to\infty} \frac{x+t\ v+B(x+tv,x+tv)-x-B(x,x)}{t} = v+2B(x,v).$$

It is necessary to assume that $f'(x)$ is invertible. That is, for each x in a suitable neighborhood of 0, the operator $v \to v+2B(x,v)$ is invertible. Then we would try to find a B such that the hypotheses of Theorem 2' hold but those of Theorem 2 fail. Without going into details, I can say that one kind of calculation leads to the conclusion that B should satisfy the following condition, which we might call *separately bounded*:

There is a k_0 *such that for every* k *there is* $\sigma(k)$ *and* $C_k > 0$ *such that*

$$\|B(x,y)\|_k \le C_k \|x\|_{k_0} \|y\|_k \qquad (x,y\in E)$$

(and then, by symmetry, the same result would hold with x,y interchanged), but that B should *not* satisfy the following condition which we might call *jointly bounded*:

There is a k_0 *such that for every* k *there is* $C_k > 0$ *such that*

$$\| B(x,y) \|_k \leq C_k \| x \|_{k_o} \| y \|_{k_o} \qquad (x,y \in E).$$

However, we have the following somewhat surprising result from the structure theory:

THEOREM 5.

In $H(\mathbb{D})$ *and its coordinate subspaces, every continuous, symmetric, bilinear, separately bounded function is jointly bounded. In* $C^\infty(T)$ *and each of its coordinate subspaces, this statement is false.*

In the light of this result, I feel quite uncertain as to the exact reason for the difference between Theorem 2 and Theorem 2'. Is the latter in some sense stronger or are the hypotheses actually equivalent? Perhaps future research will explain the matter.

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