NON-LINEAR ANALYSIS AND THE STRUCTURE OF FRÉCHET SPACES

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Summary. *These notes try to give a flavor of some recent research which has elucidated some unexpected connections between two apparently unrelated topics in Functional Analysis, namely, inverse function theorems and the structure theory of Fréchet spaces.*

This article is based on a series of lectures which I gave in December, 1981 at Università degli Studi di Lecce by invitation of Prof. V.B.Moscatelli for whose hospitality I am very grateful.

In these notes I would like to be rather informal, trying to give a flavor of some recent research which has elucidated some unexpected connections between two apparently unrelated topics in Functional Analysis. For the many details which will be omitted, I refer to standard texts and/or the references at the end.

INTRODUCTION.

A Fréchet space is a complete metrizable locally convex space. We will consider some details later and the reader can consult [3] for a basic discussion of these spaces, but for now I would like to mention three function spaces which are examples: $C^\infty(T)$, $H(\mathbb{C})$, $H(\mathbb{D})$. They are, respectively: the space of infinitely differentiable functions on the unit circle with the topology of uniform convergence of each derivative, the space of functions analytic in the complex plane with the compact - open topology and the space of functions analytic in the open unit disk in the compact - open topology.

An interesting problem, which has connections to partial differential equations and other functional equations, arises from consideration of a function $F : U \to E$ where U is a neighborhood of 0 in a Fréchet space E and $f(0) = 0$. The question is: if y is "small enougn" can we always solve the equation $f(x) = y$? Putting it another way, we ask if $f(U)$ is again a neighborhood of 0 in E. As we will see,

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serious difficulties arise when we try to study this situation in the context of a general Fréchet space.

A totally different object of investigation is the structure of Fréchet spaces. There we consider a fixed space E and try to determine (up to isomorphism) all of its subspaces and quotient spaces. There are many other similar kinds of questions and this turns out to be rich area of study.

It is a little bit surprising that there are important connections between these two areas. These are being discovered in various current research activities and it is my main purpose in these lectures to describe some of them. Thus, the discussion will be divided into three parts: inverse function theorems, structure theory, and connections.

INVERSE FUNCTION THEOREMS.

We begin with $f : U \to E$ with $f(0) = 0$ and we want to solve $f(x) = y$ for small y. Of course, there are important related questions. Is the solution unique? Does it depend continuously on parameters? And so on. There are interesting things to say about such questions but, in these lectures, I will consider only the existence problem.

Our basic approach to solving $f(x) = y$ will be Newton's method. This works equally well when E is 1-dimensional, n-dimensional or even an infinite dimensional Banach space. The following picture describes the 1-dimensional situation but leads to formulas which work in the more general context: