

ON A SPECIAL ALGEBRA WITH TWO OPERATIONS

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In [S] J. Szép suggested the study of a special algebra; the aim of this work is to analyze such an algebra.

Let $S(\cdot, \times)$ be an algebra where " \cdot " is a group operation (instead of $a \cdot b$ we shall write ab) and " \times " is a semigroup operation with an idempotent element e ; moreover

$$(\alpha) \quad \forall a, b, c \in S : (a \times b)c = ac \times bc, \quad c(a \times b) = ca \times c^{-1}a$$

where c^{-1} is the inverse of c in $S(\cdot)$.

The following results hold:

1) $S(\times)$ is an idempotent semigroup. In fact, let 1 denote the unit element of $S(\cdot)$; since $e \times e = e$, we have $a = 1 \cdot a = ee^{-1}a = (e \times e)c^{-1}a = (1 \times 1)a = a \times a$ for every $a \in S$.

$$2) \quad \text{For every } a \in S : a^2 = a \cdot a = a(a \times a) = a^2 \times a^{-1}a = a^2 \times 1.$$

3) Let $G_1 = \{x \in S \mid x \times 1 = x\}$, $G_2 = \{x \in S \mid 1 \times x = x\}$; then the following result holds : $u \in G_1 \implies 1 \times u = 1$, $u \in G_2 \implies u \times 1 = u^2$.

$$\text{In fact } u \in G_1 \implies u \times 1 = u \implies u^{-1}(u \times 1) = u^{-1}u \implies 1 \times u = 1;$$

$$u \in G_2 \implies 1 \times u = u \implies u(1 \times u) = u^2 \implies u \times 1 = u^2.$$

4) For every $u \in S : u \in G_2 \implies u^2 = 1$. In fact (see 3), 1))

$$u \in G_2 \implies u^2 \times u = (u \times 1) \times u = u \times (1 \times u) = u \times u = u$$

and hence

$$u \in G_2 \implies u^2 \times u = u \implies (u \times 1)u = u \implies u^2 \cdot u = u \implies u^2 = 1.$$

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5) $G_1 \cap G_2 = \{1\}$ (see 3))

6) The sets G_1, G_2 are normal subgroup of $S(\cdot)$.

The following steps prove the assertion:

(i) if $u, v \in G_1$ then $u \times 1 = u$, $v \times 1 = v$ so $uv = (u \times 1)v = uv \times v = uv \times (v \times 1) = (u \times 1)v \times 1 = uv \times 1$ and hence $uv \in G_1$. (Similarly $u, v \in G_2 \implies uv \in G_2$)

(ii) $u \in G_1 \implies 1 \times u = 1 \implies (1 \times u)u^{-1} = u^{-1} \implies u^{-1} \times 1 = u^{-1} \implies u^{-1} \in G_1$;
 $u \in G_2 \implies u^{-1} \in G_2$ (see 4)).

(iii) For every $a \in G_1$, $b \in S: b^{-1}ab \times 1 = b^{-1}(a \times 1)b \times 1 = (b^{-1}a \times b)b \times 1 = (b^{-1}ab \times b^2) \times 1 = b^{-1}ab \times (b^2 \times 1) = b^{-1}ab \times b^2 = b^{-1}ab$ and hence $b^{-1}ab \in G_1$.

For every $a \in G_2$, $b \in S: b^{-1}ab = b^{-1}(1 \times a)b = (b^{-1} \times ba)b = 1 \times b a b \in G_2$.

7) $S = G_1 \cdot G_2$. In fact for $x \in S$ we consider the element $a = x \times 1 \in G_1$.

So it follows that $a = (1 \times x^{-1})x$, i.e. $1 \times x^{-1} = ax^{-1}$ and hence, since $1 \times x^{-1} \in G_2$, $x = (1 \times x^{-1})^{-1} \cdot a = a \cdot (1 \times x^{-1})^{-1} \in G_1 \cdot G_2$.

From 5), 6), 7) it follows that

8) $G = G_1 \otimes G_2$ (where \otimes denotes the direct product)

9) $1 \times g_1 g_2 = g_2$, $g_1 g_2 \times 1 = g_1$ where $g_1 \in G_1$, $g_2 \in G_2$. In fact, since

$$(g_1^{-1} g_2^{-1} \times 1)g_1 g_2 = (1 \times g_1 g_2) \in G_2, \text{ it follows } g_1^{-1} g_2^{-1} \times 1 \in g_1^{-1} G_2.$$

But $g_1^{-1} g_2^{-1} \times 1 \in G_1$ and so (by 5) $g_1^{-1} g_2^{-1} \times 1 = g_1^{-1}$. This means that

$$1 \times g_1 g_2 = (g_1^{-1} g_2^{-1} \times 1)g_1 g_2 = g_1^{-1} g_1 g_2 = g_2 \quad (\text{Likewise for } g_1 g_2 \times 1 = g_1).$$

10) If $a = g_1 g_2$, $b = h_1 h_2$, where $g_1, h_1 \in G_1, g_2, h_2 \in G_2$ then $a \times b = g_1 g_2 \times h_1 h_2 =$

$$= (1 \times h_1 h_2 g_2^{-1} g_1^{-1})g_1 g_2 = h_2 g_2^{-1} g_1 g_2 = h_2 g_1 = g_1 h_2.$$

So we have the following

THEOREM 1. For the algebra $S(\cdot, \times)$ the following properties hold

$$1) \quad S = G_1 \otimes G_2 \quad \text{and} \quad g^2 = 1 \quad \forall g \in G_2 .$$

$$2) \quad g_1 g_2 \times h_1 h_2 = g_1 h_2 \quad (g_1, h_1 \in G_1, \quad g_2, h_2 \in G_2) .$$

Conversely we can prove

THEOREM 2. Let the group $S(\cdot)$ be the direct product of two subgroups G_1, G_2 such that $g^2 = 1$ for every $g \in G_2$. A semigroup operation " \times " with an idempotent element exists in S such that

$$\forall a, b, c \in S : (a \times b) c = a c \times b c, \quad c(a \times b) = c a \times c^{-1} b .$$

Proof. A few calculations show that the required operation is defined as follows:

$$g_1 g_2 \times h_1 h_2 = g_1 h_2 \quad \text{for every} \quad g_1, h_1 \in G_1, \quad g_2, h_2 \in G_2 .$$

REMARK. Finally we observe that theorems analogous to theorems 1 and 2 can be proved if in place of (α) one has

$$(β) \quad \forall a, b, c \in S : (a \times b) c = a c \times b c \quad c(a \times b) = c^{-1} a \times c b$$

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- [S] J. Szép : *On a finite algebra with two operations.*
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