2 - FRAMES AND THE REPRESENTATION OF $T\mathbb{E}$.

In this section we are dealing with the first order derivatives of the frame and tangent spaces.

Frame velocity and jacobians.

1 The velocity of the frame is the vector field on $\mathbb{E}$ constituted by the velocities of the world-lines of the frame. Hence it is the first derivati- ve of the notion with respect to time. On the orther hand, the jacobians are the first derivatives with respect to event. We consider only free en- tities, for simplicity of notions, leaving to the reader to write them in the complete form.

DEFINITION.

a) The (FREE) VELOCITY-FUNDAMENTAL FORM - of $\mathbb{P}$ is the map

$$D_1\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}} .$$

The (FREE) VELOCITY-EULERIAN FORM - of $\mathbb{P}$ is the map

$$\bar{P} \equiv D_1\tilde{P} \circ j : \mathbb{E} \to \mathbb{E}$$

b) The (FREE) JACOBIAN-FUNDAMENTAL-EULERIAN FORM - of $\mathbb{P}$ is the map

$$D_2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}} .$$

The (FREE) JACOBIAN-EULERIAN-EULERIAN FORM - of $\mathbb{P}$ is the map

$$\hat{P} \equiv D_2\tilde{P} \circ j : \mathbb{E} \to \bar{\mathbb{E}}^* \times \bar{\mathbb{E}} .$$

The (FREE) SPATIAL JACOBIAN-FUNDAMENTAL- EULERIAN FORM - of $\mathbb{P}$ is the map

$$\check{P} \equiv \check{D}_2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{E}}$$

The (FREE) SPATIAL JACOBIAN-LAGRANGIAN-LAGRANGIAN FORM, RELATIVE TO THE INITAL TIME $\tau \in \mathbb{T}$ AND TO THE FINAL TIME $\tau' \in \mathbb{T}$, of $\mathbb{P}$ - is the map

$$\check{P}_{(\tau'\tau)} \equiv D\tilde{P}_{(\tau'\tau)} : \mathbb{S}_\tau \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} .$$

We will denote by

$$\bar{\mathbf{P}} : \mathbb{E} \to T\mathbb{E} \quad , \qquad \hat{\mathbf{P}} : T\mathbb{E} \to T\mathbb{E} \qquad \check{\mathbf{P}} : \mathbb{T} \times \check{T\mathbb{E}} \to \check{T\mathbb{E}}$$

the maps associated with $\bar{P}$, $\hat{P}$, $\check{P}$.

We will write

$$\check{x}_p \equiv \hat{\mathbf{P}}(x) \, , \qquad \forall\, x \in T\mathbb{E} \, ,$$

$$\check{u}_P \equiv \hat{P}(e)(u), \qquad \forall\, x \in T_e\mathbb{E} \, .$$

2 We get immediate important properties of these maps

PROPOSITION.

We have

a) $\underline{t} \circ D_1\tilde{P} = 1$

b) $\underline{t} \circ D_2\tilde{P} = 0$

Hence we can write

$$D_1\tilde{P} : \mathbb{T} \times \mathbb{E} \to \mathbb{U} \qquad , \qquad \bar{D}_2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{S}}$$

$$\bar{P} : \mathbb{E} \to \mathbb{U} \qquad \check{P} : E \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \quad , \qquad \hat{P} : \mathbb{E} \to \mathbb{E}^* \otimes \bar{\mathbb{S}}$$

Moreover, all the previous maps are expressible by $\tilde{P}$, $\bar{P}$ and $\check{P}$:

c) $D_1\tilde{P} = \bar{P} \circ \tilde{P}$;

e) $\hat{P} = id_{\bar{\mathbb{E}}} - t \otimes \bar{P}$,

hence $\ddot{P}$ is a projection operator $\bar{\mathbb{E}} \to \bar{\mathbb{S}}$;

d) $D_2\tilde{P} = \check{P} \circ (\hat{P} \circ \pi^2)$;

f) $\check{D}_2\tilde{P}_{\tau'}|\mathbb{S}_\tau = \check{P}_{(\tau',\tau)}$

We have also the group properties

g) $(\check{P}_{(\tau'',\tau')} \circ \tilde{P}_{(\tau',\tau)}) \circ \check{P}_{(\tau',\tau)} = \check{P}_{(\tau'',\tau)}$;

h) $$\check{P}_{(\tau,\tau)} = id_{\bar{\mathbb{S}}} \;;$$

hence $\check{P}_{(\tau',\tau)}$ preserves the orientation of $\mathbb{S}$

i) $$\det \check{P}_{(\tau',\tau)} > 0.$$

We have

e) $$\bar{P} = \delta x_0 \,, \qquad \hat{P} = Dx^i \otimes \delta x_i$$

PROOF.

a) and b) follow from (II,1,10 a), by derivation with respect to τ and e.

c) follows from (II,1,10 c), by derivation with respect to τ and taking $\sigma \equiv \tau$,

d) follows from (II,1,10 b), by derivation with respect to e.

e) follows from (II,1,10 c), by derivation with respect to e.

f) follows from definitions.

g) and h) follows from (II,1,6).

i) $\check{P}_{(\tau',\tau)}(e)$ is an isomorphism, hence $\det \check{P}_{(\tau',\tau)}(e) \neq 0$;

det $\check{P}_{(\tau,\tau)}(e) = 1$, for (h), and $\check{P}_{(\tau',\tau)}(e)$ is continuous with respect to τ', for (f) .

Representation of $T\mathbb{P}$.

3 In order to get the space $T\mathbb{P}$ handy, it is useful to regard it as a quotient. In this way we could view $T\mathbb{P}$ as a quotient space $T\mathbb{E}_{/\mathbb{P}}$. But a reduced representation by means of $\check{T}\mathbb{E}_{/\mathbb{P}}$ is more simple, for the equivalence classes have a unique representative for each time $\tau \in \mathbb{T}$.

PROPOSITION.

Let $v \in T\mathbb{P}$. Then

$$\mathbb{C}_v \equiv \check{T}p^{-1}(v) = (T_2P)_v\,(\mathbb{T}) \hookrightarrow \check{T}\mathbb{E}$$

is a C^∞ submanifold.

Then we get a partition of $\check{T}\mathbb{E}$, given by

$$\check{T}\mathbb{E} = \bigsqcup_{v \in T\mathbb{P}} \mathbb{C}_v ,$$

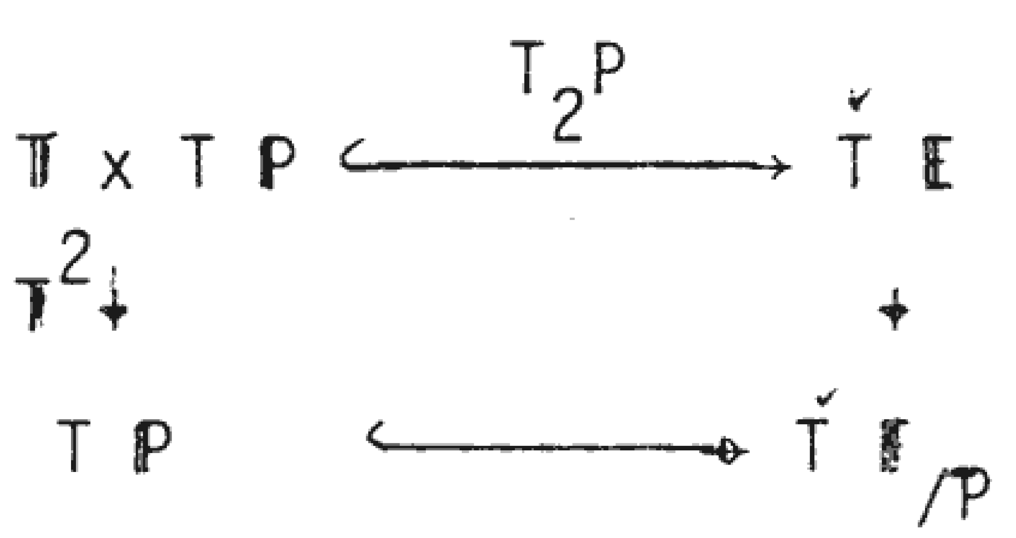
and the quotient space $\check{T}\mathbb{E}_{/\mathbb{P}}$, which has a natural C^∞ structure and whose equivalence classes are characterized by

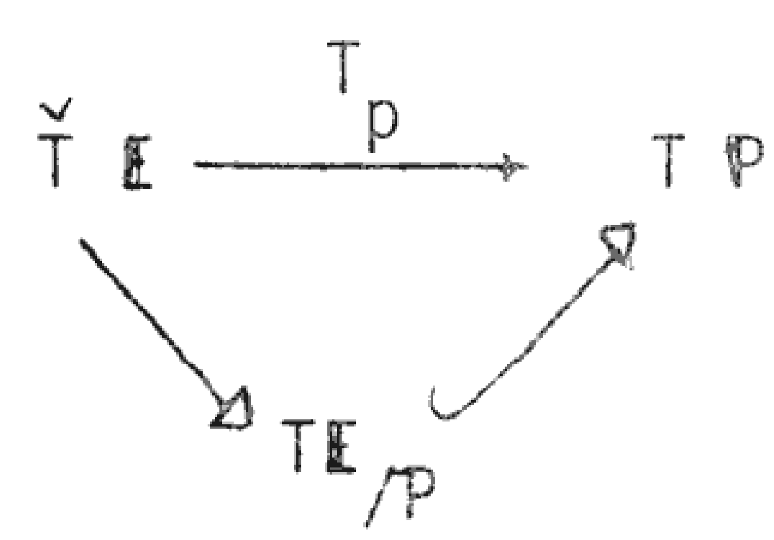
$$[e,u] = [e',u'] \iff p(e) = p(e')\ ,\ \check{P}(t(e'),e)(u) = u' . \qquad \text{(b)}$$

We get a natural C^∞ diffeomorphism between $T\mathbb{P}$ and $\check{T}\mathbb{E}_{/\mathbb{P}}$ given by the unique maps

$$T\mathbb{P} \to \check{T}\mathbb{E}_{/\mathbb{P}} \qquad \text{and} \qquad \check{T}\mathbb{E}_{/\mathbb{P}} \to T\mathbb{P} ,$$

which make commutative the two following diagrams, respectively,





PROOF.

(a) follows from (II,1,11).

(b) follows from

$$[e,u] = [e',u'] \iff (e',u') \in (T_2P)_{Tp(e,u)} \iff (e'u') = TP_{(t(e),t(e'))}(e,u).$$

The C^∞ structure on $T\mathbb{E}_{/\mathbb{P}}$ is induced by the charts adapted to $(\mathbb{T}_q)_{q \in \mathbb{P}}$.

We will often make the identification

$$T\mathbb{P} \cong \check{T}\mathbb{E}_{/\mathbb{P}} ,$$

which is very useful in calculations.

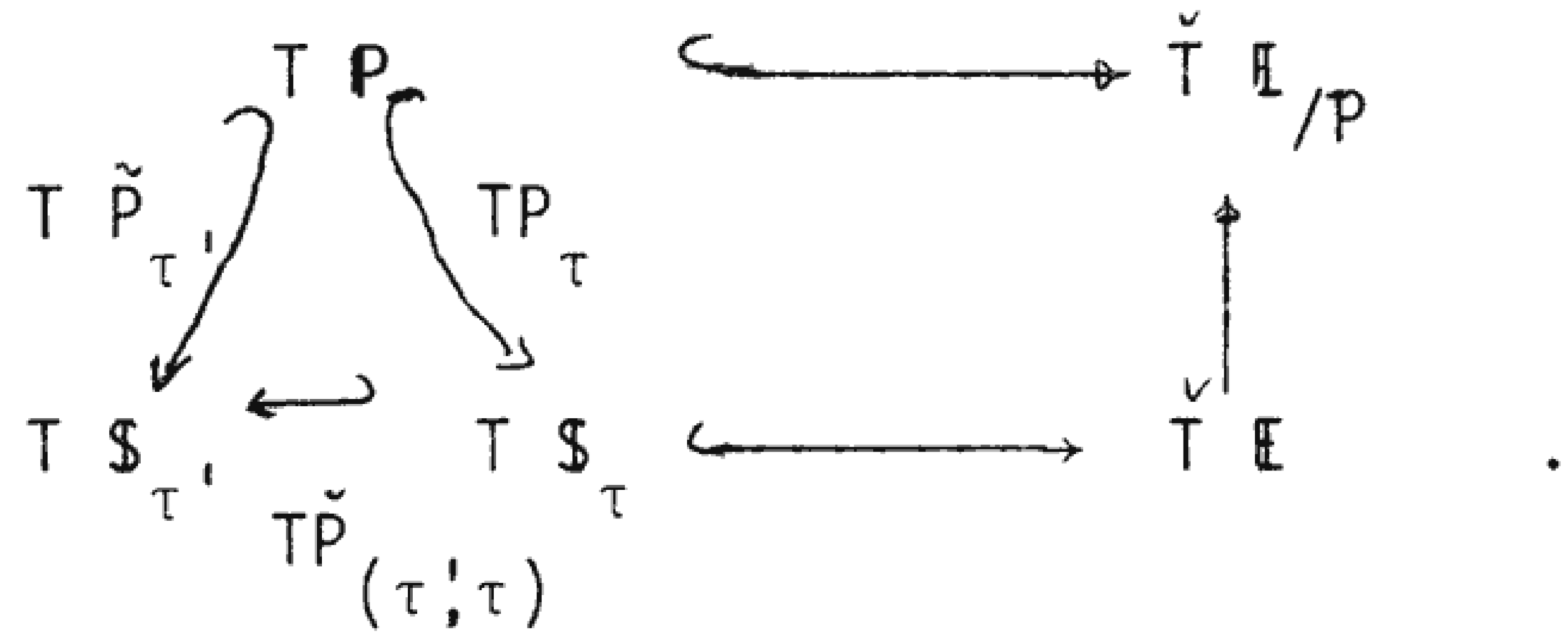
4 Choicing a time $\tau \in \mathbb{T}$ and taking, for each equivalence class, its re presentative at the time τ, we get a second interesting representation of $T\mathbb{P}$.

PROPOSITION.

The maps $\widetilde{TP}_\tau$ and Tp_τ are inverse C^∞ diffeomorphisms

$$\widetilde{TP}_\tau : T\mathbb{P} \to T\mathbb{S}_\tau \equiv T\mathbb{S}_\tau \equiv \mathbb{S}_\tau \times \bar{\mathbb{S}} , \quad T\, p_\tau : T\, \mathbb{S}_\tau \equiv \mathbb{S}_\tau \times \bar{\mathbb{S}} \to T\mathbb{P} \quad .$$

5 The relation between the different representations of $T\mathbb{P}$ is shown by the following commutative diagram



6 Taking into account the identification $T\mathbb{P} \cong T\mathbb{E}_{/P}$, we get the following expression of T_p and TP.

PROPOSITION.

a) $T\, p\, (e,u) = [e,\, \hat{P}(e)(u)]$

b) $T\, P\, (\tau,\lambda;[e,u]) = (\check{P}(\tau,e), \lambda\, \bar{P}(\tilde{P}(\tau,e) + \check{P}(\tau,e)(u))$

PROOF.

a) The following diagram is commutative

$$\begin{array}{ccccc} & & \overbrace{\hphantom{\qquad\qquad\qquad\qquad\qquad\qquad}}^{T_2 P} & & \\ \mathbb{T} \times T\mathbb{E} & \xrightarrow{id_{\mathbb{T}} \times Tp} & \mathbb{T} \times T\mathbb{P} & \xrightarrow{T_2 P} & \check{T}\mathbb{E} \\ \uparrow & & \downarrow & & \downarrow \\ T\mathbb{E} & \xrightarrow[Tp]{} & T\mathbb{P} & \xrightarrow[\cong]{} & \check{T}\mathbb{E}_{/\mathbb{P}} \end{array}$$

Hence we get

$$Tp(e,u) = (T_2\tilde{P})(t(e);e,u) \ .$$

b) The following diagram is commutative

$$\begin{array}{ccccc} T\mathbb{T} \times \check{T}\mathbb{E} & \hookrightarrow & T\mathbb{T} \times T\mathbb{E} & \xrightarrow{T\tilde{P}} & T\mathbb{E} \\ \downarrow & & \downarrow T(id_{\mathbb{T}} \times p) & \nearrow_{TP} & \\ T\mathbb{T} \times \check{T}\mathbb{E}_{/\mathbb{P}} & \xrightarrow[\cong]{} & T\mathbb{T} \times T\mathbb{P} & & \end{array}$$

Hence, we get

$$TP(\tau,\lambda;\ [e,u]) = T\check{P}(\tau,\lambda;\ TP_{(\tau,t(e))}(e,u)) \ .$$

<u>Frame vertical and horizontal spaces.</u>

8 The bundle $\Pi \equiv (\mathbb{E},p,\mathbb{P})$ induces two useful spaces.

DEFINITION.

The FRAME VERTICAL TANGENT SPACE is

$$\check{T}_p\mathbb{E} \equiv \text{Ker } Tp \hookrightarrow T\mathbb{E} \ .$$

The FRAME HORIZONTAL TANGENT SPACE, or FRAME PHASE SPACE is

$$\mathring{T}_p\mathbb{E} \equiv T\mathbb{E}_{/\check{T}_p\mathbb{E}} \ .$$

9 We have several representations of these spaces.

PROPOSITION.

a) We have $\check{T}_{\mathcal{P}}\mathbb{E} \equiv \mathrm{Ker}\ Tp = \mathrm{Im}\ T_1\mathcal{P}$.

Hence $\check{T}_{\mathcal{P}}\mathbb{E}$ is the subspace of $T\mathbb{E}$ generated by the velocity of $\mathcal{P}$

$$\check{T}_{\mathcal{P}}\mathbb{E} = \{(e,u)\in T\mathbb{E} \mid u = \lambda\,\bar{P}(e)\} .$$

$\check{T}_{\mathcal{P}}\mathbb{E}$ is the C^∞ submanifold of $T\mathbb{E}$ characterized by

$$\dot{x}^i = 0 .$$

Moreover, the maps

$$T_1 P : T\,\mathbb{T} \times \mathbb{P} \to \check{T}_{\mathcal{P}}\mathbb{E} \qquad (\tau,\lambda;q) \mapsto (\mathcal{P}(\tau,q),\lambda\bar{P}(P(\tau,q)))$$

and

$$(Tt,\check{p}) : \check{T}_{\mathcal{P}}\mathbb{E} \to T\,\mathbb{T} \times \mathbb{P} \qquad (e,\lambda\bar{P}(e)) \mapsto (t(e),\lambda;p(e))$$

are inverse C^∞ diffeomorphisms;

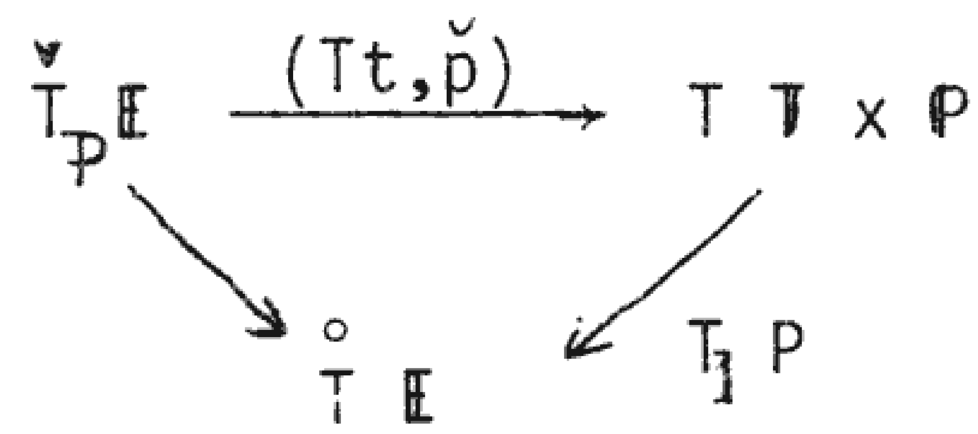
the maps

$$\check{T}_{\mathcal{P}}\mathbb{E} \to \mathring{T}\,\mathbb{E} \qquad \text{and} \qquad \mathring{T}\,\mathbb{E} \to \check{T}_{\mathcal{P}}\mathbb{E} ,$$

given by $(e,\lambda\bar{P}(e)) \mapsto (e,\lambda)$ and $(e,\lambda) \mapsto (e,\lambda\bar{P}(e))$

are inverse C^∞ diffeomorphisms;

the following diagram is commutative



b) The charts adapted to $\{\mathbb{T}_q\}_{q\in\mathbb{P}}$ induce a C^∞ atlas on $\mathring{T}_{\mathcal{P}}\mathbb{E}$.

Hence $\mathring{T}_{\mathcal{P}}\mathbb{E}$ is the space

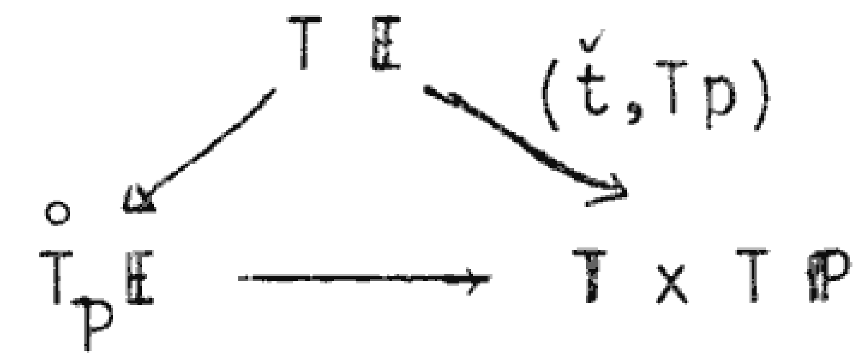
$$\mathring{T}_{\mathcal{P}}\mathbb{E} = \{[e,v]\}_{(e,v)\in T\mathbb{E}} = \{[e,u]\}_{(e,u)\in \check{T}\mathbb{E}} = \{(e,u + \check{T}_{\mathcal{P}e}\mathbb{E})\} .$$

Moreover, the maps

$$\mathbb{T} \times T\,\mathbb{P} \to \mathring{T}_{\mathcal{P}}\mathbb{E} \qquad \text{and} \qquad \mathring{T}_{\mathcal{P}}\mathbb{E} \to \mathbb{T} \times T\,\mathbb{P}$$

induced by the diagrams

$$\mathbb{T} \times T\,\mathbb{P} \xrightarrow{T_2P} T\,\mathbb{E} \to \mathring{T}_{\mathcal{P}}\mathbb{E} \qquad \text{and}$$



and given by

$$(\tau,[e,u]) \mapsto [\tilde{P}(\tau,e),\check{P}(\tau,e)(u)] \quad \text{and} \quad [e,v] \mapsto (t(e),[e,\hat{P}(e,(v)])$$

are inverse C^∞ diffeomorphisms;

the maps

$$\mathring{T}_{\mathcal{P}}\mathbb{E} \to \check{T}\,\mathbb{E} \qquad \text{and} \qquad \check{T}\,\mathbb{E} \to \mathring{T}_{\mathcal{P}}\mathbb{E}$$

given by $[e,v] \mapsto (e,\hat{P}(e)(v))$ and $(e,u) \mapsto [e,u]$

are inverse C^∞ diffeomorphisms;

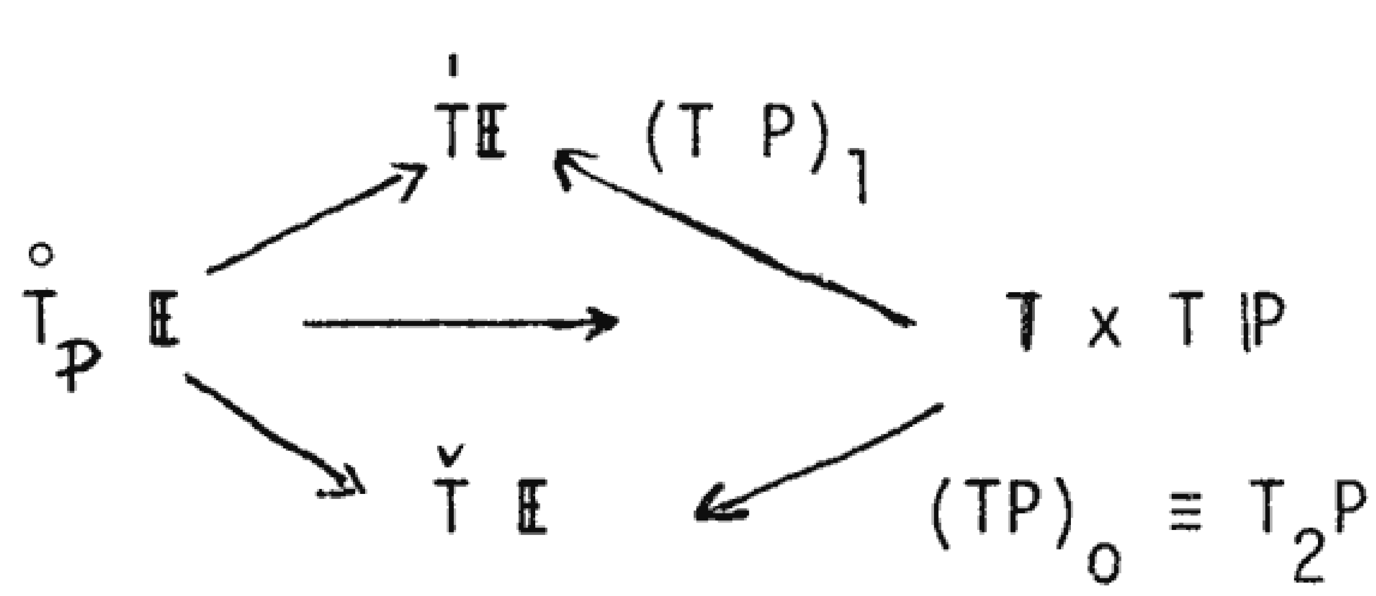
the maps

$$\mathring{T}_{\mathcal{P}}\mathbb{E} \to \dot{T}\,\mathbb{E} \qquad \text{and} \qquad \dot{T}\,\mathbb{E} \to \mathring{T}_{\mathcal{P}}\mathbb{E}$$

given by $[e,v] \to (e,\hat{P}(e)(v)+\bar{P}(e))$ and $(e,w) \mapsto [e,w]$

are inverse C^∞ diffeomorphisms;

the following diagram is commutative



We will often make the identifications

$$\check{T}_{\mathcal{P}}\mathbb{E} \cong T\,\mathbb{T} \times \mathbb{P} \cong \overset{\circ}{T}\,\mathbb{E}$$

$$\overset{\circ}{T}_{\mathcal{P}}\mathbb{E} \cong \mathbb{T} \times T\,\mathbb{P} \cong \check{T}\,\mathbb{E} \cong \overset{\prime}{T}\,\mathbb{E}$$

Frame metric function.

10 We get a "time depending" Riemannian structure on $\mathbb{P}$, induced by the family of diffeomorphisms

$$T\,\mathbb{P} \longrightarrow T\,\mathbb{S}_{\tau}$$

DEFINITION.

The FRAME TIME DEPENDING METRIC FUNCTION is the function

$$g_{\mathcal{P}} : \mathbb{T} \times T\,\mathbb{P} \longrightarrow \mathbb{R}$$

given by the composition

$$\mathbb{T} \times T\,\mathbb{P} \to \check{T}\,\mathbb{E} \xrightarrow{g} \mathbb{R}\,,$$

i.e.

$$g_{\mathcal{P}}(\tau, [e,u]) \equiv \frac{1}{2}(\check{P}(\tau,e)(u))^2 \qquad \dot{-}$$

Taking into account $\mathbb{T} \times T\,\mathbb{P} \cong \mathbb{V}$, we will write also

$$g_{\mathcal{P}} : \mathbb{V} \longrightarrow \mathbb{R}\ .$$

11 PROPOSITION.

We have
$$g_{\mathcal{P}} = \frac{1}{2}\, g_{ij}\, \dot{x}^i \dot{x}^j \qquad \dot{-}$$

Representation of $T\mathbb{E}$.

12 Most of the previous results can be sumarized in the following fundamental theorem, which gives the representation of $T\mathbb{E}$ induced by the frame.

THEOREM.

The map $\qquad T\mathbb{E} \to \mathring{T}_P\mathbb{E} \times_{\mathbb{E}} \mathring{T}\mathbb{E}$,

given by the natural projections, is a C^∞ diffeomorphism.

The map $\qquad \check{T}\mathbb{E} \oplus_{\mathbb{E}} \check{T}_P\mathbb{E} \to T\mathbb{E}$,

given by the natural inclusions, is a C^∞ diffeomorphism.

The maps

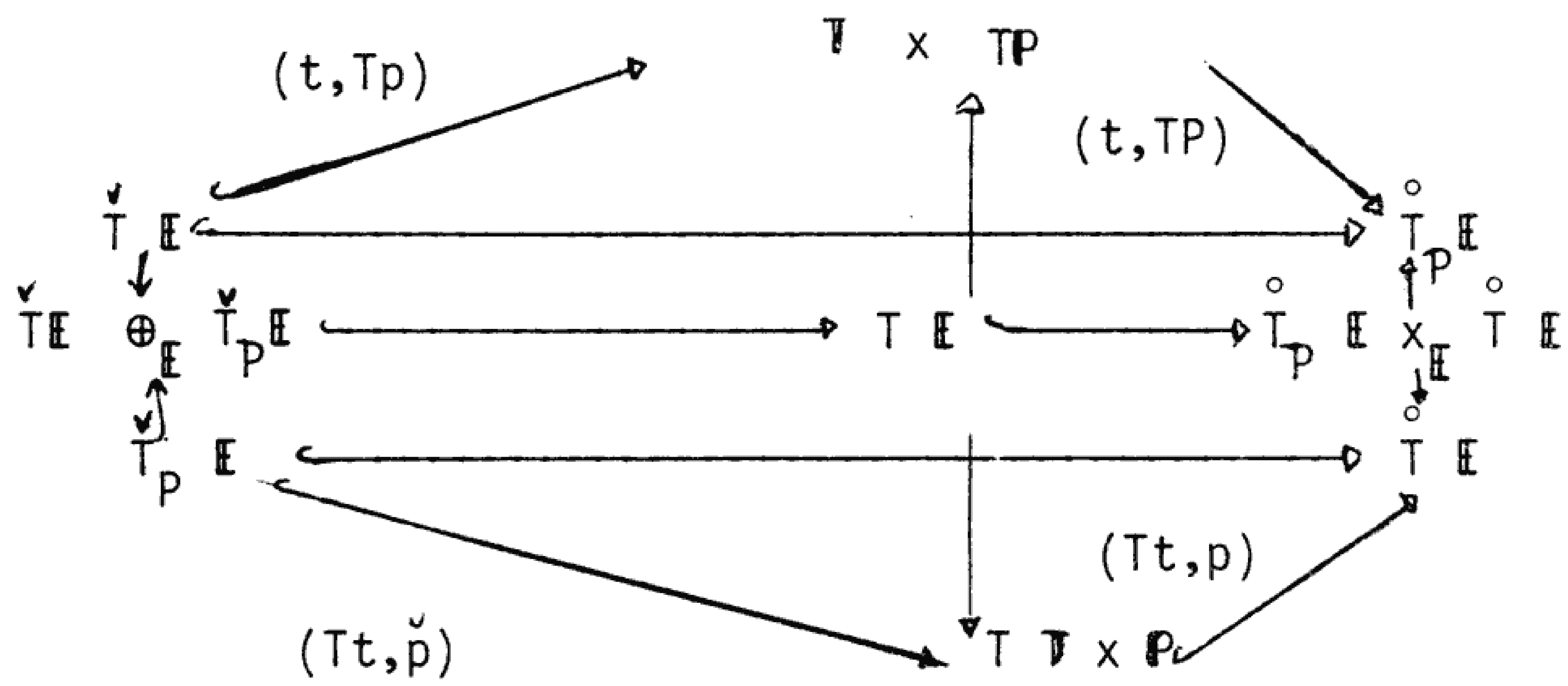
$t(t,p) : T\mathbb{E} \to T\mathbb{T} \times T\mathbb{P}$ $\qquad$ and $\qquad$ $TP : T\mathbb{T} \times T\mathbb{P} \to T\mathbb{E}$

are inverse C^∞ diffeomorphisms.

Moreover we have the C^∞ diffeomorphisms

$\mathring{T}_P\mathbb{E} \to \mathbb{T} \times TP \to T\mathbb{E}$ $\qquad$ and $\qquad$ $\mathring{T}\mathbb{E} \to T\,T \times P \to T_P\mathbb{E}$.

Hence, the relation among the previous three representations of $T\mathbb{E}$ is given by the following commutative diagram



The maps

$$T\mathbb{E} \to \check{T}\mathbb{E} \to \mathbb{T} \times T\mathbb{P} \to \mathring{T}_P\mathbb{E}$$

are given by

$$(e,u) \mapsto (e,\hat{P}(e)(u)) \mapsto (t(e),[e,\hat{P}(e)(u)]) \mapsto [e,u]$$

The map

$$T\mathbb{E} \to \check{T}_{\mathcal{P}}\mathbb{E} \to T\,\mathbb{T} \times P \to \overset{\circ}{T}\,\mathbb{E}$$

are given by

$$(e,u) \to (e,u^\circ\bar{P}(e)) \to (t(e),u^\circ,[e]) \to (e,u^\circ) \quad .$$

The choice of the most convenient representation depends on circumstances. Some of these have a theoretical relevance, other a computational advantage. So, for esplicite calculations, we will generally use the following identifications

$$T\,\mathbb{P} \cong \check{T}\mathbb{E}_{/\mathcal{P}} \quad , \qquad \text{and} \qquad \mathbb{T} \times T\,\mathbb{P} \cong T_{\mathcal{P}}\mathbb{E} \quad .$$

Notice that in the decomposition of the vector field $x : \mathbb{E} \to T\mathbb{E}$

$$x = x^\circ\,\bar{P} + \check{x}_{\mathcal{P}} : E \to \check{T}_P\mathbb{E} + \check{T}\,\mathbb{E}$$

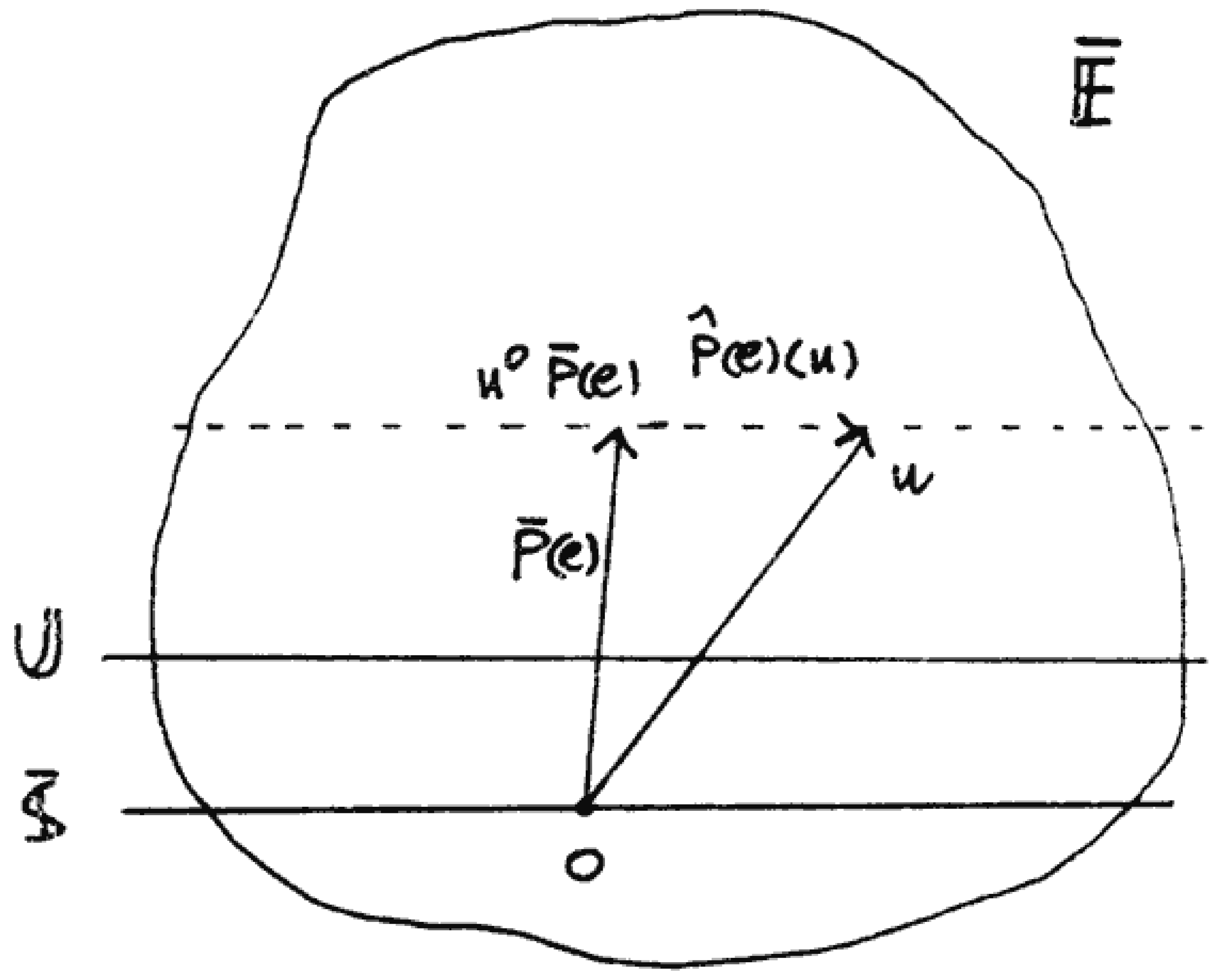
the component x° is absolute, but the space $\check{T}_{\mathcal{P}}\mathbb{E}$ is frame depending, and the space $\check{T}\mathbb{E}$ is absolute, but the component $\check{x}_P$ is frame depending.

Physical description.

$\bar{P}$ is the field of velocity the frame continuum. $\hat{P}$ is the spatial projection operator induced by the velocity and $\check{P}$ is the infinitesimal displacement generated by the continuum motion on spatial vectors.

We identify (at the first order) each vector of $T_q\mathbb{P}$ with a strip having as first side the world-line q and as second side another world-line.

We can describe the situation by a picture.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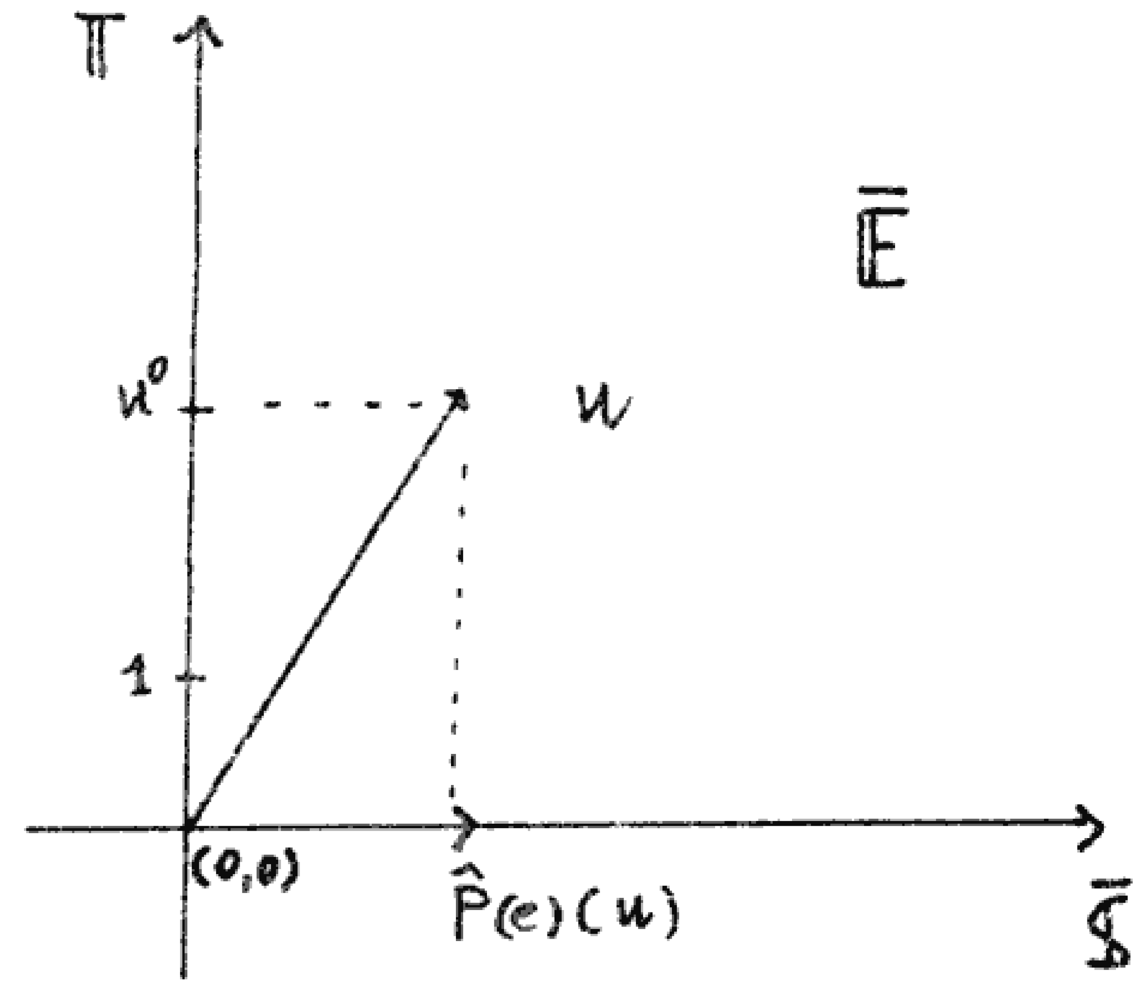
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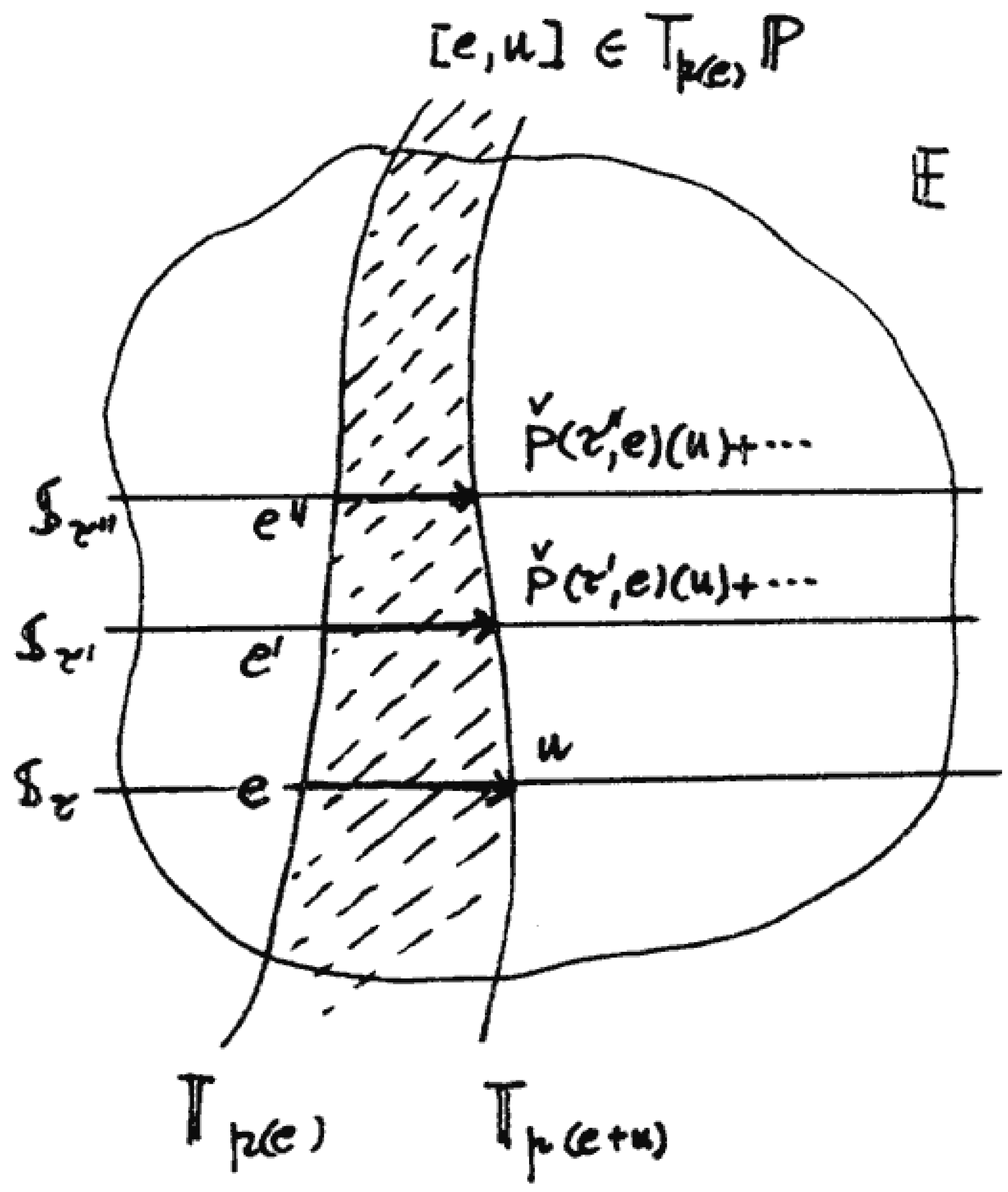
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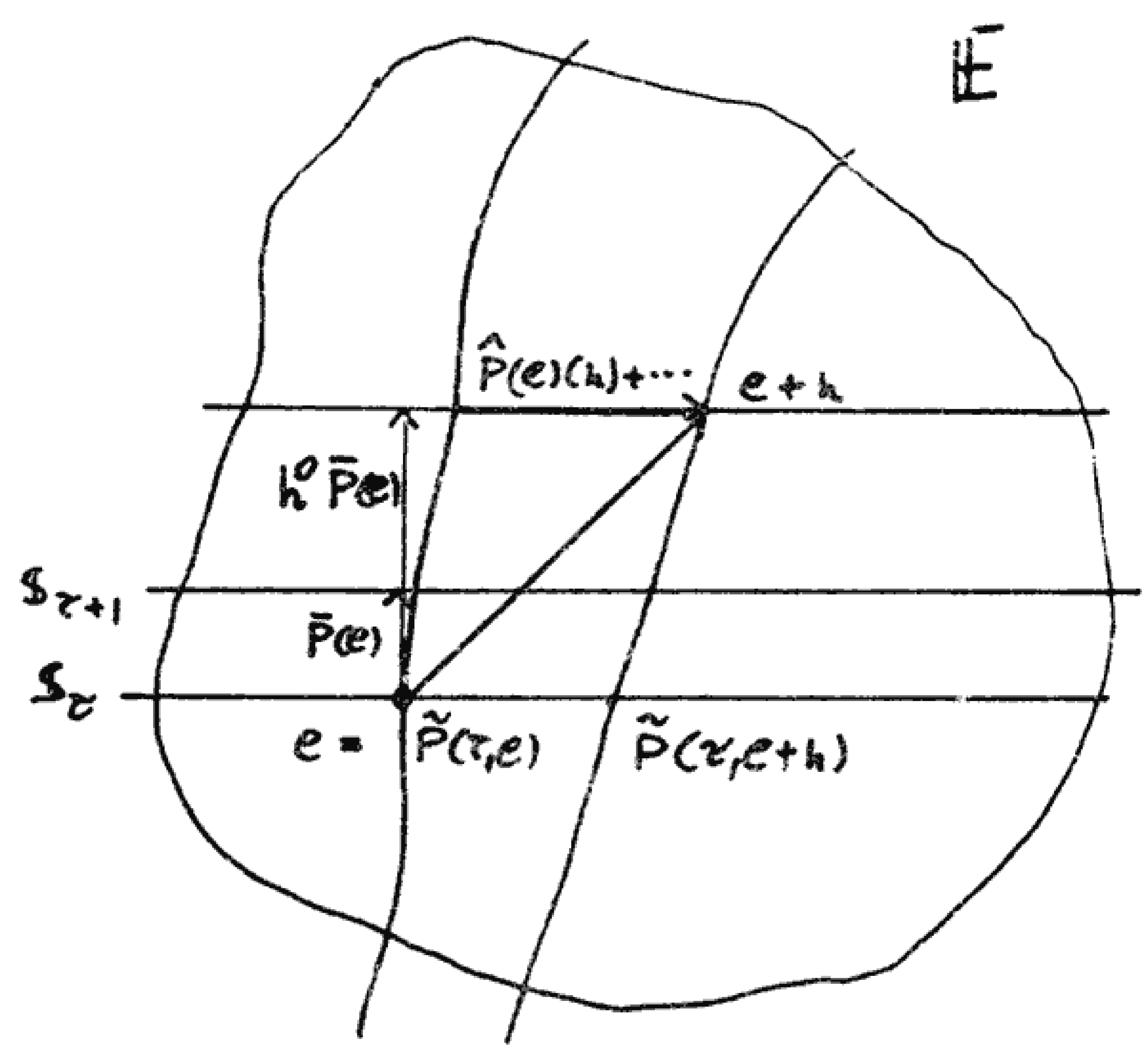
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