II CHAPTER

FRAMES OF REFERENCE

Here we study the absolute kinematics of a continuum, which, viewed as a frame of reference, determines positions, the splitting of event space into space-time and the consequent splitting of velocity space. We analyse the positions space and its structures as the time-depending metric, the time-depending affine connection and the Coriolis map. Finally we make a classification of frames.

I FRAMES AND THE REPRESENTATION OF $\mathbb{E}$.

Frames, positions and adapted charts.

1 The basic elements of observed kinematics are frames, constituted by a reference continuum, whose particles determine positions on $\mathbb{E}$.

For simplicity of notations, we consider only global frames, leaving to the reader the obvious generalization to local frames.

DEFINITION.

A FRAME (OF REFERENCE) is a couple

$$\mathcal{P} \equiv \{\mathbf{P}, \{\mathbf{T}_q\}_{q\in\mathbf{P}}\}$$

where $\mathbf{P}$ is a set and, $\forall\, q \in \mathbf{P}$, $\mathbf{T}_q$ is a world line, such that

a)(*) $$\mathbb{E} = \bigsqcup_{q\in\mathbf{P}} \mathbb{T}_q \;;$$

b) $\forall\, e \in \mathbb{E}$, there exists a neighbourhood U of e and a C^∞ chart

$$x \equiv \{x^\circ, x^i\} : U \to \mathbb{R} \times \mathbb{R}^3$$

adapted to the family of submanifolds $\{\mathbf{T}_q\}_{q\in\mathbf{P}}$.

$\mathbf{P}$ is the POSITION SPACE; each $q\in\mathbf{P}$ is a POSITION; the map

$$p : \mathbb{E} \to \mathbf{P}$$

given by $e \mapsto$ the unique $q\in\mathbf{P}$, such that $e \in \mathbf{T}_q$,

is the POSITION MAP; if $e \in \mathbb{E}$, then $p(e) \in \mathbf{P}$ is the POSITION of e .

(*) It sufficies to assume $E = \bigcup_{p\in P} \mathbb{T}_q$. In fact, taking into account b) and that each $\mathbb{T}_q$ is open and connected, we see that, if $q \neq q'$, then $\mathbb{T}_q \cap \mathbb{T}_{q'} = \emptyset$.

Henceforth we assume a frame $\mathcal{P}$ to be given.

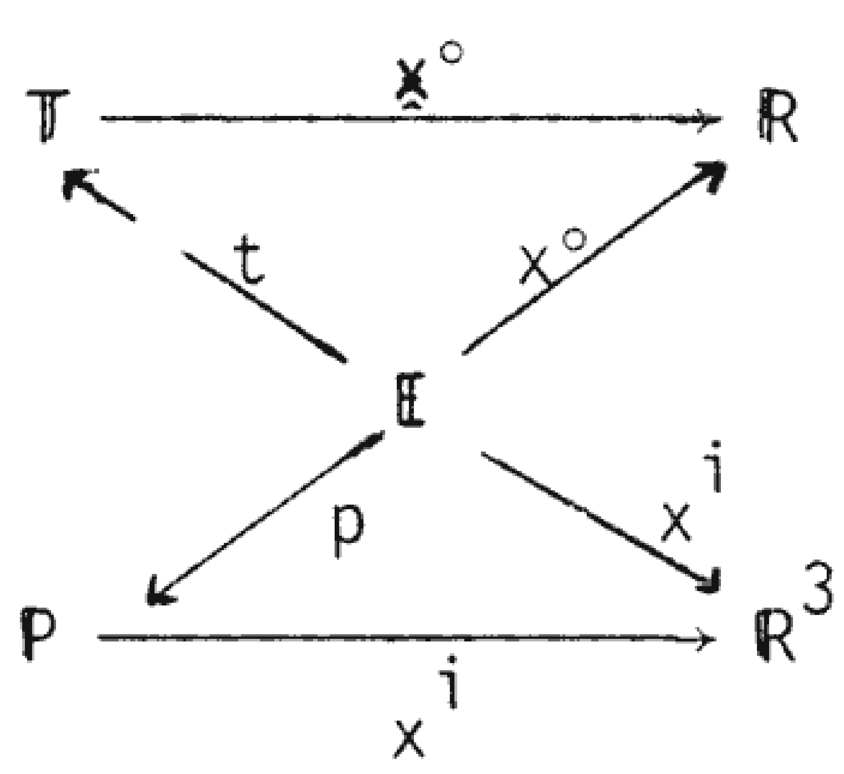
2 Calculations develop in an easier way if performed with respect to a chart adapted to $\mathcal{P}$. For simplicity of notations, we consider only global charts, leaving to the reader the obvious generalization to local charts, our considerations being essentially local.

DEFINITION.

A CHART ADAPTED TO $\mathcal{P}$ is a chart

$$\{x^{\circ}, x^{i}\} : \mathbb{E} \to \mathbb{R} \times \mathbb{R}^{3} ,$$

such that it is special and it factorizes through $\mathbb{P}$, i.e. such that the following diagram is commutative



,

where $\hat{x}^{\circ} : \mathbb{T} \to \mathbb{R}$ is a normal oriented cartesian chart
Charts adapted to $\mathcal{P}$ exist by definition 1.
Hencefort we assume a chart x adapted to $\mathcal{P}$ to be given.

<u>Representation of the position space</u> $\mathbb{P}$.

3 $\mathbb{P}$ results naturally into a C^{∞} manifold.

PROPOSITION.

There is a unique C^{∞} structure on $\mathbb{P}$, such that the map $p : \mathbb{E} \to \mathbb{P}$ is C^{∞} . Namely it is induced by the charts adapted to $\{\mathbb{T}_q\}_{q \in \mathbb{P}}$.

PROOF.

<u>Unicity</u>. If $y : V \subset \mathbb{P} \to \mathbb{R}^{3}$ is a chart which makes p C^{∞}

and if $x : U \subset \mathbb{E} \to \mathbb{R} \times \mathbb{R}^3$ is a C^∞ chart adapted to $\{\mathbb{T}_q\}_{q\in\mathbb{P}}$, then the map (defined locally)

$$\mathbb{R}^3 \hookrightarrow \mathbb{R} \times \mathbb{R}^3 \xrightarrow{x^{-1}} \mathbb{E} \xrightarrow{p} \mathbb{P} \xrightarrow{y} \mathbb{R}^3 ,$$

which is the change from x to y, is C^∞ .

<u>Existence</u>. The change of charts on $\mathbb{P}$ induced by charts adapted to $\{\mathbb{T}_q\}_{q\in\mathbb{P}}$ is C^∞ $\dot{-}$

4 We get a first immediate representation of $\mathbb{P}$.

The frame $\mathcal{P}$ determines a partition of $\mathbb{E}$ into the equivalence classes $\{\mathbb{T}_q\}_{q\in\mathbb{P}}$.

Then we get the natural identification of $\mathbb{P}$ with the quotient space $\mathbb{E}/\mathcal{P}$

$$\mathbb{P} \overset{\sim}{=} \mathbb{E}/\mathcal{P},$$

by writing $$q \overset{\sim}{=} p^{-1}(q) \equiv \mathbb{T}_q$$

and $$[e] = q = [e'] \iff p(e) = q = p(e') .$$

We will ofter identify $\mathbb{P}$ and $\mathbb{E}/\mathcal{P}$.

5 Choicing a time $\tau \in \mathbb{T}$ and taking, for each equivalence class, its representative, at the time τ, we get a second interesting representation of $\mathcal{P}$.

For this purpose, let us introduce three maps related with $\mathcal{P}$.

DEFINITION.

Let $\tau, \tau' \in \mathbb{T}$.

Then we define the three maps

a) $P_\tau : \mathbb{P} \to \mathbb{S}_\tau$

given by $q \mapsto$ the unique $e \in \mathbb{S}_\tau \cap \mathbb{T}_q$;

b) $p_\tau \equiv p_{|\mathbb{S}_\tau} : \mathbb{S}_\tau \to \mathbb{P}$;

c) $\tilde{P}_{(\tau',\tau)} \equiv P_{\tau'} \circ p_\tau : \mathbb{S}_\tau \to \mathbb{S}_{\tau'}$.

6 Then we see that $\mathbb{P}$ is diffeomorphic (not canonically) to a 3-dimensional affine space.

PROPOSITION.

The maps $\tilde{P}_\tau$ and p_τ are inverse C^∞ diffeomorphisms:

$$\tilde{P}_\tau : \mathbb{P} \to \mathbb{S}_\tau \quad , \quad p_\tau : \mathbb{S}_\tau \to \mathbb{P} .$$

Moreover we have

$$\tilde{P}_{(\tau'',\tau')} \circ \tilde{P}_{(\tau',\tau)} = P_{(\tau'',\tau)}$$

and

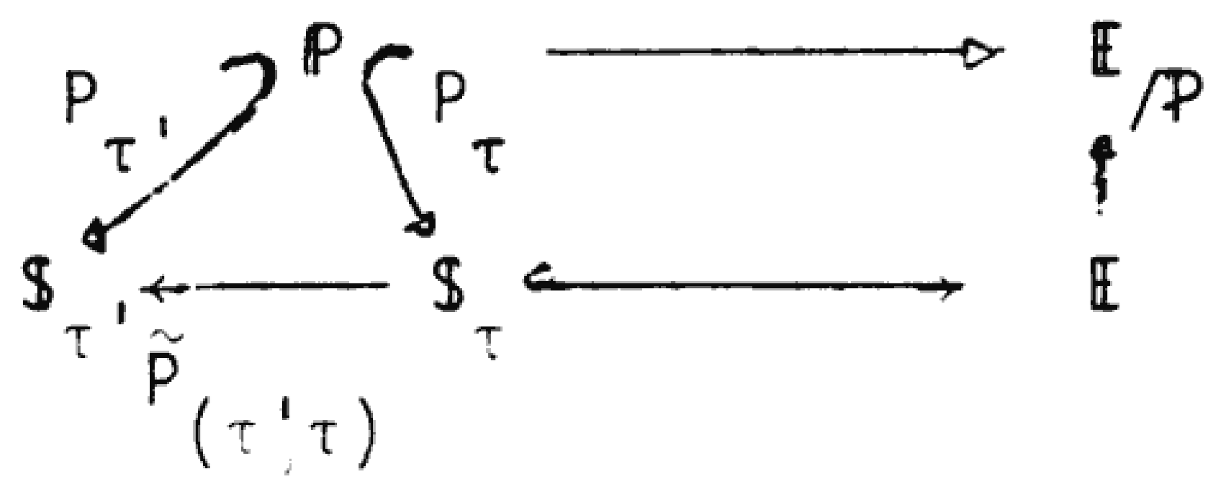
$$\tilde{P}_{(\tau,\tau)} = id_{\mathbb{S}_\tau}$$

hence $\tilde{P}_{(\tau',\tau)}$ is a C^∞ diffeomorphism .

PROOF.

P_τ and p_τ ave inverse bijections. Moreover, p_τ , which is the composition $\mathbb{S}_\tau \hookrightarrow \mathbb{E} \to \mathbb{P}$, is C^∞ and det $DP_\tau = \det(\partial y_i \circ \hat{x}^j \circ p_\tau) \neq 0$, where y is a special chart .

7 The relation among the different representation of $\mathbb{P}$ is shown by the following commutative diagram



<u>Frame motion.</u>

8 We need a further map given by the motions associated to the world lines of $\mathcal{P}$.

DEFINITION.

The MOTION of $\mathcal{P}$ is the map

$$P : \mathbb{T} \times \mathbb{P} \to \mathbb{E}$$

given by $$(\tau, q) \mapsto \text{the unique } e \in \mathbb{S}_\tau \cap \mathbb{T}_q$$ $\dot{-}$

Thus P is the union of the family of maps $\{P_\tau\}_{\tau \in \mathbb{T}}$ previously introduced; on the other hand, P is the union of the family of maps $\{P_q\}_{q \in \mathbb{P}}$, constituted by the motions associated with the world-lines of $\mathcal{P}$. The motion P characterizes the frame $\mathcal{P}$.

9 For calculations it is more advantageous a further map, substantially equivalent to P, which relates affine spaces.

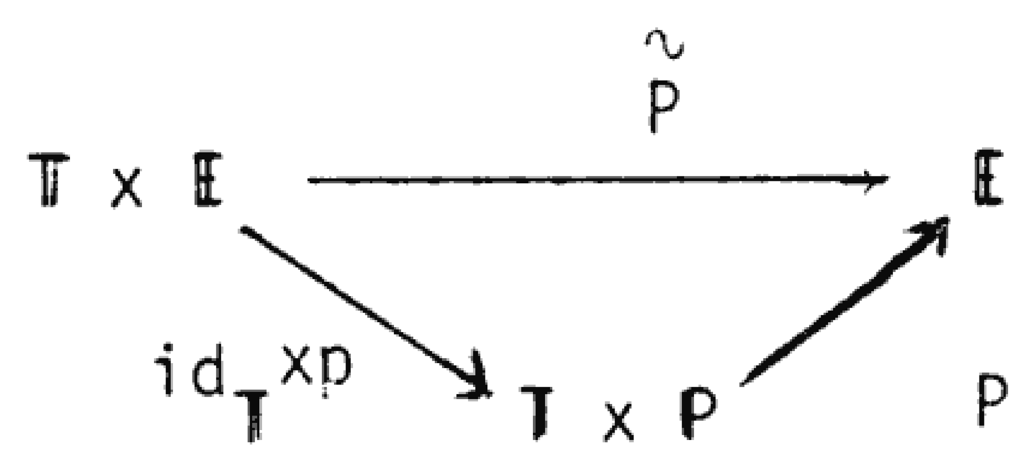
DEFINITION.

We define the map

$$\tilde{P} \equiv P \circ (id_{\mathbb{T}} \times p) : \mathbb{T} \times \mathbb{E} \to \mathbb{E} \,,$$

given by $$(\tau, e) \mapsto P(\tau, p(e))$$ $\dot{-}$

Thus the following diagram is commutative by definition



10 The following immediate formulas will be used in calculations.

PROPOSITION.

We have

a) $t(\tilde{P}(\tau,e)) = \tau$, i.e. $t \circ \tilde{P} = id_{\mathbb{T}}$;

b) $\tilde{P}(t(e),e) = e$, i.e. $P \circ j = id_{\mathbb{E}}$;

c) $\tilde{P}(\tau,\tilde{P}(\sigma,e)) = \tilde{P}(\tau,e)$.

$\tilde{P}$ characterizes the frame $\mathbf{P}$.

We have $x^{\circ} \circ \tilde{P} = \underset{\wedge}{x}^{\circ}$, $x^{i} \circ \tilde{P} = x^{i}$.

<u>Rappresentation of $\mathbb{E}$</u> .

11 The frame $\mathbf{P}$ determines the splitting of the event space in space-time.

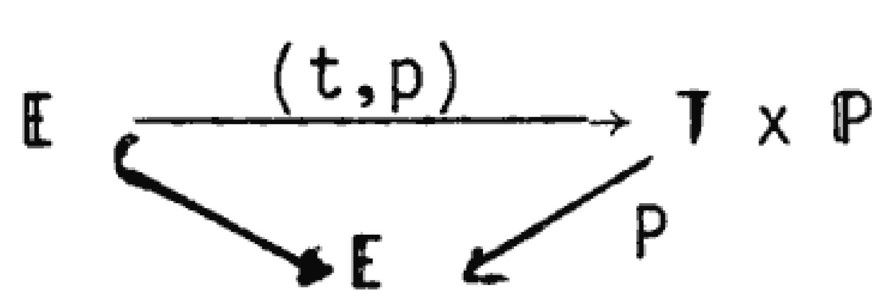
THEOREM.

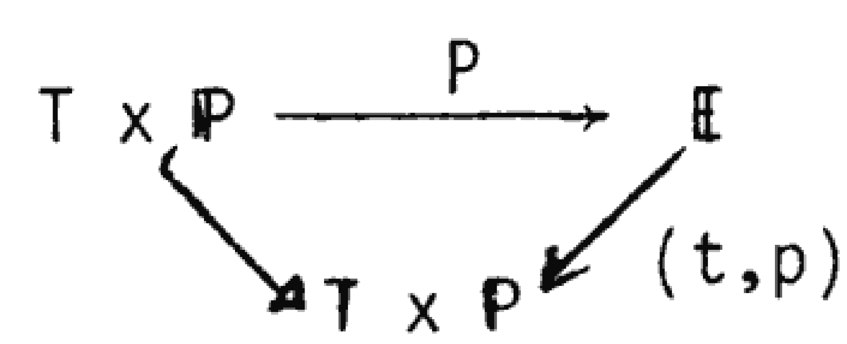
The maps

$(t,p) : \mathbb{E} \to \mathbf{T} \times \mathbf{P}$ and $P : \mathbf{T} \times \mathbb{P} \to \mathbb{E}$

are inverse C^{∞} diffeomorphisms.

Namely the following diagrams are commutative





Hence $(\mathbb{E},p,\mathbf{P})$ results into a C^{∞} bundle, with fiber $\mathbf{T}$.

PROOF.

P and (t,p) are inverse bijections. Moreover (t,p) is C^∞ and $\det D(t,p) \neq 0$.

12 DEFINITION.

The FRAME BUNDLE is

$$\equiv (\mathbb{E},p,\mathbb{P}) \quad .$$

Thus we have two bundle structures on $\mathbb{E}$, namely

$\eta \equiv (\mathbb{E},t,\mathbb{T})$, wich has an absolute basis $\mathbb{T}$ and a non canonical fiber diffeomorfic to $\mathbb{P}$ or to $\mathbb{S}_\tau$, $\forall\ \tau\in\mathbb{T}$,

$\pi \equiv (\mathbb{E},p,\mathbb{P})$, wich has a frame depending basis $\mathbb{P}$, diffeomorfic to $\mathbb{S}_\tau$, $\forall\ \tau\in\mathbb{T}$, and an absolute fiber $\mathbb{T}$.

The frame bundle π characterizes the frame $\mathcal{P}$.

Physical description.

A frame $\mathcal{P}$ is a set $\mathbb{P}$ of particles, never meeting, filling, at each time $\tau\in\mathbb{T}$, the whole space $\mathbb{S}_\tau$, with a C^∞ flow, hence first a frame is a continuum and we study the absolute kinematics of its particles.

Such a continuum can be viewed as a frame of reference. In fact it determines a partition of $\mathbb{E}$ in positions. Each position in the setof all events touched by the same frame particle. Under this aspect we can identify the set of positions with the setof particles $\mathbb{P}$.

We can describe the frame, its motion, the positions and the splitting of $\mathbb{E}$ into the space-time $\mathbb{T} \times \mathbb{P}$, by a picture. Notice that we can consider only the differentiable properties induced by the paper to $\mathbb{P}$, in the picture : in fact the affine and metrical properties of $\mathbb{P}$ are time depending.