b/ for $k=1,2,\dots,n$ the functions φ_k are continuous and for fixed $x_1,\dots,x_{k-1}, x_{k+1},\dots,x_n$ they are concave in x_k. Under these assumptions the game has at least one equilibrium point.

Proof. See J.B. Rosen [8] . ■

Remark. If we assume that the functions φ_k are strictly concave in x_k, then the uniqueness of the equilibrium point in general is not true /see Example 4./. For the uniqueness of the equilibrium point of n-person games J.B. Rosen [8] gave sufficient conditions, but the assumptions of the next paragraphs are independent of the conditions introduced by J.B. Rosen.

2. The solution of a special class of concave games

Let us assume that for $k=1,2,\dots,n$

$$X_k = \left\{ \underline{x}_k \mid \underline{x}_k \in R^{m_k} , \ \underline{h}_k(\underline{x}_k) \geqq \underline{0} \right\},$$

where

a/ $\mathcal{D}(\underline{h}_k) = R^{m_k}$, $\mathcal{R}(\underline{h}_k) \subset R^{\ell_k}$, the components of $\underline{h}_k$ are concave, continuously differentiable functions;

b/ X_k is bounded, and in each point of X_k the Kuhn-Tucker regularity condition holds /see G. Hadley [3]/ ;

c/ φ_k is continuous, concave in $\underline{x}_k$ for fixed $\underline{x}_1,\dots,\dots,\underline{x}_{k-1}, \underline{x}_{k+1},\dots,\underline{x}_n$ and continuously differentiable with respect to $\underline{x}_k$.

Lemma 4. The game $\Gamma = (n; X_1,\dots,X_n ; \varphi_1,\dots,\varphi_n)$ has at least one equilibrium point.

Proof. It is obvious that all conditions of the Nikaido--Isoda theorem are satisfied. ■

Let $k=1,2,\dots,n$ and for fixed strategy vector $\underline{x}^* = (x_1^*, x_2^*,\dots,x_n^*)$ consider the mathematical programming problem

$$\frac{\underline{h}_k(\underline{x}_k) \geqq \underline{0}}{\varphi_k(\underline{x}_1^*,\dots,\underline{x}_k,\dots,\underline{x}_n^*) \rightarrow \max .} \tag{8}$$

Lemma 5. A vector $\underline{x}^* = (\underline{x}_1^*,\dots,\underline{x}_n^*)$ is an equilibrium point if and only if for $k=1,2,\dots,n$ $\underline{x}_k^*$ is an optimal solution of the problem (8) .

Proof. a/ If $\underline{x}_k^*$ is a feasible solution, then the constraint implies that $\underline{x}_k^* \in X_k$, thus $\underline{x}^* = (\underline{x}_1^*,\dots,\underline{x}_n^*)$ is a strategy vector. If $\underline{x}_k^*$ is an optimal solution, then for any feasible solution $\underline{x}_k \in X_k$, $\varphi_k(\underline{x}_1^*,\dots,\underline{x}_k^*,\dots,\underline{x}_n^*) \geqq \varphi_k(\underline{x}_1^*,\dots,\underline{x}_k,\dots,\underline{x}_n^*)$. Thus $\underline{x}^*$ is an equilibrium point.

b/ If $\underline{x}^*$ is an equilibrium point, then inequalities (1) imply that the components $\underline{x}_k^*$ are optimal solutions of the problems (8) . ■

Lemma 6. A vector $\underline{x}^* = (\underline{x}_1^*,\dots,\underline{x}_n^*)$ is an equilibrium point if and only if for $k=1,2,\dots,n$ there exists a vector $\underline{u}_k \in R^{\ell_k}$ such that

$$\begin{gathered} \underline{u}_k \geqq \underline{0} \\ \nabla_k \varphi_k(\underline{x}^*) + \underline{u}_k^T \nabla_k \underline{h}_k(\underline{x}_k^*) = \underline{0}^T \\ \underline{h}_k(\underline{x}_k^*) \geqq \underline{0} \\ \underline{u}_k^T \underline{h}_k(\underline{x}_k^*) = 0 \end{gathered} \tag{9}$$

/where $\nabla_k \varphi_k$ is the gradient vector of φ_k with respect to $\underline{x}_k$ and $\nabla_k \underline{h}_k$ is the Jacobian matrix of $\underline{h}_k$/.

Proof. Under the assumptions given above, problem (8) is a concave programming problem. It is known that the Kuhn-Tucker equations and inequalities (9) are sufficient and necessary conditions for the optimality of a vector $\underline{x}_k^*$ $(k=1,2,\dots,n)$ /see Hadley [3] / . ■

To the sake of simple notations let

$$\underline{\Psi}_k(\underline{x}, \underline{u}_k) = \nabla_k f_k(\underline{x}) + \underline{u}_k^T \nabla_k \underline{h}_k(\underline{x}_k),$$

where $\underline{x} = (\underline{x}_1, \dots, \underline{x}_n)$.

Now we can prove our main theorem.

Theorem 2. A vector $\underline{x}^* = (\underline{x}_1^*, \dots, \underline{x}_n^*)$ is an equilibrium point if and only if there exists a vector $\underline{u}^* = (\underline{u}_1^*, \dots, \underline{u}_n^*)$ such that $(\underline{x}^*, \underline{u}^*)$ is an optimal solution of the mathematical programming problem

$$\left.\begin{aligned} \underline{u}_k &\geqq \underline{0} \\ \underline{\Psi}_k(\underline{x}, \underline{u}_k) &= \underline{0}^T \\ \underline{h}_k(\underline{x}_k) &\geqq \underline{0} \end{aligned}\right\} \quad (k=1,2,\dots,n) \qquad (10)$$

$$\overline{\sum_{k=1}^{n} \underline{u}_k^T \underline{h}_k(\underline{x}_k) \longrightarrow \min.}$$

Proof. a/ Let $\underline{x}^*$ be an equilibrium point /Lemma 4. implies that there exists at least one equilibrium point./ Then there exists a vector $\underline{u}^* = (\underline{u}_1^*, \dots, \underline{u}_n^*)$ such that the equations and inequalities are satisfied for $\underline{u}_k = \underline{u}_k^*$, thus the value of the objective function of the programming problem (10) is zero for $\underline{u}_k = \underline{u}_k^*$, $\underline{x}_k = \underline{x}_k^*$. For arbitrary feasible solution $(\underline{x}, \underline{u})$

of (1o) the objective function value is nonnegative, thus $(\underline{x}^*, \underline{u}^*)$ is an optimal solution.

b/ Let $(\underline{x}^*, \underline{u}^*)$ be an optimal solution of (1o). Since it is a feasible solution, each term of the objective function is nonnegative, conseqently the value of the objective function is nonnegative. But for the equilibrium point of the game /which exists/ the objective function has zero value, therefore the optimality of $(\underline{x}^*, \underline{u}^*)$ implies that the objective function at the point $(\underline{x}^*, \underline{u}^*)$ must have zero value. Since all terms are nonnegative in the objective function, all terms are equal to zero. Thus the equations and inequalities (9) are valid for $\underline{x} = \underline{x}^*$, $\underline{u} = \underline{u}^*$, consequently Lemma 6. implies that $\underline{x}^*$ is an equilibrium point. ∎

<u>Remark 1.</u> Problem (1o) is a mathematical programming problem which can be solved by numerical methods /e.g. cutting plane or gradient type algorithms, see G. Hadley [3]/.

<u>Remark 2.</u> In the special case of n=2 and $\varphi_2 = -\varphi_1$ problem (1o) was discovered by M.D.Canon [2].

Finally we will shaw well-known algorithms can be derived from the above general method as special cases.

<u>General polyhedral games</u>

Using the notations of Example 3. we have

$$\nabla_k \varphi_k (\underline{x}^*) = \underline{a}_k (\underline{x})^T$$

$$\nabla_k \underline{h}_k (\underline{x}_k) = - \underline{\underline{A}}_k ,$$

since

$$\varphi_k(\underline{x}^*) = \underline{a}_k(\underline{x})^T \underline{x}_k$$

$$\underline{h}_k(\underline{x}_k) = \underline{b}_k - \underline{\underline{A}}_k \underline{x}_k .$$

Thus problem (1o) has the form:

$$\left.\begin{array}{c} \underline{u}_k \geqq \underline{0} \\ \underline{a}_k(\underline{x})^T - \underline{u}_k^T \underline{\underline{A}}_k = \underline{0}^T \\ \underline{b}_k - \underline{\underline{A}}_k \underline{x}_k \geqq \underline{0} \end{array}\right\} \quad (k=1,2,\dots,n) \tag{11}$$

$$\sum_{k=1}^{n} \underline{u}_k^T (\underline{b}_k - \underline{\underline{A}}_k \underline{x}_k) \longrightarrow \min.$$

Let us observe that the second constraint implies that

$$\underline{a}_k(\underline{x})^T = \underline{u}_k^T \underline{\underline{A}}_k ,$$

and by using the fact that $\varphi_k(\underline{x}) = \underline{a}_k(\underline{x})^T \underline{x}_k$ we can write problem (11) in a more convenient form:

$$\left.\begin{array}{c} \underline{u}_k \geqq \underline{0} \\ \underline{a}_k(\underline{x})^T - \underline{u}_k^T \underline{\underline{A}}_k = \underline{0}^T \\ \underline{b}_k - \underline{\underline{A}}_k \underline{x}_k \geqq \underline{0} \end{array}\right\} \quad (k=1,2,\dots,n)$$

$$\sum_{k=1}^{n} \left(\underline{u}_k^T \underline{b}_k - \varphi_k(\underline{x})\right) \longrightarrow \min. \tag{12}$$

As a special case let $n=2$. Since

$$\underline{a}_1(\underline{x}) = \underline{\underline{A}}\, \underline{x}_2, \qquad \underline{a}_2(\underline{x}) = \underline{\underline{B}}^T \underline{x}_1 ,$$

where $\underline{\underline{A}} = \left(a_{i_1 i_2}^{(1)}\right)$ and $\underline{\underline{B}} = \left(a_{i_1 i_2}^{(2)}\right)$,

problem (12) can be rewritten as

$$
\begin{aligned}
\underline{u}_1 &\geqq \underline{0} \\
\underline{u}_2 &\geqq \underline{0} \\
\underline{x}_2^T \underline{\underline{A}} - \underline{u}_1^T \underline{\underline{A}}_1 &= \underline{0}^T \\
\underline{x}_1^T \underline{\underline{B}} - \underline{u}_2^T \underline{\underline{A}}_2 &= \underline{0}^T \qquad (13) \\
\underline{b}_1 - \underline{\underline{A}}_1 \underline{x}_1 &\geqq \underline{0} \\
\underline{b}_2 - \underline{\underline{A}}_2 \underline{x}_2 &\geqq \underline{0}
\end{aligned}
$$

$$\sum_{k=1}^{2} \left(\underline{u}_k^T \underline{b}_k\right) - \underline{x}_1^T \left(\underline{\underline{A}} + \underline{\underline{B}}\right) \underline{x}_2 \longrightarrow \min ,$$

which is a quadratic programming problem with linear constraints. Let us observe that the unknown vector $(\underline{x}_1, \underline{x}_2, \underline{u}_1, \underline{u}_2)$ is $m_1 + m_2 + \ell_1 + \ell_2$ dimensional. In a further special case when $\underline{\underline{B}} = - \underline{\underline{A}}$ /zero-sum case/, problem (13) is a linear programming problem, which can be solved by the simplex method.

Mixed extension of finite games

As we have seen in Example 3, in our case

$$\underline{\underline{A}}_k = \begin{pmatrix} -\underline{\underline{I}} \\ \underline{1}^T \\ -\underline{1}^T \end{pmatrix}, \qquad \underline{b}_k = \begin{pmatrix} \underline{0} \\ 1 \\ -1 \end{pmatrix}.$$

Let us write the vectors $\underline{u}_k$ in block form corresponding to the special block form of $\underline{\underline{A}}_k$ and $\underline{b}_k$, then we have

$$\underline{u}_k = \begin{pmatrix} \underline{v}_k \\ \alpha_k \\ \beta_k \end{pmatrix},$$

where $\underline{v}_k \in R^{m_k}$, α_k and β_k are scalers. Using these special notations problem (12) can be written in the form

$$\left.\begin{aligned} \underline{v}_k &\geqq \underline{0} \\ \alpha_k &\geqq 0 \\ \beta_k &\geqq 0 \\ \underline{a}_k(\underline{x})^T + \underline{v}_k^T - \alpha_k \underline{1}^T + \beta_k \underline{1}^T &= \underline{0}^T \\ \underline{x}_k &\geqq \underline{0} \\ \underline{1}^T \underline{x}_k &= 1 \end{aligned}\right\} \quad (k=1,2,\dots,n) \tag{14}$$

$$\sum_{k=1}^{n} \left(\alpha_k - \beta_k - \varphi_k(\underline{x}) \right) \to \min.$$

Let us observe that the nonnegative vector $\underline{v}_k$ appears only in the fourth constraint and we can introduce the new variable $\gamma_k = \alpha_k - \beta_k$, which is not necessarily nonnegative. Then we get by multiplying the objective function by -1 the following problem:

$$\left.\begin{aligned} \underline{a}_k(\underline{x})^T &\leqq \gamma_k \underline{1}^T \\ \underline{x}_k &\geqq \underline{0} \\ \underline{1}^T \underline{x}_k &= 1 \end{aligned}\right\} \quad (k=1,2,\dots,n) \tag{15}$$

$$\sum_{k=1}^{n} \left(\varphi_k(\underline{x}) - \gamma_k \right) \to \max$$

which is the method of H. Mills [6].

Bimatrix games

From the previous case the bimatrix games can be obtained by choosing $n=2$. Simple calculations show that

$$\underline{a}_1 (\underline{x}) = \underline{\underline{A}}\, \underline{x}_2 , \qquad \underline{a}_2 (\underline{x}) = \underline{\underline{B}}^T \underline{x}_1 ,$$

thus problem (15) can be written as

$$\begin{aligned} \underline{\underline{A}}\, \underline{x}_2 &\leqq \gamma_1 \underline{1} \\ \underline{\underline{B}}^T \underline{x}_1 &\leqq \gamma_2 \underline{1} \\ \underline{x}_1 &\geqq \underline{0} \\ \underline{x}_2 &\geqq \underline{0} \\ \underline{1}^T \underline{x}_1 &= 1 \\ \underline{1}^T \underline{x}_2 &= 1 \\ \hline \underline{x}_1^T (\underline{\underline{A}} + \underline{\underline{B}})\, \underline{x}_2 - \gamma_1 - \gamma_2 &\longrightarrow \max , \end{aligned} \tag{16}$$

which is a quadratic programming problem with linear constraints and it was discovered by O.L. Mangasarian and H. Stone [5]. For matrix games $\underline{\underline{B}} = - \underline{\underline{A}}$, thus problem (16) is a linear programming problem, which can be separated with respect to the variables $(\underline{x}_1, \gamma_2)$ and $(\underline{x}_2, \gamma_1)$, and so problem (16) can be reduced for two linear programming problems

$$\begin{aligned} \underline{\underline{A}}\, \underline{x}_2 &\leqq \gamma_1 \underline{1} \\ \underline{x}_2 &\geqq \underline{0} \\ \underline{1}^T \underline{x}_2 &= 1 \\ \hline \gamma_1 &\longrightarrow \min \end{aligned} \tag{17}$$

and

$$\begin{aligned} - \underline{\underline{A}}^T \underline{x}_1 &\leqq \gamma_2 \underline{1} \\ \underline{x}_1 &\geqq \underline{0} \\ \underline{1}^T \underline{x}_1 &= 1 \\ \hline \gamma_2 &\longrightarrow \min , \end{aligned} \tag{18}$$

where the problems have $m_2 + 1$ and $m_1 + 1$ variables, respectively.