INTRODUCTION.- By using the definitions of o-regular and completely o-regular functions from a topological space S to a finite directed Graph G (see Background we go on (see [3],[4] and [5]) with the study of normalization theorems for regular homotopy. To this purpose, given a partition P of S, we introduce the definitions of quasi-constant and weakly quasi-constant function with respect to P (see Definitions 4 and 10). Then, by using also the first normalization theorem (see [4], Theorem 12) we prove that any o-regular function from a compact space S to G is completely o-homotopic and weakly quasi-constant w.r.t. a suitable partition P. (The second normalization theorem) (see Theorem 3).

The previous theorem can be refined when S is a compact triangulable space, proving that any o-regular function from S to G is completely o-homotopic to a function pre-cellular w.r.t. a suitable decomposition (*) C of S. (The third normalization theorem) (see Theorem 6).

Moreover we prove that between two pre-cellular functions which are o-homotopic, there exists also a homotopy which is pre-cellular w.r.t. a suitable decomposition of SxI. (The third normalization theorem for homotopies) (see Theorem 8).

Then all the previous results are generalized to the case between pairs of topological spaces and of graphs (see § 5,6) and to the case between (n+1)-tuples (see § 7).

At least we apply the results to the case of n-dimensional groups of regular homotopy and we obtain that in any class of regular homotopy group there exists a loop which is a pre-cellular function w.r.t. a suitable triangulation (subdivision into cubes) of I^n. With references to this, we remark that the subdivisions into cubes are useful to give a combinatorial interpretation of homotopy groups by blocks of vertices:

(*) For simplicity, we consider the finite decompositions C of S by (open) CW-complexes which satisfy the condition that for all $\sigma \in C$, $\bar{\sigma}$ is a subcomplex of C.

(see [10]).

The previous results will be used in a next paper to prove that regular homotopy groups are isomorphic to the classical homotopy groups ot the polyhedron $|K_G|$ of the simplicial complex K_G associated with G, whose simplexes are given by the totally headed subsets of G.

0) Background.

Let X be a non-empty subset of a finite directed graph G. A vertex of X is called a *head* of X in G if it is a predecessor of all the other vertices of X. We denote by $H_G(X)$ the set of the heads of X in G. X is called *headed* if $H(X) \neq \emptyset$ and *totally headed* if all the non-empty subsets of X are headed.

Given a function $f\colon S \to G$, where S is a topological space, we denote by capital letter V the set of all the f-counterimages of $v \in G$, and, if we want to emphasize the function f, we write $V^f = f^{-1}(v)$.

We call *image-envelope* of a point $x \in S$ by f, and we denote by $\langle f(x)\rangle$, the set of vertices, such that the closure of their f-counterimages includ the point i.e. $v \in \langle f(x)\rangle \Leftrightarrow x \in \overline{V}^f$.

A function $f\colon S \to G$ is called *o-regular*, if, for all different $v,w \in G$, such that v is not a predecessor of w, it is $V \cap \overline{W} = \emptyset$. We proved that f is o-regular iff:

i) $\langle f(x)\rangle$ is headed, $\forall x \in S$;

ii) $f(x) \in H(\langle f(x)\rangle)$, $\forall x \in S$. (See [5], Proposition 2).

So it is natural to define a more restrictive class of functions by sayi that a function $f\colon S \to G$ is *completely o-regular* (or simply *c. o-regular*) if

i') $\langle f(x)\rangle$ is totally headed, $\forall x \in S$;

ii') $f(x) \in H(\langle f(x)\rangle)$, $\forall x \in S$.

Afterwards we also consider functions satisfying only condition *i'*, whic we call *completely quasi regular* functions. In [5] we proved that a completely quasi regular function can be replaced by a c. o-regular one by constructing the o-patterns of the function (where an *o-pattern* of a function $f\colon S \to G$ is a function $g\colon S \to G$ such that $g(x) \in H(\langle f(x)\rangle)$, $\forall x \in S$). In the case of pairs of topological spaces S, S' and of pairs of graphs G, G in [5] in order to introduce the o-patterns, we gave the definition of *balanced* function i.e. of a function $f\colon S,S' \to G,G'$ such that $\langle f(x')\rangle = \langle f'(x')\rangle$, $\forall x' \in S'$. With reference to this we remember that if the subspace S' is open in S, all the functions are balanced.

1) Enlargability of sets in a uniform space.

DEFINITION 1. - *Let $(S,\mathcal{W})$ be a uniform space, where the filter $\mathcal{W}$ is the uniformity of S. Given a vicinity $W \in \mathcal{W}$, we put $W(x) = \{y \in S \ / \ (x,y) \in W\}$, $\forall x \in S$, and* $W(X) = \bigcup_{x \in X} W(x)$, $\forall X \subset S$.

' REMARK. - If (S,d) is a metric space the subsets $W^{\varepsilon} = \{(p,q) \in S \times S \ /$

$p,q)<\varepsilon\}$, $\varepsilon>0$, constitute a basis of the uniformity induced by the tric d.

DEFINITION 2. - *Let $(S,\mathcal{W})$ be a uniform space and W a vicinity of $\mathcal{W}$. ien n subsets $X_1,\dots,X_n$ of S are called* W-enlargable *if $W(X_1)\cap\dots\cap$ $(X_n) = \emptyset$.*

REMARK. If $X_1,\dots,X_n$ are W-enlargable, then all the m-tuples $(m>n)$, otained by adding any n-m subsets of S, are still W-enlargable .

DEFINITION 3. - *Let (S,d) be a metric space and $X_1,\dots,X_n$ subsets of . We call* enlargability *of the n-tuple $X_1,\dots,X_n$, and we denote by* nl$(X_1,\dots,X_n)$ *the non-negative real number r such that:*

$$W^{\varepsilon}(X_1)\cap\dots\cap W^{\varepsilon}(X_n)\begin{cases} = \emptyset, & \forall\varepsilon\leqslant r\\ \neq\emptyset, & \forall\varepsilon> r.\end{cases}$$

REMARK 1. - If $\overline{X}_1\cap\dots\cap\overline{X}_n\neq\emptyset$, we put $enl(X_1,\dots,X_n) = 0$, while if one at least among the X_i is empty, we put $enl(X_1,\dots,X_n)$ = diameter of S.

REMARK 2. - Let $X_1,\dots,X_m$ be a m-tuple of subsets of S, obtained by adding to the n-tuple $X_1,\dots,X_n$ any $m-n$ subsets of S, then $enl(X_1,\dots,X_n)\leqslant enl(X_1,\dots,X_m)$.

REMARK 3. - Let $X_1\neq\emptyset$, $X_2\neq\emptyset$. It results $enl(X_1,X_2)\leqslant d(X_1,X_2)\leqslant 2enl(X_1,X_2)$. In fact if we put $d(X_1,X_2)=\eta$, for all ε there exist $x\in X_1$ and $y\in X_2$ such that $d(x,y)<\eta+\varepsilon$. Hence it is $W^{\eta+\varepsilon}(X_1)\cap W^{\eta+\varepsilon}(X_2)\neq\emptyset$, i.e. $enl(X_1,X_2)<\eta+\varepsilon = d(X_1,X_2)+\varepsilon$. Since ε is arbitrary, it follows $enl(X_1,X_2)\leqslant d(X_1,X_2)$.
Moreover let $r = enl(X_1,X_2)$. For all $\varepsilon>0$ it is $W^{r+\varepsilon}(X_1)\cap W^{r+\varepsilon}(X_2)\neq\emptyset$. Then there exist $z\in W^{r+\varepsilon}(X_1)\cap W^{r+\varepsilon}(X_2)$, $x_1\in X_1$ and $x_2\in X_2$ such that $d(X_1,X_2)\leqslant d(x_1,x_2)\leqslant d(x_1,z)+d(x_2,z)\leqslant 2r+2\varepsilon = 2enl(X_1,X_2)+2\varepsilon$. Since ε is arbitrary, it follows $d(X_1,X_2)\leqslant 2enl(X_1,X_2)$. We remark that it may be $d(X_1,X_2)<2enl(X_1,X_2)$. In fact if $S=\{x_1,x_2\}$ is the discrete metric space, where $d(x_1,x_2) = 1$, it is $enl(\{x_1\},\{x_2\}) = 1$.

PROPOSITION 1. - *Let S be a compact space and the filter $\mathcal{W}$ the uniformity of S (*). If, for n subsets $X_1,\dots,X_n$ of S, it results $\overline{X}_1\cap\dots\cap\overline{X}_n$*

(*) We remark that in a compact space there exists only one uniformity compatible with the topology (see [2] , Cap. 2, §4, n° 1).

$= \emptyset$, *then there exists a vicinity* $W \in \mathcal{W}$ *such that* $X_1, \ldots, X_n$ *are* W-*enlargable.*

Proof. - We suppose all the sets X_i are non-empty, otherwise the proposition is trivial. Since S is compact, $\forall i = 1, \ldots, n$, the family $\{W(\overline{X}_i)\}$, $\forall W \in \mathcal{W}$, constitute a basis of the neighbourhoods filter of $\overline{X}_i$ (see [2], Cap. 2, § 4, n° 3); moreover, since S is normal, the neighbourhoods filter of $\overline{X}_i$ is closed. Consequently, $\{W(\overline{X}_1) \cap \ldots \cap W(\overline{X}_n)\}$ $\forall W \in \mathcal{W}$ is the basis of a closed filter $\mathfrak{F}$. Now, if $\mathfrak{F}$ is the null filter, there exists $W \in \mathfrak{F}$ such that $W(\overline{X}_1) \cap \ldots \cap W(\overline{X}_n) = \emptyset = W(X_1) \cap \ldots \cap W(X_n)$, i.e. $X_1, \ldots, X_n$ are W-enlargable. Otherwise, since S is compact, there exists a point x adherent to $\mathfrak{F}$, and since $\mathfrak{F}$ is a closed filter, $x \in W(\overline{X}_1) \cap \ldots \cap W(\overline{X}_n)$, $\forall W \in \mathcal{W}$. Then it is $x \in W(\overline{X}_i)$, $\forall W \in \mathcal{W}$, $i = 1, \ldots, n$. As the sets $W(\overline{X}_i)$ constitute a basis of the neighbourhoods filter of $\overline{X}_i$, it follows $x \in \overline{X}_i$, $i = 1, \ldots, n$, i.e. $x \in \overline{X}_1 \cap \ldots \cap \overline{X}_n$. Contradiction

COROLLARY 2. - *Let* S *be a compact metric space and* $X_1, \ldots, X_n$ *subsets of* S *such that* $\overline{X}_1 \cap \ldots \cap \overline{X}_n = \emptyset$, *then it is* $enl(X_1, \ldots, X_n) > 0$. □

2) *The second normalization theorem.*

DEFINITION 4. - *Let* A *be a non-empty set,* G *a finite graph and* $P = \{X_j\}$, $j \in J$, *a partition of* A. *A function* $f: A \rightarrow G$ *is called* quasi constant with respect to P (w.r.t. P) *or* P-constant *if the restrictions of* f *to each* X_j *are constant functions. Moreover, if* A *is a topological space,* $f: A \rightarrow G$ *is called* weakly quasi-constant w.r.t. P *or* **weakly** P-costant *if the restrictions of* f *to the interior of every* X_j *are constant.*

REMARK. - If $P' = \{X'_k\}$, $k \in K$, is a partition of A finer than P, i.e. if all the $X_i \in P$ are the union of elements $X'_k \in P'$, then the function f is obviously quasi-contant also w.r.t. P'.

DEFINITION 5. - *Let* $(S, \mathcal{W})$ *be a uniform space and* W *a vicinity of* $\mathcal{W}$. *A subset* X *of* S *is called* small of order W *or a* W-subset *if* $X \times X \subseteq W$. *Moreover a family* $\mathcal{X} = \{X_j\}$, $j \in J$, *is called* small of order W *or a* W-family *if* $X_j \times X_j \subseteq W$, $\forall j \in J$.

REMARK 1. - If W is closed and $\{X_j\}$, $j \in J$, is a W-family, $\{\overline{X}_j\}$, $j \in J$, is a W-family.

REMARK 2. - If S is metric, small of order W^{ε} is the same as saying that the diameter of X is $< \varepsilon$ and, respectively, the mesh of the family $\mathcal{X}$ is $< \varepsilon$.

THEOREM 3. - (The second normalization theorem). *Let S be a compact space, the filter $\mathcal{W}$ the uniformity of S, G a finite directed graph and $f: S \to G$ a completely o-regular function from S to G. Then there exists a vicinity $W \in \mathcal{W}$ such that, for all the W-partitions $P = \{X_j\}$, $j \in J$, there exists a function $h: S \to G$ which is completely o-regular, weakly P-constant and completely o-homotopic to f.*

Proof. - Consider all the n-tuples $a_1,\dots,a_n$, $n \geqslant 2$, non-headed in G. Since f is c. o-regular, it follows $\bar{A}_1 \cap \dots \cap \bar{A}_n = \emptyset$. By Proposition 1, for every n-tuple $a_1,\dots,a_n$, there exists a vicinity $V^{(a_1,\dots,a_n)} \in \mathcal{W}$ such that $A_1,\dots,A_n$ are $V^{(a_1,\dots,a_n)}$-enlargable. Then we put $V = \bigcap V^{(a_1,\dots,a_n)}$ and consider a simmetric vicinity $W \in \mathcal{W}$ such that $W \circ W \subseteq V$. Now, if $P = \{X_j\}$, $j \in J$, is a W-partition, we can define a relation $g: S \to G$, by putting, as constant value, for every $X_j \in P$, any vertex of $H(\{f(X_j)\})$. We prove that g satisfies the following conditions:

i) g is a function. We have only to state that, for all X_j, the set $\{f(X_j)\} = \{a_1,\dots,a_n\}$ is headed. Suppose it is non-headed, and let $x_1,\dots,x_n \in X_j$ be, such that $f(x_1) = a_1,\dots, f(x_n) = a_n$. Since $X_j \times X_j \subseteq W$ it follows $(x_r,x_s) \in W$, $r,s = 1,\dots,n$, and also $x_1 \in W(x_1) \cap \dots \cap W(x_n) \subseteq V(A_1) \cap \dots \cap V(A_n)$. Contradiction.

ii) g is completely quasi-regular, i.e. $\forall x \in S$ the image-envelope $\langle g(x) \rangle$ is totally-headed. Suppose there exists $x \in S$ and a n-tuple $a_1,\dots,a_n \in \langle g(x) \rangle$ non-headed. Then it results $x \in \bar{A}_1^g \cap \dots \cap \bar{A}_n^g$ and so $W(x) \cap A_i^g \neq \emptyset$, $\forall i = 1,\dots,n$. Hence in $W(x)$ there are n points $x_1,\dots,x_n$ such that $x_i \in A_i^g$, $\forall i = 1,\dots,n$. But, from the definition of g, there exist n elements $X_i \in P$ and n points y_i such that $g(x_i)=a_i=f(y_i)$, $\forall i = 1,\dots,n$ where $x_i, y_i \in X_i$. Since P is a W-partition, we have $(x_i,y_i) \in W$. Therefore by $(x,x_i) \in W$, $(x_i,y_i) \in W$ and $W \circ W \subseteq V$, $\forall i=1,\dots,n$, it results $x \in V(y_1) \cap \dots \cap V(y_n) \subseteq V(A_1) \cap \dots \cap V(A_n)$. Contradiction.

iii) The function $F: S \times I \to G$, given by:

$$F(x,t) = \begin{cases} f(x) & \forall x \in S, \quad \forall t \in [0,\frac{1}{2}[\\ g(x) & \forall x \in S, \quad \forall t \in [\frac{1}{2},1] \end{cases}$$

is completely quasi-regular. This is true $\forall x \in S$, $\forall t \neq \frac{1}{2}$, since f and g are completely quasi-regular functions. We have to prove this also $\forall x \in S$, $t = \frac{1}{2}$, i.e. that $\langle F(x,t) \rangle = \langle f(x) \rangle \cup \langle g(x) \rangle$ is totally headed. Suppose $x \in S$ and let $a_1,\dots,a_n \in \langle f(x) \rangle \cup \langle g(x) \rangle$ be a n-tuple non-headed.

We can order the a_i in such a way that $a_1,\ldots,a_p \in \langle f(x)\rangle$ and $a_{p+1},\ldots,$ $a_n \in \langle f(x)\rangle - \langle g(x)\rangle$. Therefore it is $x\in \overline{A}^f_1 \cap\ldots\cap \overline{A}^f_p$, and so $W(x)\cap A^f_i \neq \emptyset$, $\forall i = 1,\ldots,p$. Hence there are in $W(x)$ p points $x_1,\ldots,x_p$ such that $x_i \in A^f_i$, $\forall i = 1,\ldots,p$. Then it is $x\in W(x_1)\cap\ldots\cap W(x_p) \subseteq$ $V(A^f_1)\cap\ldots\cap V(A^f_p)$. Moreover it is $x\in \overline{A}^g_{p+1}\cap\ldots\cap \overline{A}^g_n$, and by *ii)* it follows $x\in V(A^f_{p+1})\cap\ldots\cap V(A^f_n)$. Hence we obtain the contradiction $x\in V(A^f_1)\cap\ldots\cap V(A^f_n)$.

Now if we consider any o-pattern h of g, we obtain the sought functio In fact we have:

i') $h: S\to G$ is completely o-regular (see [5], Proposition 7).

ii') h is weakly p-constant by the definition of o-pattern of a quasi-constant function.

iii') h is completely o-homotopic to f. Since the homotopy F is completely quasi-regular by iii), there exists an o-pattern E of F (which is completely o-regular by [5], Proposition 7). Moreover we can choose E such that $E(x,0) = f(x)$, $E(x,1) = h(x)$, $\forall x\in S$, since f and h are completely o-regular i.e. $f(x)\in H(\langle f(x)\rangle) = H(\langle F(x,0)\rangle)$ and $h(x)\in H(\langle g(x)\rangle) = H(\langle F(x,1)\rangle)$, $\forall x\in S$. Then h is completely o-homotopic to f by E. □

REMARK 1. If W is a closed set, we can give the function g, by choosing as constant image of $X_j\in P$ any vertex of $H(\{f(\overline{X}_j)\})$.

REMARK 2. - *If S is a compact metric space, we can determine a real positive number r and choose partitions P with mesh $< r$.* In fact, we have just to calculate $enl(A_1,\ldots,A_n)$, $\forall n$-tuple $a_1,\ldots,a_n$ non-headed; so the real number r is given by $\frac{1}{2}\, inf(enl(A_1,\ldots,A_n))$.

REMARK 3. - If G is an undirected graph, the function g can be choosen quasi-constant. Moreover if S is a compact metric space, by Remark to Definition 2, we have just to consider the couples of non-adjacent vertices a_h,a_k and then to find the distances $d(A_h,A_k)$ rather than the enlargabilities $enl(A_h,A_k)$. Consequently, if we put $r' = inf(d(A_h,A_k))$ and $r = \frac{1}{2} inf(enl(A_h,A_k))$, since by Remark 3 to Definition 3 it follows $r'\leqslant 4r$, we can choose a covering $P = \{X_j\}$, $j\in J$, with mesh $< \frac{r'}{4}$. So we obtain again Property 7 of [8].

3) The third normalization theorem.

By comparing the second normalization theorem for directed and undirected graphs, we remark an asymmetry since for the former we are able to construct a q.constant function, while for the latter we obtain only a weakly q.constant function Nevertheless, by choosing a particular compact space S, also for directed graphs we obtain results similar to those for undirected graphs. For this purpose we consider the compact triangulable spaces and its finite decompositions C by (open) CW-complexes (see [13], Cap. VII) which satisfy the condition:

(1) $\forall \sigma \in C$, $\bar{\sigma}$ is a subcomplex of C, i.e. $\forall \tau \in C,\ \tau \cap \bar{\sigma} \neq \emptyset \implies \tau \in \bar{\sigma}$. (*)

DEFINITION 6. - *Let C be a finite cellular complex and D a subset of cells of C. We denote by $|C|$ a realization of C and by $|D|$ the subspace of C constituted by the points of the cells of D.*

REMARK. - Nevertheless, if there is no ambiguity, we denote by σ both a cell and the subspace $|\sigma|$. So, for example, we write $\bar{\sigma}$ rather than $|\bar{\sigma}|$.

DEFINITION 7. - *Let D be a non-empty subset of cells of a finite complex C. We call* star of a point $x \in |D|$ *w.r.t. D, and write $st_D(x)$, the set of the cells of D whose closure in $|C|$, and therefore in $|D|$, includes x. Moreover we call* star of a subset $X \subset |D|$ *w.r.t. D, and write $st_D(X)$, the set of the cells of D, whose closure has a non-empty intersection with X. Similarly we can define the* star $st_D(\sigma)$ of a cell *of D and the* star $st_D(D')$ of a subset D' *of D. Then, if $D = C$, simply we write $st(x)$, $st(X)$ ..., rather than $st_C(x)$, $st_C(X)$, ..., .*

REMARK 1. - *The stars are open sets in $|D|$.* In fact their complements are closed in $|D|$, since if for a cell τ it is $\tau \subset |D|$, also it follows $\bar{\tau} \subset |D|$. Then, if $D = C$, the complements of the stars are subcomplexes of C.

REMARK 2. - If x is any point of a cell $\sigma \in D$, then $st_D(x) = st_D(\sigma)$. In fact in $|D|$ it results $x \in \bar{\tau} \iff \sigma \subset \bar{\tau}$.

DEFINITION 8. - *Let D be a subset of cells of a finite cellular complex C. A cell $\tau \in D$ is said to be* maximal *in D if it is $\tau = st_D(\tau)$.*

REMARK. - *A cell is maximal in D iff it is an open set in $|D|$.* Consequently the cells maximal in a star are the cells maximal in C which are included in the star.

DEFINITION 9. - *Let D be a subset of cells of a finite complex C, x a point of $|D|$ and X a subset of $|D|$. We denote by $st^m_D(x)$ (resp.*

(*) We add (1), since we consider cellular subdivisions (triangulations and subdivisions into cubes) of this kind. Nevertheless we can obtain the same results also

$st_D^m(X)$) *the set of the maximal cells of* D, *whose closure includes* x (*resp. has non-empty intersection with* X). *If* $D = C$ *simply we write* $st^m(x)$ *and* $st^m(X)$, *rather than* $st_C^m(x)$ *and* $st_C^m(X)$.

REMARK. - *Let* x *be any point of a cell* $\sigma \in D$, *then obviously it results* $st_D^m(x) = st_D^m(\sigma)$.

DEFINITION 10. - *Let* C *be a finite cellular complex and* G *a finite graph. A function* $f\colon |C| \to G$ *is called* quasi-constant w.r.t. C *or* C-constant *if* f *is quasi-constant w.r.t. the partition determined by the cellular decomposition of* $|C|$. *Then, if* D *is a non-empty subset of cells of* C, *the function* $f\colon |C| \to G$ *is called* properly quasi-constant in D w.r.t. C *or* properly C-constant *in* D, *if, for all the cells* σ *non-maximal in* D, *there exists a cell* $\tau \in D$ (*different from* σ), *such that:*
i) the restrictions of f *to* σ *and to* τ *are identical.*
ii) $\sigma \subset \bar{\tau}$.
At least if $D = C$ *the function* $f\colon |C| \to G$ *is called* properly quasi-constant w.r.t. C *or* properly C-constant.

REMARK. - A function $f\colon |C| \to G$ (properly) C-constant is also (properly) quasi-constant w.r.t. a cellular decomposition C' finer than C.

PROPOSITION 4. - *Let* C *be a finite cellular complex,* D *a subset of cells of* C, G *a finite graph and* $f\colon |C| \to G$ *a* C-*constant function. Then it results* $\langle f(x) \rangle = f(st(x))$, $\forall x \in |C|$.
Moreover, the function f *is properly* C-*constant in* D *iff it is* $f(st_D(\sigma)) = f(st_D^m(\sigma))$, $\forall \sigma \in D$.
At least, if $D = C$, *the previous relation is equivalent to* $\langle f(x) \rangle = f(st^m(x))$, $\forall x \in |C|$.

Proof. - i) Let v be any vertex of G and σ any cell of C, then it follows:
$v \in \langle f(x) \rangle \iff x \in \overline{V}^f \iff \exists \sigma \,/\, x \in \bar{\sigma}$ and $f(\sigma) = v \iff \exists \sigma / \sigma \in st(x)$ and $f(\sigma) = v \iff v \in f(st(x))$.
ii) If it is $f(st_D(\sigma)) = f(st_D^m(\sigma))$, $\forall \sigma \in D$, the function f is properly C-constant in D, since, $\forall \sigma \in D$, from $f(\sigma) \in f(st_D^m(\sigma))$ we obtain there exists in D a maximal cell τ such that $\sigma \in \bar{\tau}$ and $f(\sigma) = f(\tau)$. The converse follows from the definition of properly quasi-constant function.
iii) By Remark 2 to Definition 7, by Remark to Definition 9 and by i), the condition $\langle f(x) \rangle = f(st^m(x))$, $\forall x \in |C|$, is equivalent to $f(st(\sigma)) = f(st^m(\sigma))$, $\forall \sigma \in C$. □

In order to employ briefer notations, we give the following:

DEFINITION 11. - *Let C be a finite cellular complex and G a finite directed graph. A completely o-regular function $f:|C|\to G$, which is properly C-constant is called a function* pre-cellular w.r.t. C *or a* C-pre-cellular *function.*

PROPOSITION 5. - *Let C be a finite cellular complex and G a finite directed graph. Then every C-pre-cellular function $f:|C|\to G$ is characterized, up to complete o-homotopy, by the restriction of f to the set of the maximal cells of C.*

Proof. -Let $g:|C|\to G$ be a C-pre-cellular function which takes the same values as f on all the maximal cells of C. By Proposition 4 it results $\langle f(x)\rangle = f(st^m(x)) = g(st^m(x)) = \langle g(x)\rangle$, $\forall x\in|C|$. Since g is c. o-regular, it is $g(x)\in H(\langle g(x)\rangle) = H(\langle f(x)\rangle)$, i.e. g is an o-pattern of f and then g is c.o-homotopic to f. (See [5], Proposition 7). □

THEOREM 6. - (The third normalization theorem). *Let S be a compact triangulable space, G a finite directed graph and $f: S\to G$ a completely o-regular function. Then, for every finite cellular decomposition C of S with suitable mesh, there exists a C-pre-cellular function $h: S\to G$ which is completely o-homotopic to f.*

Proof. -Let C be a cellular decomposition of S with mesh $< r$, where $r = \frac{1}{2}inf(enl(A_1,\dots,A_n))$, $\forall a_1,\dots,a_n$ non-headed n-tuple of G (see Remark 2 to Theorem 3). Then we construct the function g by choosing, $\forall \sigma_i\in C$, a vertex in $H(\{f(\bar\sigma_i)\})$ rather than in $H(\{f(\sigma_i)\})$ (see Remark 1 to Theorem 3). Hence, $\forall x\in|C|$, it is $H(g(st^m(x)))\subseteq H(\langle g(x)\rangle)$. Given, indeed, a vertex $a\in H(g(st^m(x)))$ and a cell $\tau\in st^m(x)$ such that $g(\tau) = a$, i.e. $a\in H(\{f(\bar\tau)\})$, we prove that a is a predecessor of all the vertices of $\langle g(x)\rangle$. In fact if $b\in\langle g(x)\rangle$ and a is not a predecessor of b, b is the image of a non-maximal cell σ, while, by definition of g, we have $b\in H(\{f(\bar\sigma)\})$. Since $\sigma\subset\bar\tau$, and also $\bar\sigma\subset\bar\tau$, it is $b\in f(\bar\tau)$. Hence a is not a head of $f(\bar\tau)$. Contradiction.
By remarking that, $\forall x\in\sigma$, it is $g(st^m(x)) = g(st^m(\sigma))$, we can define the o-pattern h in the following way:

$$h(\sigma) = \text{ a vertex of } H(g(st^m(\sigma))),\quad \forall\sigma\in C.$$

The function h is properly C-constant since, if τ is a maximal cell, from $g(st^m(\tau)) = \{g(\tau)\}$ it results $h(\tau) = g(\tau)$. Hence, by definition, we have $h(\sigma)\in g(st^m(\sigma)) = h(st^m(\sigma))$, $\forall\sigma\in C$. □

REMARK. - If G is an undirected graph, it is not necessary to contruct also the o-pattern to obtain a properly quasi-constant functio[n]. In this case the condition is reduced to $h(\sigma)$ = a vertex of $g(st^m(\sigma))$.

4) *The third normalization theorem for homotopies.*

Let $e,f: S \to G$ be two functions pre-cellular w.r.t. two finite decompositions C and K of S and $F: S\times I \to G$ a complete o-homotopy between e and f. Then, for every sufficently fine finite cellular decomposition Γ of $S\times I$, by Theorem 6, the function F can be replaced by a Γ-pre-cellular function $\hat{h}: S\times I \to G$. In order that the function $\hat{h}$ may also be a homotopy between e and f, the restrictions of $\hat{h}$ to $S\times\{0\}$ and $S\times\{1\}$ must coincide with e and f. Hence it is necessary that $\hat{h}$ characterizes on $S\times\{0\}$ and $S\times\{1\}$ two decompositions $\tilde{C}$ and $\tilde{K}$ finer than C and K, since e and f are properly quasi-constant (see Remark to Definition 10). Nevertheless, as, for example, the value of the function $\hat{h}$ on $S\times\{0\}$ depends from the value assumed by the function F on the maximal cells of the star $st(\tilde{C})$, in general the restriction $\hat{h}/_{|\tilde{C}|}$ is different from e. Consequently, at first, we must replace the homotopy F by a homotopy M given by:

$$M(x,t) = \begin{cases} e(x) & \forall x \in S, \quad \forall t \in \left[0,\frac{1}{3}\right] \\ F(x,3t-1) & \forall x \in S, \quad \forall t \in \left[\frac{1}{3},\frac{2}{3}\right] \\ f(x) & \forall x \in S, \quad \forall t \in \left[\frac{2}{3},1\right] \end{cases}$$

Then we have to costruct suitable cellular decompositions of the three cylinders $S\times\left[0,\frac{1}{3}\right]$, $S\times\left[\frac{1}{3},\frac{2}{3}\right]$ and $S\times\left[\frac{2}{3},1\right]$.

PROPOSITION 7. - *Let S be a compact triangulable space, C a finite cellular decomposition of S, G a finite graph and $e: S\to G$ a properly C-constant function. If we consider the decomposition $L = \{\{0\},]0,1[, \{1\}\}$ of I and the product decomposition $\Gamma = C\times L$ of the cylinder $S\times I$, then the function $F: S\times I \to G$, given by $F(x,t) = e(x)$, $\forall x\in S$, $\forall t\in I$, is properly Γ-constant .*

Proof. - We have only to remark that a cell τ is maximal in Γ iff $\tau = \tau'\times]0,1[$, where τ' is a maximal cell in C. Then it results $F(\tau) = e(\tau')$. □

REMARK. - Since the restrictions $F/_{S\times\{0\}}$ and $F/_{S\times\{1\}}$ coincide with e, they are obviously C-constant.

So we obtain:

THEOREM 8. -(The third normalization theorem for homotopies). *Let S be a compact triangulable space, G a finite directed graph, C,D two finite decompositions of S and e,f: S→G two functions pre-cellular w.r.t. C and D respectively, which are completely o-homotopic. Then, from any finite cellular decomposition* Γ_2 *of* $S\times\left[\frac{1}{3},\frac{2}{3}\right]$ *of suitable mesh which induces on the bases* $S\times\left\{\frac{1}{3}\right\}$ *and* $S\times\left\{\frac{2}{3}\right\}$ *decompositions* $\widetilde{C}$ *and* $\widetilde{D}$ *finer than C and D, we obtain a finite cellular decomposition* Γ *of* $S\times I$ *and a homotopy between f and g which is a* Γ*-pre-cellular function.*

Proof. - Let $F: S\times I\to G$ be a complete o-homotopy between e and f. We define the complete o-homotopy $M: S\times I\to G$ between e and f as in the introduction of this paragraph. Then, if we consider the restriction of M to $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, we can determine the real number r, upper bound of the mesh. Now if Γ_2 is a finite cellular decomposition, satisfying the conditions of the theorem and with mesh $< r$, we can consider the cellular decomposition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ of the cylinder $S\times I$, such that:

i) Γ_1 is the product decomposition $\widetilde{C}\times L_1$ of $S\times\left[0,\frac{1}{3}\right]$, where $L_1 = \left\{\{0\}, \left]0,\frac{1}{3}\right[, \left\{\frac{1}{3}\right\}\right\}$.

ii) Γ_3 is the product decomposition $\widetilde{D}\times L_3$ of $S\times\left[\frac{2}{3},1\right]$, where $L_3 = \left\{\left\{\frac{2}{3}\right\}, \left]\frac{2}{3},1\right[, \{1\}\right\}$.

Then we define the function $\hat{g}: S\times I\to G$, given by:

$$\hat{g}(\sigma) = \begin{cases} M(\sigma), & \forall\sigma\in \Gamma-\Gamma_2, \\ \text{a vertex of } H(\{M(\bar{\sigma})\}), & \forall\sigma\in\Gamma_2. \end{cases}$$

Afterwards, by Theorem 6, we construct the o-pattern $\hat{h}$ of $\hat{g}$, by choosing as element of $H(\hat{g}(st^m(\sigma)))$, the value $\hat{g}(\sigma) = M(\sigma)$ if $\sigma\in \Gamma-\Gamma_2$. By construction $\hat{h}$ is a Γ-pre-cellular function. Hence $\hat{h}$ is the sought homotopy since $\hat{h}/_{S\times\{0\}} = e$ and $\hat{h}/_{S\times\{1\}} = f$. □

REMARK. - The finite cellular decomposition Γ induces on the bases $S\times\{0\}$ and $S\times\{1\}$ the decompositions $\widetilde{C}$ and $\widetilde{D}$.

5) The second normalization theorem between pairs.

Given a set A, a non-empty subset A' of A, a finite graph G and a subgraph G' of G, we can generalize Definition 4, by considering function $f: A,A'\to G,G'$ which are quasi-constant w.r.t. a partition $P = \{X_j\}$, $j\in J$, of A. In this case it follows that the image of every X_j, such that $X_j\cap A' \neq \emptyset$, necessary is a vertex of G'. Moreover, if A is a topo logical space and A' a subspace of A, we can also generalize the definition of weakly P-constant. So we have:

PROPOSITION 9. - *Let S be a compact space, the filter $\mathcal{W}$ the uniformity of S, S' a closed subspace of S, U a closed neighbourhood of S', G a finite directed graph, G' a subgraph of G and $f: S,U \to G,G'$ a completely o-regular function. If we choose in $\overset{\circ}{U}$ a closed neighbourhood K of S', we can determine a vicinity $W \in \mathcal{W}$ such that, for all the W-partitions $P = \{X_j\}$, $j \in J$, there exists a function $h: S,\overset{\circ}{K} \to G,G'$, which is completely o-regular, weakly P-constant and completely o-homotopic to $f: S,S' \to G,G'$.*

Proof. - At first there exists a closed neighbourhood K of S', included in U, since S is normal. Then, by following the proof of Theorem 3, we determine a vicinity $V \in \mathcal{W}$ such that $V(A_1^f) \cap \ldots \cap V(A_n^f) = \emptyset$, $\forall n$-tuple $a_1, \ldots, a_n$ non-headed of G. Moreover, if $\mathcal{W}'$ is the trace filter of $\mathcal{W}$ on $U \times U$, we obtain, as before, a vicinity $Z' \in \mathcal{W}'$ such that $Z'(A'^{f'}_1) \cap \ldots \cap Z'(A'^{f'}_m) = \emptyset$, $\forall m$-tuple $a'_1, \ldots, a'_m$ non-headed of G'. Since $Z' \in \mathcal{W}'$, necessarily it is $Z' = V_1 \cap (U \times U)$, where $V_1 \in \mathcal{W}$. Then we choose a symmetric vicinity $W \in \mathcal{W}$ such that $W \circ W \subseteq V \cap V_1$ and $W(K) \subset U$. Now, given a W-partition $P = \{X_j\}$, $j \in J$, of S, we define a relation $g: S,\overset{\circ}{K} \to G,G'$, by putting, for every X_j, $j \in J$, the constant value:

$$g(X_j) = \begin{cases} \text{a vertex of } H_G(\{f(X_j)\}) & \text{if } X_j \cap K = \emptyset, \\ \text{a vertex of } H_{G'}(\{f'(X_j)\}) & \text{if } X_j \cap K \neq \emptyset. \end{cases}$$

We verify that g satisfies the following conditions:

i) g is a function. In fact it results:

a) $\forall X_j \,/\, X_j \cap K = \emptyset$, the set $\{f(X_j)\}$ is headed in G. For proving this we go on as in i) of the proof of Theorem 3.

b) $\forall X_j \,/\, X_j \cap K \neq \emptyset$, the set $\{f'(X_j)\}$ is headed in G'. At first we prove that $X_j \subseteq U$. Let $z \in X_j \cap K$, $\forall y \in X_j$ it is $(z,y) \in X_j \times X_j \subseteq W$, i.e: $X_j \subseteq W(z) \subseteq W(K) \subseteq U$. Then, if we go on as in i) of Theorem 3, we obtain that $\{f'(X_j)\}$ is headed in G'. Moreover, we remark that the vertex $g(x)$, chosen in $H_{G'}(\{f'(X_j)\})$, is also an element of $H_G(\{f(X_j\})$, since $f(X_j) = f'(X_j)$.

From a) and b) it follows that there exists $g(x)$, for every $x \in S$; hence g is a function.

ii) and *iii) The function $g: S,\overset{\circ}{K} \to G,G'$ and the homotopy $F: S \times I, \overset{\circ}{K} \times I \to G,G$ between f and g given by:*

$$F(x,t) = \begin{cases} f(x) & \forall x \in S, \quad \forall t \in \left[0, \frac{1}{2}\right[\\ g(x) & \forall x \in S, \quad \forall t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

are completely quasi-regular functions.

a) $g: S\to G$ and $F: S\times I\to G$ are c. quasi-regular functions. We obtain this result as in ii) and iii) of Theorem 3.

b) The restrictions $g': \mathring{K}\to G'$ and $F': \mathring{K}\times I\to G'$ are c. quasi-regular. At first we observe that, by the definition of g, it is $g(K)\subset G'$ and then $F(K\times I)\subset G'$. Secondly we go on as in ii) and iii) of Theorem 3, by choosing, $\forall x'\in \mathring{K}$, the neighbourhood $W(x')\cap \mathring{K}$, rather than $W(x')$, and by using the vicinity Z' rather than V. Then, for example, if we suppose that the m-tuple $a_1',\ldots,a_m'\in \langle g'(x')\rangle$ is non-headed, we obtain the contradiction $x'\in Z'(A'^{f'}_1)\cap\ldots\cap Z'(A'^{f'}_m)$.

From a) and b) it follows ii) and iii).

Now if we consider any o-pattern h of g, we obtain the sought function. In fact we have:

i') $h: S,\mathring{K}\to G,G'$ is completely o-regular (see [5], Proposition 15).

ii') h is weakly P-constant by the definition of o-pattern of a quasi-constant function.

iii') h is completely o-homotopic to $f: S,S'\to G,G'$. Since the homotopy $F: S,\mathring{K}\to G,G'$ is c.quasi-regular by iii) and $\mathring{K}$ is open, there exists an o-pattern E (which is c.o-regular by [5], Proposition 15) of F. We can choose E such that $E(x,0) = f(x)$ and $E(x,1) = h(x)$, $\forall x\in S$, for f and g are c.o-regular i.e.:

a) $f(x)\in H_G(\langle f(x)\rangle) = H_G(\langle F(x,0)\rangle)$ and $h(x)\in H_G(\langle g(x)\rangle) = H_G(\langle F(x,1)\rangle)$, $\forall\ x\in S$.

b) $f'(x)\in H_{G'}(\langle f'(x)\rangle) = H_{G'}(\langle F'(x,0)\rangle)$ and $h'(x)\in H_{G'}(\langle g(x)\rangle) = H_{G'}(\langle F'(x,1)\rangle)$, $\forall x\in\mathring{K}$.

Hence the o-pattern $h(x) = E(x,1)$ is c.o-homotopic to f by E. □

REMARK. - If S is a compact metric space, we can determine a positive real number r and choose partitions P with mesh $< r$. In fact, we put $\varepsilon_1 = inf(enl(A^f_1,\ldots,A^f_n))$, $\forall n$-tuple $a_1,\ldots,a_n$ non-headed of G and $\varepsilon_2 = inf(enl(A'^{f'}_1,\ldots,A'^{f'}_m))$, $\forall m$-tuple $a_1',\ldots,a_m'$ non-headed of G' and we choose ε_3 such that $W^{\varepsilon_3}(K)\subset U$. Then the real number r is given by $inf(\frac{\varepsilon_1}{2},\frac{\varepsilon_2}{2},\ \varepsilon_3)$.

THEOREM 10. -(The second normalization theorem between pairs). *Let S be a compact space, the filter $\mathcal{W}$ the uniformity of S, S' a closed subspace of S, G a finite directed graph, G' a subgraph of G and $f:S,S'\to G,G'$ a completely o-regular function. Then we can determine a closed neighbourhood K of S' and a vicinity $W\in\mathcal{W}$ such that, for all the W-partitions $P = \{X_j\}$, $j\in J$, there exists a function $h: S,\mathring{K}\to G,G'$, which is completely o-regular, weakly P-constant and completely o-homotopic* *to f.*

Proof. - By Proposition 28 of [5] and Theorem 16 of [4] there exists a closed neighbourhood U of S' and an extension $k: S,U \to G,G'$ which is c.o-regular and such that $k:S,S' \to G,G'$ is c.o-homotopic to f. Then we obtain the result by using Proposition 9 for the function $k: S,U \to G,G'$.

REMARK. - If G is an undirected graph, the function g can be choosen quasi-constant. Moreover if S is a compact metric space, we have only to consider the couples of vertices rather than the n-tuples and to determine $\varepsilon_1 = inf(d(A^f_i,A^f_j))$, $\forall$ couple a_i, a_j of non-adjacent vertices of G, $\varepsilon_2 = inf(d(A^{f'}_r,A^{f'}_s))$, $\forall$ couple a_r,a_s of non-adjacent vertices of G'. Then, if we put $r' = inf(\varepsilon_1, \varepsilon_2)$, as in Remark 3 to Theorem 3, we can choose a covering $P = \{X_j\}$, $j \in J$, with mesh $< \frac{r'}{4}$ (see [8], Corollary 8).

6) The third normalization theorem between pairs.

Now we consider pairs of spaces given by a finite cellular complex C and by a subcomplex C' of C; it follows that $|C'|$ is a closed subspace of $|C|$. Since we use completely o-regular functions $f:|C|,|C'| \to G,G'$ balanced by the open set $|st(C')|$ (see [5], Definitions 6 and 12), we put:

DEFINITION 12. - *Let C be a finite complex, C' a subcomplex of C, G a finite graph and G' a subgraph of G. A function $f:|C|,|C'| \to G,G'$ is called* pre-cellular w.r.t. *C,C' or C,C'*-pre-cellular *if:*
i) $f:|C|,|st(C')| \to G,G'$ is completely o-regular.
ii) $f:|C| \to G$ is properly C-constant.
iii) $f:|C| \to G$ is properly C-constant in C'.

THROREM 11. - (The third normalization theorem between pairs). *Let S be a compact triangulable space, S' a closed triangulable subspace of S, G a finite directed graph, G' a subgraph of G and $f:S,S' \to G,G'$ a completely o-regular function. Then for every finite cellular decomposition C,C' of the pair S,S', with suitable mesh, there exists a function $h: S,S' \to G,G'$ which is C,C'-pre-cellular and completely o-homotopic to f.*

Proof. - By proceeding as in the proof of Theorem 10, at first we consider an extension $k: S,U \to G,G'$, where U is a closed neighbourhood of S'. Then, by Remark to Proposition 9, we determine a positive real number $r = inf(\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}, \varepsilon_3)$, where $\varepsilon_1 = inf(enl(A_1^k,\dots,A_n^k))$, $\forall n$-tuple $a_1,\dots,a_n$ non-headed of G, $\varepsilon_2 = inf(enl(A_1'^{k'},\dots,A_m'^{k'}))$, $\forall m$-tuple $a_1',\dots,a_m'$ non-headed of G', ε_3 is such that $W^{\varepsilon_3}(S') \subset U$. Since we can use $|st(C')|$ as an open neighbourhood of S', now it is not necessary to construct, as in Proposition 9, a closed neighbourhood K of S', included in U, and to consider the interior $\mathring{K}$.

Then, if C,C' is a finite decomposition of S,S' with mesh $< r$, it results $|\overline{st(C')}| \subseteq W^r(S')$, since all the cells have diameter $< r$. Afterwards, we construct the c.quasi-regular function $g: |C|, |st(C')| \to G,G'$ by putting, $\forall \sigma \in C$, (see Proposition 9 and Remark 1 to Theorem 3):

$$g(\sigma) = \begin{cases} \text{a vertex of } H_G(\{k(\tilde{\sigma})\}) & \text{if } \sigma \in C - st(C') \\ \text{a vertex of } H_{G'}(\{k(\tilde{\sigma})\}) & \text{if } \sigma \in st(C'). \end{cases}$$

To construct a c.o-regular o-pattern h, we must separate the cells of C w.r.t. $st(C')$ as before. Moreover, to obtain h properly quasi-constant, we must separate the cells of C w.r.t. C' in the following way:

a) cells included in $C-C'$: $\begin{cases} 1) \text{ cells } \tau \text{ maximal in } C \\ 2) \text{ cells } \sigma \text{ non-maximal in } C \end{cases}$

b) cells included in C': $\begin{cases} 1) \text{ cells } \tau \text{ maximal in } C \\ 2) \text{ cells } \tau' \text{ maximal in } C' \text{ and non-maximal in } C \\ 3) \text{ cells } \sigma' \text{ non-maximal in } C'. \end{cases}$

Now (see Theorem 6), by induction, we construct the o-pattern h, by putting at the first step:

i) $h(\tau) = g(\tau)$

ii) $h(\sigma) =$ a vertex of $H_\Gamma(g(st^m(\sigma)))$ where $\begin{cases} \Gamma = G & \text{if } \sigma \in C-st(C') \\ \Gamma = G' & \text{if } \sigma \in st(C') \end{cases}$

iii) $h(\tau') =$ a vertex of $H_{G'}(g(st^m(\tau')))$.

If we define, as before, the images of the cells maximal in C', at the second and last step, we put:

$h(\sigma') =$ a vertex of $H_{G'}(h(st^m_{C'}(\sigma')))$.

Hence $h: |C|, |st(C')| \to G,G'$ is the sought function. □

REMARK. - If G is an undirected graph, it is not necessary to construct the extension of the function $f: |C|, |C'| \to G,G'$. In fact, if we determine the upper bound $\frac{r'}{4}$ of the mesh as in Remark to Theorem 10, and, consequently, if we consider the cellular decomposition C,C', we can obtain the strongly regular function $g: S, |st(S')| \to G,G'$, by putting, $\forall \sigma \in C$:

$$g(\sigma) = \begin{cases} \text{a vertex of } f(\tilde{\sigma}) & \text{if } \sigma \in C-st(C') \\ \text{a vertex of } f(\tilde{\sigma}) \cap \sigma' & \text{if } \sigma \in st(C'). \end{cases}$$

Moreover, in the construction of the o-pattern h, we have only to separate the cells w.r.t. C and C'.

Theorem 8 can be generalized by:

THEOREM 12. - (The third normalization theorem for homotopies of functions between pairs). *Let S be a compact triangulable space, S' a closed triangulable subspace of S, G a finite directed graph, G' a subgraph of G, C,C' and D,D' two finite cellular decompositions of S,S' and e,f: $S,S' \to G,G'$ two functions pre-cellular w.r.t. C,C' and D,D' respectively, which are completely o-homotopic. Then, from any finite cellular decomposition Γ_2, Γ'_2 of the pair $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, $S'\times\left[\frac{1}{3},\frac{2}{3}\right]$ of suitable mesh, which induces on the pairs of bases $S\times\left\{\frac{1}{3}\right\}$ and $S'\times\left\{\frac{2}{3}\right\}$ decompositions $\widetilde{C}$, $\widetilde{C}'$ and $\widetilde{D}$,$\widetilde{D}'$ finer than C,C' and D,D', we obtain a finite cellular decomposition Γ,Γ' of the pair $S\times I$, $S'\times I$ and a homotopy between e and f which is a Γ,Γ'-pre-cellular function.*

Proof. - Since $|st(C')|$ and $|st(D')|$ are rispectively balancers (see [5], Definition 12) of e and f in S', the open set $U = |st(C')| \cap |st(D')|$ is a common balancer of e and f. Now let F: $S\times I$, $S'\times I \to G,G'$ be a complete o-homotopy between e and f and, by Proposition 30 of [5] we can construct a closed neighbourhood V of $S'\times I$ and a c. o-regular function $\hat{k}$: $S\times I, V \to G,G'$, which is a homotopy between e and f. Then, the c.o-homotopy $\hat{k}$ can be replaced by the c.o-homotopy M given by:

$$M(x,t) = \begin{cases} e(x) & \forall x \in S, \quad \forall t \in \left[0,\frac{1}{3}\right] \\ F(x,3t-1) & \forall x \in S, \quad \forall t \in \left[\frac{1}{3},\frac{2}{3}\right] \\ f(x) & \forall x \in S, \quad \forall t \in \left[\frac{2}{3},1\right] \end{cases}$$

and, by considering the restriction of M to $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, we determine the real number r, upper bound of the mesh (see the proof of Theorem 11). Moreover, if Γ_2, Γ'_2 is a cellular decomposition, which satisfies the conditions of the theorem and with mesh $< r$, we can construct the cellular decomposition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, $\Gamma' = \Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$ of the pair of cylinders $S\times I$, $S'\times I$, where $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma'_2$ are the product decompositions, respectively, of $\widetilde{C}\times L_1$, $\widetilde{C}'\times L_1$, $\widetilde{D}\times L_3$, $\widetilde{D}'\times L_3$ (see Theorem 8).

Then we define the function $\hat{g}$: $S\times I$, $S'\times I \to G,G'$ by putting:

$$\hat{g}(\sigma) = \begin{cases} M(\sigma), & \forall \sigma \in \Gamma - \Gamma_2 \\ \text{a vertex of } H_G(\{M(\widetilde{\sigma})\}) & \text{if } \sigma \in \Gamma_2 - st_{\Gamma_2}(\Gamma'_2) \\ \text{a vertex of } H_{G'}(\{M(\widetilde{\sigma})\}) & \text{if } \sigma \in st_{\Gamma_2}(\Gamma'_2). \end{cases}$$

Hence, by Theorem 11, we construct the o-pattern $\hat{h}$ of $\hat{g}$, by choosing, if $\sigma \in \Gamma - \Gamma_2$, as value of $\hat{h}(\sigma)$, the value $\hat{g}(\sigma) = M(\sigma)$. In this way

$\hat{h}$ coincides with M on $S\times\left[0,\frac{1}{3}\right]$ and $S\times\left[\frac{2}{3},1\right]$. □

REMARK. - If G is an undirected graph, it is not necessary to construct the extension $\hat{k}$ of the function F. (See Remark to Theorem 11).

7) Case of n subspaces and n subgraphs.

The previous results can be easily generalized to the case between $(n+1)$-tuples (see [3], §8b and [5], §11).

Let S be a compact topological space, G a finite directed graph, $S_1,\dots,S_n$ closed subspaces of S and $G_1,\dots,G_n$ subgraphs of G, such that S_j is a subspace of S_i and G_j a subgraph of G_i, $\forall i,j = 1,\dots,n$, $j>i$. In this case we have to consider functions $f\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ between $(n+1)$-tuples and their restrictions $f_1\colon S_1\to G_1$, $\dots$, $f_n\colon S_n\to G_n$.

7a) Given a c.o-regular function $f\colon S,S_1,\dots,S_n\to G,G_1,\dots,G_n$, where S is compact and $S_1,\dots,S_n$ are closed subspaces, by [5], §11.6, we can construct n closed neighbourhoods U_i of S_i, $i=1,\dots,n$ and a c.o-regular extension $k\colon S,U_1,\dots,U_n\to G,G_1,\dots,G_n$ such that $k\colon S,S_1,\dots,S_n\to G,G_1,\dots,G_n$ is c.o-homotopic to f. Now, for all the pairs U_i,S_i, $i=1,\dots,n$, we determine a closed neighbourhood K_i of S_i, included in $\mathring{U}_i$. Then, if the filter $\mathcal{W}$ is the uniformity of S, by following the proof of Proposition 9, we can obtain:

i) a vicinity $V\in\mathcal{W}$ such that $V(A_1^k)\cap\dots\cap V(A_n^k)\neq\emptyset$, $\forall r$-tuple $a_1,\dots,a_r$ non-headed of G;

ii) $\forall i=1,\dots,n$ a vicinity Z_i of the trace-filter $\mathcal{W}_i$ of $\mathcal{W}$ on $U_i\times U_i$, such that $Z_i(A_1^{k_i})\cap\dots\cap Z_i(A_s^{k_i})=\emptyset$, $\forall s$-tuple $a_1,\dots,a_s$ non-headed of G_i, and, consequently, we obtain a vicinity $V_i\in\mathcal{W}$ / $Z_i = V_i\cap(U_i\times U_i)$.
At least, we choose a symmetric vicinity W, such that $W\circ W\subset V\cap V_1\cap\dots\cap V_n$ and $W(K_i)\subseteq U_i$, $i=1,\dots,n$.
Given, now, a W-partition $P=\{X_j\}$, $j\in J$, of the space S, we define a relation $g\colon S,\mathring{K}_1,\dots,\mathring{K}_n\to G,G_1,\dots,G_n$ by putting, $\forall X_j$, $j\in J$, the constant value:

$$g(X_j)=\begin{cases}\text{a vertex of } H_G(\{f(X_j)\}) & \text{if } X_j\cap K_1=\emptyset\\ \text{a vertex of } H_{G_1}(\{f_1(X_j)\}) & \text{if } X_j\cap K_1\neq\emptyset \text{ and } X_j\cap K_2=\emptyset\\ \dots\dots\dots & \\ \text{a vertex of } H_{G_n}(\{f_n(X_j)\}) & \text{if } X_j\cap K_n\neq\emptyset.\end{cases}$$

Similarly to Proposition 9, we verify that g is c. quasi-regular and that every o-pattern h of g is c.o-homotopic to f. Hence we can give:

THEOREM 13. - (The second normalization theorem between n-tuples). *Let $S,S_1,\dots,S_n$ be a $(n+1)$-tuple of topological spaces, where S is compact and $S_1,\dots,S_n$ are closed subspaces of S, $G,G_1,\dots,G_n$ a $(n+1)$-tuple of finite directed graphs and $f: S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ a completely o-regular function. Then, if the filter $\mathcal{W}$ is the uniformity of S, we can determine n closed neighbourhoods K_i of S_i, $i=1,\dots,n$ and a vicinity $W \in \mathcal{W}$ such that, for all the W-partitions $P = \{X_j\}$, $j \in J$, there exists a function $h: S, \mathring{K}_1,\dots,\mathring{K}_n \to G,G_1,\dots,G_n$ which is completely o-regular, weakly P-constant and completely o-homotopic to f.* □

REMARK 1. - If S is a compact metric space, we can determine a positive real number r and consider partitions with mesh $< r$.

REMARK 2. - If G is an undirected graph, the function g can be choosen P-constant. Moreover, since it is not necessary to replace f with an extension k, we have only to consider a symmetric vicinity W / $W \circ W \subset V \cap V_1 \cap \dots \cap V_n$.

7b) Now let $C,C_1,\dots,C_n$ be a $(n+1)$-tuple of spaces which consists of a finite cellular complex C and of n subcomplexes $C_1,\dots,C_n$. A function $f: |C|,|C_1|,\dots,|C_n| \to G,G_1,\dots,G_n$ is called *pre-cellular* w.r.t. $C,C_1,\dots,C_n$ if:

i) $f: |C|, |st(C_1)|,\dots,|st(C_n)| \to G,G_1,\dots,G_n$ is c.o-regular;
ii) $f: |C| \to G$ is properly C-constant;
iii) $f: |C| \to G$ is properly C-constant in C_1 (in $C_2,\dots,C_n$).

Now, if $f: S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ is a c.o-regular function, where S is a compact triangulable space, $S_1,\dots,S_n$ closed triangulable subspaces of S, we can consider c.o-regular extension $k: S,U_1,\dots,U_n \to G,G_1,\dots,G_n$ and determine the positive real number $r = inf(\frac{\varepsilon}{2},\frac{\varepsilon_1}{2},\dots,\frac{\varepsilon_n}{2},\eta_1,\eta_2,\dots,\eta_n)$ where $\varepsilon = inf(enl(A_1^k,\dots,A_r^k))$, $\forall r$-tuple $a_1,\dots,a_r$ non-headed of G, $\varepsilon_i = inf(enl(A_1^{k_i},\dots,A_{s_i}^{k_i}))$, $\forall$ s_i-tuple $a_1,\dots,a_{s_i}$ non-headed of G_i, and η_i are such that $W^{\eta_i}(S_i) \subset U_i$, $i=1,\dots,n$.
Given then a finite cellular decomposition $C,C_1,\dots,C_n$ of $S,S_1,\dots,S_n$ with mesh $< r$, we consider the following partition of C: $D_0 = C-st(C_1)$, $D_1 = st(C_1)-st(C_2),\dots$, $D_n = st(C_n)$ and we construct the c.quasi-regular function $g: |C|, |st(C_1)|,\dots,|st(C_n)| \to G,G_1,\dots,G_n$, by putting, $\forall D_i$,

$\forall\sigma \in D_i$, $g(\sigma)$ = a vertex of $H_{G_i}(\{k(\bar{\sigma})\})$, where $G_0 = G$. We separate the cells of C, besides using the subsets D_i, also in the following way:

i) cells included in $C-C_1$ $\begin{cases} \text{1) cells } \tau \text{ maximal in } C \\ \text{2) cells } \sigma \text{ non-maximal in } C \end{cases}$

ii) cells included in C_1-C_2 $\begin{cases} \text{1) cells } \tau \text{ maximal in } C \\ \text{2) cells } \tau_1 \text{ maximal in } C_1 \text{ and non-maximal in } C \\ \text{3) cells } \sigma_1 \text{ non-maximal in } C_1. \end{cases}$

....

n+1) cells included in C_n $\begin{cases} \text{1) cells } \tau \text{ maximal in } C \\ \text{2) cells } \tau_1 \text{ maximal in } C_1 \text{ and non-maximal in } C \\ \ldots\ldots\ldots\ldots \\ n\text{+1) cells } \tau_n \text{ maximal in } C_n \\ n\text{+2) cells } \sigma_n \text{ non-maximal in } C_n. \end{cases}$

At least, we can construct, by induction, the o-pattern h in n+1 steps by putting:

in the first step: $\begin{cases} h(\tau) = g(\tau) \\ h(\sigma) = \text{a vertex of } H_\Gamma(g(st^m(\sigma))) \\ h(\tau_1) = \text{a vertex of } H_\Gamma(g(st^m(\tau_1))) \end{cases}$

in the second step: $\begin{cases} h(\sigma_1) = \text{a vertex of } H_\Gamma(h(st^m_{C_1}(\sigma_1))) \\ h(\tau_2) = \text{a vertex of } H_\Gamma(h(st^m_{C_1}(\tau_2))) \end{cases}$

in the third step: $\begin{cases} h(\sigma_2) = \text{a vertex of } H_\Gamma(h(st^m_{C_2}(\sigma_2))) \\ h(\tau_3) = \text{a vertex of } H_\Gamma(h(st^m_{C_2}(\tau_3))) \end{cases}$

....

in the (n+1) step: $h(\sigma_n)$ = a vertex of $H_\Gamma(h(st^m_{C_n}(\sigma_n)))$,
where, $\forall\lambda \in C$, we put $\Gamma = G_i$ if $\lambda \in D_i$.

Hence we obtain:

THEOREM 14. - (The third normalization theorem between (n+1)-tuples). *Let $S, S_1, \ldots, S_n$ be a (n+1)-tuple of topological spaces, where S is a compact triangulable space, $S_1, \ldots, S_n$ are closed triangulable subspaces, $G, G_1, \ldots, G_n$ a (n+1)-tuple of finite directed graphs and $f: S, S_1, \ldots, S_n \to G, G_1, \ldots, G_n$ a completely o-regular function. Then, for every finite cellular decomposition $C, C_1, \ldots, C_n$ of $S, S_1, \ldots, S_n$ with suitable mesh, there exists a function $h: S, S_1, \ldots, S_n \to G, G_1, \ldots, G_n$ pre-cellular w.r.t. $C, C_1, \ldots, C_n$ and completely o-homotopic to f.* □

By a procedure similar to that one used in the proofs of Theorems 8 and 12, we also obtain:

THEOREM 15. - (The third normalization theorem for homotopies of functions between (n+1)-tuples). *Let* $S,S_1,\dots,S_n$ *be a (n+1)-tuple of topological spaces, where* S *is a compact triangulable space,* $S_1,\dots,S_n$ *are closed triangulable subspaces,* $G,G_1,\dots,G_n$ *a (n+1)-tuple of finite directed graphs,* $C,C_1,\dots,C_n$ *and* $D,D_1,\dots,D_n$ *two finite cellular decompositions of* $S,S_1,\dots,S_n$ *and* $e,f\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ *two functions pre-cellular w.r.t.* $C,C_1,\dots,C_n$ *and* $D,D_1,\dots,D_n$ *respectively, which are o-homotopic. Then, from any finite cellular decomposition of the (n+1)-tuple* $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, $S_1\times\left[\frac{1}{3},\frac{2}{3}\right],\dots, S_n\times\left[\frac{1}{3},\frac{2}{3}\right]$, *of suitable mesh, which induces on the bases decompositions finer than* $C,C_1,\dots,C_n$ *and* $D,D_1,\dots,D_n$, *we obtain a finite cellular decomposition* $\Gamma,\Gamma_1,\dots,\Gamma_n$ *of the (n+1)-tuple* $S\times I,S_1\times I,\dots,S_n\times I$, *and a homotopy between* e *and* f, *which is a pre-cellular function w.r.t.* $\Gamma,\Gamma_1,\dots,\Gamma_n$. □

8) Case of homotopy groups.

Since the n-cube I^n is a triangulable compact manifold, we can apply the results of the previous paragraphs to the case of absolute and relative n-dimensional groups of regular homotopy. So we can choose, as representative of any homotopy class, a loop which is pre-cellular w.r.t. a suitable cellular decomposition of I^n. Now, the cellular decompositions of I^n which are relevant for applications, are the triangulations and the subdivisions into cubes (the latter are determinated by a partition into k parts of equal size of every edge of I^n). To construct the absolute groups $Q_n(G,v)$ we consider *o-regular loops* i.e. o-regular functions $f\colon I^n,\dot{I}^n \to G,v$ where $\dot{I}^n$ is the boundary of I^n and v a vertex of G, whereas, in the case of relative groups $Q_n(G,G',v)$ we use the *o-regular relative loops*, i.e. o-regular functions $f\colon I^n,\dot{I}^n,J^{n-1} \to G,G',v$ where J^{n-1} is the union of the $(n-1)$-faces of I^n, different from the face $x_n = 0$. Since the subspaces $\dot{I}_n$, J^{n-1} are an union of faces of I^n, they are closed subspaces, which can be triangulated and subdivided into cubes. So, by applying the third normalization theorem (see Theorems 11 and 14), directly we obtain:

THEOREM 16. - *On the previous assumptions, in every o-homotopy class of the group* $Q_n(G,v)$ *(resp.* $Q_n(G,G',v)$*) there exists a loop which is pre-cellular w.r.t. a suitable triangulation (subdivision into cubes) of* I^n.

Proof. - Let α be an o-homotopy class and $f \in \alpha$ a loop. By [4], Theorem 15 and its generalization, we can replace f by a c.o-regular function $g \in \alpha$. Moreover, by Theorems 11 and 14, we can replace g by a function $h \in \alpha$ which satisfies the sought conditions, since there always exist triangulations and subdivisions into cubes with mesh $< r$, where r is a predeterminate real number. □

REMARK. - If G is a finite undirected graph, we obtain Property 13 of [8] again. Nevertheless, we remark that the meaning of properly quasi-constant function of Definition 10 is weaker than that one given there. In fact, now, the constant value of a cell σ is equal to the value of a maximal cell $\tau \in st(\sigma)$, whileas, before, the value of σ must also correspond to that one of a cell of properly upper dimension.

To obtain the third normalization theorem for homotopies, we recall that the cellular decompositions Γ_1 and Γ_2 are product decompositions. Consequently, we have:

i) To obtain a triangulation of $I^n \times I$, first we must triangulate every prism of the product. To this aim, we remark that it can be done by retaining the same triangulations $\tilde{C}$ and $\tilde{D}$ on the respective bases.

ii) Whileas, to obtain a subdivision of $I^n \times I$ into k^{n+1} cubes (where k is a multiple of 3), we must complete the subdivision of $I^n \times \left[\frac{1}{3},\frac{2}{3}\right]$ into $\frac{1}{3}k^{n+1}$ cubes, by giving a subdivision into cubes of the parallelepipeda of the product cellular decompositions Γ_1 and Γ_2.

Then we have:

THEOREM 17. - *On the previous assumptions, let f,g be two o-homotopic loops which are pre-cellular w.r.t. the triangulations T and T' (subdivisions into cubes Q and Q') of I^n. Then, between f and g there exists a homotopy which is pre-cellular w.r.t. a suitable triangulation (subdivision into cubes), which induces on $I^n \times \{0\}$ and $I^n \times \{1\}$ triangulations (subdivisions into cubes) finer than T and T' (than Q and Q').* □

REMARK 1. - If G is a undirected graph we obtain Property 14 of [8] again. Moreover now we can avoid the extension k of the c.o-regular function, by choosing as image of a cell σ, whose closure intersects the basis $S \times \{0\}$ $(S \times \{1\})$, the value of any maximal cell of $\bar{\sigma} \cap (S \times \{0\})$ $(\bar{\sigma} \cap (S \times \{1\}))$.

REMARK 2. - The subdivision into cubes is useful to obtain the regular homotopy groups by blocks of vertices of G. (See [10]).

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