base manifold is of constant curvature -1, the non-zero eigenvalues of h are ± 2, each with multiplicity n.

3. Integrals of scalar curvatures on symplectic and contact manifolds

We now want to consider a number of integral functionals defined on the set of metrics associated to a symplectic or contact structure. To begin we need to see how the set $\mathcal{A}$ of associated metrics sits in the set $\mathcal{M}$ of all Riemannian metrics with the same total volume; for a more detailed treatment see [5].

Let M be a symplectic manifold and $g_t = g + tD + O(t^2)$ be a path of metrics in $\mathcal{A}$. We will use the same letter D to denote D as a tensor field of type (1,1) and of type (0,2), $D^i{}_j = g^{ik} D_{kj}$. Now

$$g(X, JY) = \Omega(X,Y) = g_t(X, J_tY) = g(X, J_tY) + tg(X, DJ_t) + O(t^2)$$

from which

$$J = J_t + tDJ_t + O(t^2).$$

Applying J_t on the right and J of the left we have

$$J_t = J + tJD + O(t^2).$$

Squaring this yields $JDJ - D = 0$ and hence $JD + DJ = 0$. Conversely if D is a symmetric tensor field which anti-commutes with J, then $g_t = ge^{tD}$ is a path of associated metrics. We summarize this and the corresponding result in the contact case as follows (cf.[5],[6]).

Lemma 3.1. *Let M be a symplectic or contact manifold and $g \in \mathcal{A}$. A symmetric tensor field D is tangent to a path in $\mathcal{A}$ at g if and only if*

$$DJ + JD = 0 \tag{3.1}$$

in the symplectic case and

$$D\xi = 0, \; D\phi + \phi D = 0 \tag{3.2}$$

in the contact case.

Similar to the role played by Lemma 1.1 in critical point problems on $\mathcal{M}$, we have the following lemma for critical point problems on $\mathcal{A}$.

Lemma 3.2. *Let T be a second order symmetric tensor field on M. Then $\int_M T^{ij} D_{ij}\, dV_g = 0$ for all symmetric tensor fields D satifying (3.1) in the symplectic case and (3.2) in the contact case if and only if $TJ = JT$ in the symplectic case and $\phi T - T\phi = \eta \otimes \phi T\xi - (\eta \circ T\phi) \otimes \xi$ in the contact case (i.e. ϕ and T commute when restricted to the contact subbundle).*

Proof. We give the proof in the symplectic case; the proof in the contact case being similar. Let $X_1, \ldots X_{2n}$ be a local J-basis defined on a neighborhood $\mathcal{U}$ (i.e. $X_1, \ldots X_{2n}$ is an orthonormal basis with respect to g and $X_{2i} = JX_{2i-1}$) and note that the first vector field X_1 may be any unit vector field on $\mathcal{U}$. Let f be a C^∞ function with compact support in $\mathcal{U}$ and define a path of metrics $g(t)$ as follows. Make no change in g outside $\mathcal{U}$ and within $\mathcal{U}$ change g only in the planes spanned by X_1 and X_2 by the matrix

$$\begin{pmatrix} 1+tf+\frac{1}{2}t^2f^2 & \frac{1}{2}t^2f^2 \\ \frac{1}{2}t^2f^2 & 1-tf+\frac{1}{2}t^2f^2 \end{pmatrix}.$$

It is easy to check that $g(t) \in \mathcal{A}$ and clearly the only non-zero components of D are $D_{11} = -D_{22} = f$. Then $\int_M T^{ij}D_{ij}\,dV_g = 0$ becomes

$$\int_M (T^{11} - T^{22}) f\, dV_g = 0$$

Thus since X_1 was any unit vector field on $\mathcal{U}$,

$$T(X,X) = T(JX,JX)$$

for any vector field X. Since T is symmetric, linearization gives $TJ = JT$. Conversely, if T commutes with J and D anti-commutes with J, then $trTD = trTJDJ = trJTDJ = -trTD$, giving $T^{ij}D_{ij} = 0$.

Theorem 3.3 (Blair-Ianus). *Let M be a compact symplectic manifold and $\mathcal{A}$ the set of metrics associated to the symplectic form. Then $g \in \mathcal{A}$ is a critical point of $A(g) = \int_M R\,dV_g$ if and only if the Ricci operator of g commutes with the almost complex structure corresponding to g.*

Proof. The proof is again to compute $\frac{dA}{dt}$ at $t = 0$ for a path $g(t)$ in $\mathcal{A}$. Since all associated metrics have the same volume element this is easier than in the Riemannian case. In particular we have,

$$\begin{aligned} \left.\frac{dA}{dt}\right|_{t=0} &= \left.\frac{d}{dt}\int_M R_{kji}{}^k g^{ji}\,dV_g\right|_{t=0} \\ &= \int_M D_{kji}{}^k g^{ji} - R_{ji}D^{ji}\,dV_g \\ &= -\int_M R^{ji}D_{ji}\,dV_g, \end{aligned}$$

the other terms being divergences and hence contributing nothing to the integral. Setting $\left.\frac{dA}{dt}\right|_{t=0} = 0$, the result follows from Lemma 3.2.

We now review some known properties of almost Kähler manifold. First of all

$$\nabla_k J^k{}_l = 0, \tag{3.3}$$

$$(\nabla_k J_{ip})J_j{}^p = (\nabla_p J_{ij})J_k{}^p; \tag{3.4}$$

an almost Hermitian structure satisfying this last condition is call a *quasi-Kähler structure*.

The $*$-*Ricci tensor* and the $*$-*scalar curvature* are defined by

$$R^*_{ij} = R_{iklt}J^{kl}J_j{}^t, \ R^* = R^*_i{}^i.$$

The Ricci identity yields

$$\nabla_t\nabla_k J_j{}^t = (R_{kt} - R^*_{kt})J_j{}^t$$

where $R^*_{kt}J_j{}^t$ is skew-symmetric in j and k. Therefore

$$\nabla_t\nabla_k J_j{}^t + \nabla_t\nabla_j J_k{}^t = R_{kt}J_j{}^t + R_{jt}J_k{}^t. \tag{3.5}$$

The most important property of R^* is that

$$R - R^* = -\frac{1}{2}|\nabla J|^2$$

and hence $R - R^* \leq 0$ with equality holding if and only if the metric is Kähler. Thus Kähler metrics are maxima of the functional

$$K(g) = \int_M R - R^*\, dV_g$$

on $\mathcal{A}$ and the question that S. Ianus and I [8] were first interested in was whether these were the only critical points. The surprising result is that the critical point condition is again $QJ = JQ$, Q denoting the Ricci operator.

Theorem 3.4 (Blair-Ianus). *Let M be a compact symplectic manifold and $\mathcal{A}$ the set of metrics associated to the symplectic form. Then $g \in \mathcal{A}$ is a critical point of $K(g)$ if and only if $QJ = JQ$.*

Proof. To compute $\frac{dK}{dt}$ at $t = 0$, we must differentiate $R^* = R_{iklt}J^{kl}J^{it}$ along a path $g(t)$ in $\mathcal{A}$. Since Ω is fixed,

$$\left.\frac{\partial J_{ki}}{\partial t}\right|_{t=0} = 0, \quad \left.\frac{\partial J^i{}_j}{\partial t}\right|_{t=0} = -D^{im}J_{mj}, \quad \left.\frac{\partial J^{kl}}{\partial t}\right|_{t=0} = 0.$$

Then proceeding as before using (3.3)

$$\left.\frac{dK}{dt}\right|_{t=0} = \int_M [-R^{jm} + \nabla_i(J^{km}\nabla_k J^{ij}) + R^{*jm}]D_{jm}\, dV_g.$$

By Lemma 3.2, the critical point condition is that the symmetric part of the expression in brackets commutes with J. This is a long equation; some of its terms cancel by virtue of the quasi-Kähler condition (3.4) and the other terms combine by virtue of (3.5) to give the result.

The question as to whether or not on an almost Kähler manifold satisfying $QJ = JQ$ is Kählerian seems to be difficult. In [13] S. I. Goldberg showed that if J commutes with the curvature operator, then the metric is Kählerian and conjectured that a compact almost-Kähler Einstein manifold is Kählerian. K. Sekigawa [20] proved that a compact almost-Kähler Einstein manifold with non-negative scalar curvature is Kählerian.

In [9] A. J. Ledger and I proved the contact analogues of these theorems, which we present here without proof.

Theorem 3.5 (Blair-Ledger). *Let M be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $A(g) = \int_M R\,dV_g$ if and only if Q and ϕ commute when restricted to the contact subbundle.*

This integral was further studied in dimension 3 by D. Perrone [19], who gave the critical point condition as

$$\nabla_\xi h = 0.$$

To see this, recall that in dimension 3, the Ricci operator determines the full curvature tensor, i.e.

$$R(X,Y)Z = (g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y)$$
$$-\frac{R}{2}(g(Y,Z)X - g(X,Z)Y).$$

Therefore the operator l defined $lX = R(X,\xi)\xi$ by is given by

$$lX = QX - \eta(X)Q\xi + g(Q\xi,\xi)X - g(QX,\xi)\xi - \frac{R}{2}(X - \eta(X)\xi)$$

from which

$$(l\phi - \phi l)X = (Q\phi - \phi Q)X + \eta(X)\phi Q\xi - g(Q\phi X,\xi)\xi. \tag{3.6}$$

Thus the critical point condition is $l\phi - \phi l = 0$. Now recall equations (2.2) and (2.3), viz. $\frac{1}{2}(-l + \phi l\phi) = h^2 + \phi^2$ and $\nabla_\xi h = \phi - \phi h^2 - \phi l$. Applying ϕ to the first of these and adding to the second gives $\nabla_\xi h = \frac{1}{2}(l\phi - l\phi)$ and thus the critical point condition may be expressed as $\nabla_\xi h = 0$.

In the contact case the *$*$-scalar curvature* is defined by $R^* = R_{iklt}\phi^{kl}\phi^{it}$ and it was shown by Olszak [18] that

$$R - R^* - 4n^2 = -\frac{1}{2}|\nabla\phi|^2 + 2n - trh^2 \leq 0$$

with equality holding if and only if the metric is Sasakian.

Theorem 3.6 (Blair-Ledger). *Let M be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $K(g) = \int_M R - R^* - 4n^2\,dV_g$ if and only if $Q - 2nh$ and ϕ commute when restricted to the contact subbundle.*

In dimension 3, the argument giving Perrone's result gives the critical point condition as $\nabla_\xi h = -2\phi h$, a condition that will be important in the next lecture.

For contact manifolds of general odd dimension, if g is critical for both A and K, $h = 0$ so ξ is Killing. To see this note that the commutativities of Theorems 3.5 and 3.6 together imply that $\phi h = h\phi$ but $\phi h = -h\phi$ and hence $h = 0$ easily follows. As with the almost Kähler case, the question of whether or not a K-contact structure satisfying $Q\phi = \phi Q$ is Sasakian would seem to be difficult.

4. Integral of the Ricci curvature in the direction of the characteristic vector field

We devote this section to a discussion of a particular functional defined on the set of metrics associated to a contact structure. The main theorem is the following [6].

Theorem 4.1 (Blair). *Let M be a compact regular contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $L(g) = \int_M Ric(\xi)\, dV_g$ if and only if g is K-contact.*

One might conjecture this without the regularity, however we have the following counterexample: The standard contact metric structure on the tangent sphere bundle of a compact surface of constant curvature -1 is a critical point of L but is not K-contact. It is a result of Y. Tashiro [23] that the standard contact metric structure on the tangent sphere bundle of a Riemannian manifold is K-contact if and only if the base manifold is of constant curvature +1. Also recall the result of [4] that the standard contact structure of the tangent sphere bundle of a compact Riemannian manifold of non-positive constant curvature is not regular. Our second result is the following theorem [7].

Theorem 4.2 (Blair). *Let T_1M be the tangent sphere bundle of a compact Riemannian manifold (M, G) and $\mathcal{A}$ the set of all Riemannian metrics associated to its standard contact structure. Then the standard associated metric is a critical point of the functional $L(g)$ if and only if (M, G) is of constant curvature +1 or -1.*

Recall that by a K-contact structure we mean a contact metric structure for which ξ is Killing and that this is the case if and only if $h = 0$. Recall also equation (2.4), viz.

$$Ric(\xi) = 2n - trh^2.$$

Thus K-contact metrics when they occur are maxima for the function $L(g)$ on $\mathcal{A}$. Also the critical point question for $L(g)$ is the same as that for $\int_M |h|^2\, dV_g$ or $\int_M |\tau|^2\, dV_g$ where $\tau(X, Y) = (\mathcal{L}_\xi g)(X, Y) = 2g(X, h\phi Y)$. This last integral was studied by Chern and Hamilton [11] for 3-dimensional contact manifolds as a functional on $\mathcal{A}$ regarded as the set of CR-structures on M (there was an error in their calculation of the critical point condition as was pointed out by Tanno[22]).

Proof of Theorem 4.1. As with our other critical point problems, the first step is to compute $\frac{dL}{dt}$ at $t = 0$ for a path $g(t) \in \mathcal{A}$

$$\left.\frac{dL}{dt}\right|_{t=0} = \int_M (-h^i{}_m h^{mk} - R^k{}_{rs}{}^i \xi^r \xi^s + 2h^{ik}) D_{ik}\, dV_g.$$