# PREFACE

These notes contain a written version of a series of lectures on critical metrics on symplectic and contact manifolds delivered at the University of Lecce, 6-16 July 1990. The content of these lectures is a study of Riemannian metrics which are critical points of various functionals defined on various spaces of metrics, especially those associated to a symplectic or contact structure and is based for the most part on some of the author's work and joint work in this area over the last few years.

The author would like to take this opportunity to thank the University of Lecce and especially Professor Domenico Perrone for their kind invitation, support and hospitality.

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## CONTENTS

## 1. The classical integral functionals

The study of the integral of the scalar curvature, $A(g) = \int_M R\, dV_g$, as a functional on the set $\mathcal{M}$ of all Riemannian metrics of the same total volume on a compact orientable manifold $M$ is now classical. Moreover other functions of the curvature have been taken as integrands, most notably $B(g) = \int_M R^2\, dV_g$, $C(g) = \int_M |Ric|^2\, dV_g$, and $D(g) = \int_M |Riem|^2\, dV_g$, where Ric denotes the Ricci tensor and Riem denotes the full Riemannian curvature tensor; the critical point conditions for these have been computed by Berger [2]. A Riemannian metric $g$ is a critical point of $A(g)$ if and only if $g$ is an Einsten metric. Einstein metrics are critical for $B(g)$ and $C(g)$ and metrics of constant curvature and Kähler metrics of constant holomorphic curvature are critical for $D(g)$ but not necessarily conversely.

Our study in these lectures is primarily motivated by two kinds of questions. 1. Given an integral functional restricted to a smaller set of metrics, what is the critical point condition; one would expect a weaker one. The smaller sets of metrics we have in mind are the sets of metrics associated to a symplectic or contact structure. 2. Given these sets of metrics, are there other natural integrands depending on the structure as well as the curvature?

To set the stage for our study let us first prove that a Riemannian metric is critical for $A(g)$ if and only if it is Einstein. Let $M$ be a compact orientable manifold and $\mathcal{M}$ the set of all Riemannian metrics normalized by the condition of having the same total volume, usually taken to be 1, but we don't insist on the particular value in a given problem. We begin with the following lemma.

**Lemma 1.1.** *Let $T$ be a second order symmetric tensor field on $M$. Then $\int_M T^{ij} D_{ij}\, dV_g = 0$ for all symmetric tensor fields $D$ satifying $\int_M D^i_i\, dV_g = 0$ if and only if $T = cg$ for some constant $c$.*

**Proof.** Let $X, Y$ be an orthonormal pair of vector fields on a neighborhood $\mathcal{U}$ on $M$ and $f$ a $C^\infty$ function with compact support in $\mathcal{U}$. Regarding $X$ and $Y$ as part of a local orthonormal basis, define a tensor field $D$ on $M$ by $D(X,X) = f$ and $D(Y,Y) = -f$, with all other components equal to zero and $D \equiv 0$ outside $\mathcal{U}$. Then $\int_M (T(X,X) - T(Y,Y)) f\, dV_g = 0$ for any $C^\infty$ function with compact support and hence $T(X,X) = T(Y,Y)$ for every orthonormal pair $X, Y$. Therefore $T = cg$ for some function $c$ and it remains to show that $c$ is a constant. To see this let $X$ be any vector field and $D = \mathcal{L}_X g$, where $\mathcal{L}$ denotes Lie differentiation (i.e. $D$ is tangent to the orbit of $g$ under the diffeomorphism group). Then since the integral of a divergence vanishes,

$$0 = \int_M T^{ij}(\nabla_i X_j + \nabla_j X_i)\, dV_g = -2 \int_M (\nabla_i T^{ij}) X_j\, dV_g,$$

but $X$ is arbitrary so that $\nabla_i T^{ij} = 0$ from which we see that $c$ must be a constant. The converse is immediate.

Now the approach to these critical point problems is to differentiate the functional in question along a path of metrics. So let $g(t)$ be a path of metrics in $\mathcal{M}$ and

$$D_{ij} = \left.\frac{\partial g_{ij}}{\partial t}\right|_{t=0}$$

its tangent vector at $g = g(0)$. We define two other tensor fields by

$$D_{ji}{}^h = \frac{1}{2}(\nabla_j D_i{}^h + \nabla_i D_j{}^h - \nabla^h D_{ji})$$

$$D_{kji}{}^h = \nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h$$

where $\nabla$ denotes the Riemannian connection of $g(0)$ and we note that

$$D_{kji}{}^h = \left.\frac{\partial R_{kji}{}^h}{\partial t}\right|_{t=0}$$

where $R_{kji}{}^h$ denotes the curvature tensor of $g(t)$.

**Theorem 1.2.** *Let $M$ be a compact orientable $C^\infty$ manifold and $\mathcal{M}$ the set of all Riemannian metrics on $M$ with unit volume. Then $g \in \mathcal{M}$ is a critical point of $A(g) = \int_M R\, dV_g$ if and only if $g$ is Einstein.*

**Proof.** The proof is to compute $\frac{dA}{dt}$ at $t = 0$ for a path $g(t)$ in $\mathcal{M}$. First note that from $g_{ij}g^{jk} = \delta_i^k$,

$$\left.\frac{\partial g^{ij}}{\partial t}\right|_{t=0} = -D^{ij}.$$

Differentiation of the volume element gives

$$\frac{d}{dt} dV_g = \frac{d}{dt}\sqrt{det(g(t))}dx^1 \wedge \cdots \wedge dx^n = \frac{1}{2det(g(t))}\left(\frac{d}{dt}det(g(t))\right)dV_g$$

$$= \frac{1}{2}g^{ij}\left(\frac{d}{dt}g_{ij}\right)dV_g = \frac{1}{2}D_i^i dV_g.$$

Now

$$\left.\frac{dA}{dt}\right|_{t=0} = \left.\frac{d}{dt}\int_M R_{kji}{}^k g^{ji}\, dV_g\right|_{t=0}$$

$$= \int_M (D_{kji}{}^k g^{ji} - R_{ji}D^{ji} + \frac{1}{2}Rg^{ji}D_{ji})\, dV_g$$

$$= \int_M (-R^{ji} + \frac{1}{2}Rg^{ji})D_{ji}\, dV_g$$

since the integral of a divergence vanishes. On the other hand differentiation of $\int_M dV_g = 1$ gives $\int_M D_i^i\, dV_g = 0$. Thus setting $\left.\frac{dA}{dt}\right|_{t=0} = 0$ and applying the lemma, we have

$$R_{ji} - \frac{1}{2}Rg_{ji} = cg_{ji}$$

for some constant c and hence that g is Einstein.

In [15] Y. Muto computed the second derivative of $A(g)$ at a critical point and showed that the index of $A(g)$ and the index of $-A(g)$ are both positive.

Y. Muto also considered the second derivative of $D(g)$ from the following point of view. Let $\mathcal{D}$ denote the diffeomorphism group of $M$; if $f \in \mathcal{D}$, then $D(f^*g) = D(g)$ and hence we have an induced mapping $\tilde{D} : \frac{\mathcal{M}}{\mathcal{D}} \longrightarrow \mathbf{R}$. We say that a metric $g$ is a critical point of $\tilde{D}$ if its orbit under $\mathcal{D}$ is a critical point of $\tilde{D}$. As we have noted a Riemannian metric of constant curvature is a critical point of $D$; in [16] Y. Muto proved the following result.

**Theorem 1.3 (Muto).** *If $M$ is diffeomorphic to a sphere and $g$ is a metric of positive constant curvature, then the index of $D$ and the index of $\tilde{D}$ are both zero and $\tilde{D}$ has a local minimum at $g$.*

## 2. Symplectic and contact manifolds

By a *symplectic manifold* we mean a $C^\infty$ manifold $M^{2n}$ together with a closed 2-form $\Omega$ such that $\Omega^n \neq 0$. By a *contact manifold* we mean a $C^\infty$ manifold $M^{2n+1}$ together with a 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given $\eta$ there exists a unique vector field $\xi$ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ called the *characteristic vector field* of the contact structure $\eta$. A contact structure is said to be *regular* if every point has a neighborhood such that any integral curve of $\xi$ passing through the neighborhood passes through only once. The celebrated Boothby-Wang Theorem [10] states that a compact regular contact manifold is a principal circle bundle over a symplectic manifold of integral class. The Hopf fibration of an odd-dimensional sphere $S^{2n+1}$ as a principal circle over complex projective space $PC^n$ is a very well known example.

Let us now consider the Riemannian geometry of these manifolds. For a symplectic manifold $M$ let $k$ be any Riemannian metric and $X_1, \ldots, X_{2n}$ be a $k$-orthonormal basis. Consider the $2n \times 2n$ matrix $\Omega(X_i, X_j)$; it is non-singular and hence may be written as the product $GF$ of a positive definite symmetric matrix $G$ and an orthogonal matrix $F$. $G$ then defines a new metric $g$ and $F$ defines an almost complex structure $J$; checking the overlaps of local charts, it is easy to see that $g$ and $J$ are globally defined on $M$. The key point is that $\Omega(X, Y) = g(X, JY)$ where $g$ and $J$ are created simultaneously by polarization. A metric $g$ created in this way is called an *associated metric* and the set of these metrics will be denoted by $\mathcal{A}$. In particular $\mathcal{A}$ is the set of all almost Kähler metrics on $M$ which have $\Omega$ as their fundamental 2-form. We note also that all associated metrics have the same volume element $dV = \frac{1}{2^n n!}\Omega^n$.

In the contact case we have a two step process for constructing associated metrics. Starting with any Riemannian metric $k'$, define a metric $k$ by

$$k(X, Y) = k'(-X + \eta(X)\xi, -Y + \eta(Y)\xi) + \eta(X)\eta(Y).$$

$k$ is a Riemannian metric with respect to which $\eta$ is the covariant form of $\xi$. Polarizing $d\eta$ on the contact subbundle $\{\eta = 0\}$ using $k$ as in the symplectic case gives an associated metric for some constant c and hence that g is Einstein.

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$g$ and a tensor field $\phi$ of type (1,1) such that $\phi^2 = -I + \eta \otimes \xi$. As in the symplectic case $d\eta(X,Y) = g(X, \phi Y)$. We also refer to $(\eta, g)$ or $(\phi, \xi, \eta.g)$ as a *contact metric structure*. For any associated metric $dV = \frac{1}{2^n n!}\eta \wedge (d\eta)^n$.

Given a contact metric structure $(\phi, \xi, \eta.g)$ we define a tensor field $h$ by $h = \frac{1}{2}\mathcal{L}_\xi \phi$. $h$ is a symmetric operator which anti-commutes with $\phi$; $h\xi = 0$ and $h$ vanishes if and only if $\xi$ is Killing. When $\xi$ is Killing, the contact metric is said to be $K$-*contact*. We also have the following useful formulas involving $h$ on a contact metric manifold.

$$\nabla_X \xi = -\phi X - \phi h X \tag{2.1}$$

$$\frac{1}{2}\big(R(\xi, X)\xi - \phi R(\xi, \phi X)\xi\big) = h^2 X + \phi^2 X \tag{2.2}$$

$$(\nabla_\xi h)X = \phi X - h^2 \phi X - \phi R(X, \xi)\xi \tag{2.3}$$

$$Ric(\xi) = 2n - trh^2 \tag{2.4}$$

For a general reference to these ideas see [3].

We close this introduction to symplectic and contact manifolds with an example. Let $M$ be an (n+1)-dimensional $C^\infty$ manifold and $\bar\pi : TM \longrightarrow M$ its tangent bundle. If $(x^1, \ldots, x^{n+1})$ are local coordinates on $M$, set $q^i = x^i \circ \bar\pi$; then $(q^1, \ldots, q^{n+1})$ together with the fibre coordinates $(v^1, \ldots, v^{n+1})$ form local coordinates on $TM$. If $X$ is a vector field on $M$, its *vertical lift* $X^V$ on $TM$ is the vector field defined by $X^V\omega = \omega(X) \circ \bar\pi$ where $\omega$ is a 1-form on $M$, which on the left side of this equation is regarded as a function on $TM$. For an affine connection $D$ on $M$, the *horizontal lift* $X^H$ of $X$ is defined by $X^H\omega = D_X\omega$. The *connection map* $K : TTM \longrightarrow TM$ is defined by

$$KX^H = 0, \;\; K(X_t^V) = X_{\bar\pi(t)}, \;\; t \in TM.$$

$TM$ admits an almost complex structure $J$ defined by

$$JX^H = X^V, \;\; JX^V = -X^H.$$

Dombrowski [12] showed that $J$ is integrable if and only if $D$ has vanishing curvature and torsion.

If now $G$ is a Riemannian metric on $M$ and $D$ its Levi-Civita connection, we define a Riemannian metric $\bar g$ on $TM$ called the *Sasaki metric*, by

$$\bar g(X,Y) = G(\bar\pi_* X, \bar\pi_* Y) + G(KX, KY)$$

where $X$ and $Y$ are vector fields on $TM$. Since $\bar\pi_* \circ J = -K$ and $K \circ J = \bar\pi_*$, $\bar g$ is Hermitian for the almost complex structure $J$.

On $TM$ define a 1-form $\beta$ by $\beta(X)_t = G(t, \bar\pi_* X)$, $t \in TM$ or equivalently by the local expression $\beta = \sum G_{ij} v^i dq^j$. Then $d\beta$ is a symplectic structure on $TM$ and in particular $2d\beta$ is the fundamental 2-form of the almost Hermitian structure $(J, \bar g)$. Thus $TM$ has an almost Kähler structure which is Kählerian if and only if $(M, G)$ is flat (Dombrowski [12], Tachibana and Okumura [21]).

Let $\mathbf{R}$ denote the curvature tensor of $G$, $\bar{\nabla}$ the Levi-Civita connection of $\bar{g}$ and $\bar{R}$ the curvature tensor of $\bar{g}$. Complete formulas for $\bar{\nabla}$ and $\bar{R}$ can be found in [14]; here we give just two of the four formulas describing the connection.

$$(\bar{\nabla}_{X^V} Y^H)_t = -\frac{1}{2}(\mathbf{R}(X,t)Y)^H \tag{2.5}$$

$$(\bar{\nabla}_{X^H} Y^H)_t = (D_X Y)^H_t - \frac{1}{2}(\mathbf{R}(X,Y)t)^V \tag{2.6}$$

The tangent sphere bundle $\pi : T_1M \longrightarrow M$ is the hypersurface of $TM$ defined by $\sum G_{ij} v^i v^j = 1$. The vector field $N = v^i \frac{\partial}{\partial v^i}$ is a unit normal, as well as the position vector for a point $t$. The Weingarten map $A$ of $T_1M$ with respect to the normal $N$ is given by $AU = -U$ for any vertical vector $U$ and $AX = 0$ for any horizontal vector X (see e.g. [3,p.132]). Thus many computations on $T_1M$ involving horizontal vector fields can be done directly on $TM$.

Let $g'$ denote the metric on $T_1M$ induced from $\bar{g}$ on $TM$. Define $\phi'$, $\xi'$ and $\eta'$ on $T_1M$ by

$$\xi' = -JN, \;\; JX = \phi' X + \eta'(X)N.$$

$\eta'$ is the contact form on $T_1M$ induced from the 1-form $\beta$ on $TM$ as one can easily check. However $g'(X, \phi' Y) = 2d\eta'(X,Y)$, so strictly speaking $(\phi', \xi', \eta', g')$ is not a contact metric structure. Of course the difficulty is easily rectified and we shall take

$$\eta = \frac{1}{2}\eta', \;\; \xi = 2\xi', \;\; \phi = \phi', \;\; g = \frac{1}{4}g' \tag{2.7}$$

as the standard contact metric structure on $T_1M$. In local coordinates

$$\xi = 2v^i \left(\frac{\partial}{\partial x^i}\right)^H;$$

on $TM$ the vector field $v^i(\frac{\partial}{\partial x^i})^H$ is the so-called geodesic flow.

We can now compute $\nabla\xi$ in two ways, by equation (2.1) and by using (2.5) and (2.6). Comparing these we can determine the tensor field $h$ for the standard contact metric structure on $T_1M$. For a vertical vector $U$ at $t \in T_1M$ we have

$$hU_t = U_t - (\mathbf{R}_{KU,t}t)^V.$$

For a horizontal vector $X$ orthogonal to $\xi$ we have

$$hX_t = -X_t + (\mathbf{R}_{\pi_* X,t}t)^H.$$

For example, if the base manifold $(M, G)$ is of constant curvature $+1$, the structure on $T_1M$ is K-contact (Tashiro [23]). If the base manifold is flat then the non-zero eigenvalues of $h$ are $\pm 1$, each with multiplicity n, and $T_1M$ is locally $E^{n+1} \times S^n(4)$, 4 being the constant curvature of the sphere owing to the homothetic change in metric (2.7). If the base manifold is of constant curvature -1, the non-zero eigenvalues of $h$ are $\pm 2$, each with multiplicity n.

### 3. Integrals of scalar curvatures on symplectic and contact manifolds

We now want to consider a number of integral functionals defined on the set of metrics associated to a symplectic or contact structure. To begin we need to see how the set $\mathcal{A}$ of associated metrics sits in the set $\mathcal{M}$ of all Riemannian metrics with the same total volume; for a more detailed treatment see [5].

Let $M$ be a symplectic manifold and $g_t = g + tD + O(t^2)$ be a path of metrics in $\mathcal{A}$. We will use the same letter $D$ to denote $D$ as a tensor field of type (1,1) and of type (0,2), $D^i{}_j = g^{ik}D_{kj}$. Now

$$g(X, JY) = \Omega(X, Y) = g_t(X, J_tY) = g(X, J_tY) + tg(X, DJ_t) + O(t^2)$$

from which

$$J = J_t + tDJ_t + O(t^2).$$

Applying $J_t$ on the right and $J$ of the left we have

$$J_t = J + tJD + O(t^2).$$

Squaring this yields $JDJ - D = 0$ and hence $JD + DJ = 0$. Conversely if $D$ is a symmetric tensor field which anti-commutes with $J$, then $g_t = ge^{tD}$ is a path of associated metrics. We summarize this and the corresponding result in the contact case as follows (cf.[5],[6]).

**Lemma 3.1.** *Let $M$ be a symplectic or contact manifold and $g \in \mathcal{A}$. A symmetric tensor field $D$ is tangent to a path in $\mathcal{A}$ at $g$ if and only if*

$$DJ + JD = 0 \tag{3.1}$$

*in the symplectic case and*

$$D\xi = 0,\ D\phi + \phi D = 0 \tag{3.2}$$

*in the contact case.*

Similar to the role played by Lemma 1.1 in critical point problems on $\mathcal{M}$, we have the following lemma for critical point problems on $\mathcal{A}$.

**Lemma 3.2.** *Let $T$ be a second order symmetric tensor field on $M$. Then $\int_M T^{ij}D_{ij}\,dV_g = 0$ for all symmetric tensor fields $D$ satifying (3.1) in the symplectic case and (3.2) in the contact case if and only if $TJ = JT$ in the symplectic case and $\phi T - T\phi = \eta \otimes \phi T\xi - (\eta \circ T\phi) \otimes \xi$ in the contact case (i.e. $\phi$ and $T$ commute when restricted to the contact subbundle).*

**Proof.** We give the proof in the symplectic case; the proof in the contact case being similar. Let $X_1, \ldots X_{2n}$ be a local $J$-basis defined on a neighborhood $\mathcal{U}$ (i.e. $X_1, \ldots X_{2n}$ is an orthonormal basis with respect to $g$ and $X_{2i} = JX_{2i-1}$) and note that the first vector field $X_1$ may be any unit vector field on $\mathcal{U}$. Let $f$ be a $C^\infty$ function with compact support in $\mathcal{U}$ and define a path of metrics $g(t)$ as follows. Make no change in $g$ outside $\mathcal{U}$ and within $\mathcal{U}$ change $g$ only in the planes spanned by $X_1$ and $X_2$ by the matrix

$$\begin{pmatrix} 1 + tf + \frac{1}{2}t^2f^2 & \frac{1}{2}t^2f^2 \\ \frac{1}{2}t^2f^2 & 1 - tf + \frac{1}{2}t^2f^2 \end{pmatrix}.$$

It is easy to check that $g(t) \in \mathcal{A}$ and clearly the only non-zero components of $D$ are $D_{11} = -D_{22} = f$. Then $\int_M T^{ij} D_{ij}\, dV_g = 0$ becomes

$$\int_M (T^{11} - T^{22}) f\, dV_g = 0$$

Thus since $X_1$ was any unit vector field on $\mathcal{U}$,

$$T(X, X) = T(JX, JX)$$

for any vector field $X$. Since $T$ is symmetric, linearization gives $TJ = JT$. Conversely, if $T$ commutes with $J$ and $D$ anti-commutes with $J$, then $trTD = trTJDJ = trJTDJ = -trTD$, giving $T^{ij}D_{ij} = 0$.

**Theorem 3.3 (Blair-Ianus).** *Let $M$ be a compact symplectic manifold and $\mathcal{A}$ the set of metrics associated to the symplectic form. Then $g \in \mathcal{A}$ is a critical point of $A(g) = \int_M R\, dV_g$ if and only if the Ricci operator of $g$ commutes with the almost complex structure corresponding to $g$.*

**Proof.** The proof is again to compute $\frac{dA}{dt}$ at $t = 0$ for a path $g(t)$ in $\mathcal{A}$. Since all associated metrics have the same volume element this is easier than in the Riemannian case. In particular we have,

$$\begin{aligned} \left.\frac{dA}{dt}\right|_{t=0} &= \frac{d}{dt} \int_M R_{kji}{}^k g^{ji}\, dV_g \Big|_{t=0} \\ &= \int_M D_{kji}{}^k g^{ji} - R_{ji} D^{ji}\, dV_g \\ &= -\int_M R^{ji} D_{ji}\, dV_g, \end{aligned}$$

the other terms being divergences and hence contributing nothing to the integral. Setting $\left.\frac{dA}{dt}\right|_{t=0} = 0$, the result follows from Lemma 3.2.

We now review some known properties of almost Kähler manifold. First of all

$$\nabla_k J^k{}_l = 0, \tag{3.3}$$

$$(\nabla_k J_{ip}) J_j{}^p = (\nabla_p J_{ij}) J_k{}^p; \tag{3.4}$$

an almost Hermitian structure satisfying this last condition is call a *quasi-Kähler structure*.

The $*$-*Ricci tensor* and the $*$-*scalar curvature* are defined by

$$R^*_{ij} = R_{iklt} J^{kl} J_j{}^t, \ \ R^* = R^*_i{}^i.$$

The Ricci identity yields

$$\nabla_t \nabla_k J_j{}^t = (R_{kt} - R^*_{kt}) J_j{}^t$$

where $R^*_{kt} J_j{}^t$ is skew-symmetric in $j$ and $k$. Therefore

$$\nabla_t \nabla_k J_j{}^t + \nabla_t \nabla_j J_k{}^t = R_{kt} J_j{}^t + R_{jt} J_k{}^t. \tag{3.5}$$

The most important property of $R^*$ is that

$$R - R^* = -\frac{1}{2} |\nabla J|^2$$

and hence $R - R^* \leq 0$ with equality holding if and only if the metric is Kähler. Thus Kähler metrics are maxima of the functional

$$K(g) = \int_M R - R^* \, dV_g$$

on $\mathcal{A}$ and the question that S. Ianus and I [8] were first interested in was whether these were the only critical points. The surprising result is that the critical point condition is again $QJ = JQ$, $Q$ denoting the Ricci operator.

**Theorem 3.4 (Blair-Ianus).** *Let $M$ be a compact symplectic manifold and $\mathcal{A}$ the set of metrics associated to the symplectic form. Then $g \in \mathcal{A}$ is a critical point of $K(g)$ if and only if $QJ = JQ$.*

**Proof.** To compute $\frac{dK}{dt}$ at $t = 0$, we must differentiate $R^* = R_{iklt} J^{kl} J^{it}$ along a path $g(t)$ in $\mathcal{A}$. Since $\Omega$ is fixed,

$$\left.\frac{\partial J_{ki}}{\partial t}\right|_{t=0} = 0, \quad \left.\frac{\partial J^i{}_j}{\partial t}\right|_{t=0} = -D^{im} J_{mj}, \quad \left.\frac{\partial J^{kl}}{\partial t}\right|_{t=0} = 0.$$

Then proceeding as before using (3.3)

$$\left.\frac{dK}{dt}\right|_{t=0} = \int_M [-R^{jm} + \nabla_i (J^{km} \nabla_k J^{ij}) + R^{*jm}] D_{jm} \, dV_g.$$

By Lemma 3.2, the critical point condition is that the symmetric part of the expression in brackets commutes with $J$. This is a long equation; some of its terms cancel by virtue of the quasi-Kähler condition (3.4) and the other terms combine by virtue of (3.5) to give the result.

The question as to whether or not on an almost Kähler manifold satisfying $QJ = JQ$ is Kählerian seems to be difficult. In [13] S. I. Goldberg showed that if $J$ commutes with the curvature operator, then the metric is Kählerian and conjectured that a compact almost-Kähler Einstein manifold is Kählerian. K. Sekigawa [20] proved that a compact almost-Kähler Einstein manifold with non-negative scalar curvature is Kählerian.

In [9] A. J. Ledger and I proved the contact analogues of these theorems, which we present here without proof.

**Theorem 3.5 (Blair-Ledger).** *Let $M$ be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $A(g) = \int_M R\, dV_g$ if and only if $Q$ and $\phi$ commute when restricted to the contact subbundle.*

This integral was further studied in dimension 3 by D. Perrone [19], who gave the critical point condition as

$$\nabla_\xi h = 0.$$

To see this, recall that in dimension 3, the Ricci operator determines the full curvature tensor, i.e.

$$R(X,Y)Z = (g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y)$$

$$-\frac{R}{2}(g(Y,Z)X - g(X,Z)Y).$$

Therefore the operator $l$ defined $lX = R(X,\xi)\xi$ by is given by

$$lX = QX - \eta(X)Q\xi + g(Q\xi,\xi)X - g(QX,\xi)\xi - \frac{R}{2}(X - \eta(X)\xi)$$

from which

$$(l\phi - \phi l)X = (Q\phi - \phi Q)X + \eta(X)\phi Q\xi - g(Q\phi X,\xi)\xi. \tag{3.6}$$

Thus the critical point condition is $l\phi - \phi l = 0$. Now recall equations (2.2) and (2.3), viz. $\frac{1}{2}(-l + \phi l\phi) = h^2 + \phi^2$ and $\nabla_\xi h = \phi - \phi h^2 - \phi l$. Applying $\phi$ to the first of these and adding to the second gives $\nabla_\xi h = \frac{1}{2}(l\phi - l\phi)$ and thus the critical point condition may be expressed as $\nabla_\xi h = 0$.

In the contact case the *$*$-scalar curvature* is defined by $R^* = R_{iklt}\phi^{kl}\phi^{it}$ and it was shown by Olszak [18] that

$$R - R^* - 4n^2 = -\frac{1}{2}|\nabla\phi|^2 + 2n - trh^2 \leq 0$$

with equality holding if and only if the metric is Sasakian.

**Theorem 3.6 (Blair-Ledger).** *Let $M$ be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $K(g) = \int_M R - R^* - 4n^2\, dV_g$ if and only if $Q - 2nh$ and $\phi$ commute when restricted to the contact subbundle.*

In dimension 3, the argument giving Perrone's result gives the critical point condition as $\nabla_\xi h = -2\phi h$, a condition that will be important in the next lecture.

For contact manifolds of general odd dimension, if $g$ is critical for both $A$ and $K$, $h = 0$ so $\xi$ is Killing. To see this note that the commutativities of Theorems 3.5 and 3.6 together imply that $\phi h = h\phi$ but $\phi h = -h\phi$ and hence $h = 0$ easily follows. As with the almost Kähler case, the question of whether or not a K-contact structure satisfying $Q\phi = \phi Q$ is Sasakian would seem to be difficult.

## 4. Integral of the Ricci curvature in the direction of the characteristic vector field

We devote this section to a discussion of a particular functional defined on the set of metrics associated to a contact structure. The main theorem is the following [6].

**Theorem 4.1 (Blair).** *Let $M$ be a compact regular contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $L(g) = \int_M Ric(\xi)\, dV_g$ if and only if $g$ is $K$-contact.*

One might conjecture this without the regularity, however we have the following counterexample: The standard contact metric structure on the tangent sphere bundle of a compact surface of constant curvature -1 is a critical point of $L$ but is not K-contact. It is a result of Y. Tashiro [23] that the standard contact metric structure on the tangent sphere bundle of a Riemannian manifold is K-contact if and only if the base manifold is of constant curvature +1. Also recall the result of [4] that the standard contact structure of the tangent sphere bundle of a compact Riemannian manifold of non-positive constant curvature is not regular. Our second result is the following theorem [7].

**Theorem 4.2 (Blair).** *Let $T_1M$ be the tangent sphere bundle of a compact Riemannian manifold $(M, G)$ and $\mathcal{A}$ the set of all Riemannian metrics associated to its standard contact structure. Then the standard associated metric is a critical point of the functional $L(g)$ if and only if $(M, G)$ is of constant curvature +1 or -1.*

Recall that by a K-contact structure we mean a contact metric structure for which $\xi$ is Killing and that this is the case if and only if $h = 0$. Recall also equation (2.4), viz.

$$Ric(\xi) = 2n - trh^2.$$

Thus K-contact metrics when they occur are maxima for the function $L(g)$ on $\mathcal{A}$. Also the critical point question for $L(g)$ is the same as that for $\int_M |h|^2\, dV_g$ or $\int_M |\tau|^2\, dV_g$ where $\tau(X,Y) = (\mathcal{L}_\xi g)(X,Y) = 2g(X, h\phi Y)$. This last integral was studied by Chern and Hamilton [11] for 3-dimensional contact manifolds as a functional on $\mathcal{A}$ regarded as the set of CR-structures on M (there was an error in their calculation of the critical point condition as was pointed out by Tanno[22]).

**Proof of Theorem 4.1.** As with our other critical point problems, the first step is to compute $\frac{dL}{dt}$ at $t = 0$ for a path $g(t) \in \mathcal{A}$

$$\left.\frac{dL}{dt}\right|_{t=0} = \int_M (-h^i{}_m h^{mk} - R^k{}_{rs}{}^i \xi^r \xi^s + 2h^{ik}) D_{ik}\, dV_g.$$

Thus if $g(0)$ is a critical point, Lemma 3.2 gives

$$R(X,\xi)\xi = -\phi^2 X - h^2 X + 2hX \tag{4.1}$$

as the critical point condition. Using equation (2.3) this becomes

$$(\nabla_\xi h)X = -2\phi h X. \tag{4.2}$$

From this we see that the eigenvalues of $h$ are constant along the integral curves of $\xi$ and that for an eigenvalue $\lambda \neq 0$ and unit eigenvector $X$, $g(\nabla_\xi X, \phi X) = -1$.

If now $M$ is a regular contact manifold, then $M$ is a principal circle bundle with $\xi$ tangent to the fibres; locally $M$ is $\mathcal{U} \times S^1$ where $\mathcal{U}$ is a neighborhood on the base manifold. Since $h\phi + \phi h = 0$, we may choose an orthonormal $\phi$-basis of eigenvectors of $h$ at some point of $\mathcal{U} \times S^1$ say, $X_{2i-1}$, $X_{2i} = \phi X_{2i-1}$, $\xi$. Since the eigenvalues are constant along the fibre, we can continue this basis along the fibre with at worst a change of orientation of some of the eigenspaces when we return to the starting point. Thus if $Y$ is a vector field along the fibre, we may write

$$Y = \sum_i (\alpha_{2i-1} X_{2i-1} + \beta_{2i} X_{2i}) + \gamma\xi$$

where the coefficients are periodic functions.

Now suppose that the critical point $g$ is not a K-contact metric. Since $\phi$ and $h$ anti-commute, we may assume that all the $\lambda_{2i-1}, i = 1, \ldots, n$ are non-negative. Also from equation (4.2) it is easy to see that if some of the $\lambda_{2i-1}$ vanish, the zero eigenspace of $h$ is parallel along $\xi$ and hence we may choose the corresponding $X_{2i-1}$ and $X_{2i}$ parallel along a fibre. Again since $M$ is regular we may choos a vector field $Y$ on $\mathcal{U} \times S^1$ such that at least some $\alpha_{2i-1} \not\equiv 0$ for some $\lambda_{2i-1} \neq 0$ and $Y$ is horizontal, i.e. $\eta(Y) = 0$, and projectable, i.e. $[\xi, Y] = 0$. Writing $Y = \sum_i (\alpha_{2i-1} X_{2i-1} + \beta_{2i} X_{2i})$ along a fibre we have using (2.1)

$$0 = [\xi, Y] = \nabla_\xi Y - \nabla_Y \xi$$

$$= \sum_i \big((\xi\alpha_{2i-1})X_{2i-1} + \alpha_{2i-1}\nabla_\xi X_{2i-1} + (\xi\beta_{2i})X_{2i} + \beta_{2i}\nabla_\xi X_{2i}$$

$$+\alpha_{2i-1}X_{2i} + \lambda_{2i-1}\alpha_{2i-1}X_{2i} - \beta_{2i}X_{2i-1} + \lambda_{2i-1}\beta_{2i}X_{2i-1}\big).$$

Taking components we have

$$0 = \xi\alpha_{2j-1} + \sum_i \alpha_{2i-1} g(\nabla_\xi X_{2i}, X_{2j}) + \lambda_{2j-1}\beta_{2j},$$

$$0 = \xi\beta_{2j} + \sum_i \beta_{2i} g(\nabla_\xi X_{2i}, X_{2j}) + \lambda_{2j-1}\alpha_{2j-1}.$$

Multiplying the first of these by $\beta_{2j}$, the second by $\alpha_{2j-1}$ and summing on $j$ we have

$$\xi\Big(\sum_j \alpha_{2j-1}\beta_{2j}\Big) = -\sum_j \lambda_{2j-1}(\alpha_{2j-1}^2 + \beta_{2j}^2) \leq 0.$$

Thus $\sum_j \alpha_{2j-1}\beta_{2j}$ is a non-increasing, non-constant function along the integral curve, contradicting its periodicity.

In preparation for a sketch of the proof of Theorem 4.2 we state the following lemma of Cartan.

**Lemma 4.3.** *Let $(M,G)$ be a Riemannian manifold, $D$ the Levi-Civita connection of $G$ and $\mathbf{R}$ its curvature tensor. Then $(M,G)$ if locally symmetric if and only if*

$$(D_X\mathbf{R})(Y,X,Y,X) = 0$$

*for all orthonormal pairs $\{X,Y\}$.*

**Proof of Theorem 4.2.** As we have seen the tangent sphere bundle, $T_1M$, inherits a contact structure from the symplectic structure on $TM$ and a natural associated metric from the Sasaki metric on $TM$. After computing $(R(U,\xi)\xi)_t, t \in T_1M$ for a vertical tangent vector $U$, we consider the critical point condition (4.1) and compare horizontal and vertical parts. This yields for any orthonomal pair $\{X,t\}$ on the base manifold $(M,G)$

$$(D_t\mathbf{R})(X,t)t = 0 \tag{4.3}$$

and

$$\mathbf{R}(\mathbf{R}(X,t)t,t)t = X, \tag{4.4}$$

see [7] for more details. From (4.3) and Lemma 4.3 we see that $(M,G)$ is locally symmetric.

Now working on $(M,G)$, for each unit tangent vector $t \in T_mM$, let $[t]^\perp$ denote the subspace of $T_mM$ orthogonal to $t$ and define a symmetric linear transformation $L_t : [t]^\perp \longrightarrow [t]^\perp$ by $L_tX = \mathbf{R}(X,t)t$. Then from (4.4) we have that $(L_t)^2 = I$ and hence that the eigenvalues of $L_t$ are $\pm 1$. Now $M$ is irreducible, for if $M$ had a locally Riemannian product structure, choosing $t$ to one factor and $X$ tangent to the other we would have $\mathbf{R}(X,t)t = 0$, contradicting the fact that the only eigenvalues of $L_t$ are $\pm 1$. However the sectional curvature of an irreducible locally symmetric space does not change sign. Thus if for some $t$, $L_t$ had both +1 and -1 occuring as eigenvalues, there would be sectional curvatures equal to +1 and -1. Consequently only one eigenvalue can occur and hence $(M,G)$ is a space of constant curvature +1 or -1.

Conversely if $(M,G)$ has constant curvature $c$, let $U$ be a vertical vector tangent to $T_1M$ and $X$ a horizontal vector orthogonal to $\xi$. Then at a point $t$, $hU_t = (1-c)U_t$, $hX_t = (c-1)X_t$, $(R(\xi,U)\xi)_t = -c^2U_t$ and $(R(\xi,X)\xi)_t = (3c^2-4c)X_t$. Substituting these into the critical point condition (4.1) we see that it is satisfied when $c = \pm 1$.

**Remarks: I.** Recently Mr. S. R. Deng has begun the study of the second variation for the functional $L(g)$.

**Proposition 4.4 (Deng).** *Let $g \in \mathcal{A}$ be a critical point of $L(g)$, then at $g$, $\frac{d^2L}{dt^2}$ is nonpositive.*

**II.** Clearly in dimension 3 by Perrone's form of the critical point condition for $A$ and the form (4.2) for $L$, we see that if $g$ is a critical point for both of them, $g$ is a K-contact metric. Now the *Webster scalar curvature* $W$ on a 3-dimensional contact metric manifold is defined by

$$W = \frac{1}{8}(R + \frac{1}{4}|\tau|^2 + 2);$$

by virtue of (2.4) and $\frac{1}{4}|\tau|^2 = |h|^2$, $W$ becomes

$$W = \frac{1}{8}(R - Ric(\xi) + 4).$$

Chern and Hamilton [11] studied the functional $E_W(g) = \int_M W\, dV_g$ for 3-dimensional contact manifolds as a functional on $\mathcal{A}$ regarded as the set of CR-structures on M and proved the following Theorem.

**Theorem 4.5 (Chern-Hamilton).** *Let $M$ be a compact 3-dimensional contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $E_W(g) = \int_M W\, dV_g$ if and only if $g$ is K-contact.*

An alternate proof was given by D. Perrone [19]. In view of the work we have done so far we can prove this theorem as follows.

**Proof of Theorem 4.5.** Clearly it is enough to consider $\int_M R - Ric(\xi)\, dV_g$ and having computed the derivatives of each term separately we see that

$$\frac{d}{dt}\int_M R - Ric(\xi)\, dV_g \Big|_{t=0} = \int_M (-R^{ki} + h^i{}_m h^{mk} + R^k{}_{rs}{}^i \xi^r \xi^s - 2h^{ik}) D_{ik}\, dV_g.$$

Thus the critical point condition is

$$(Q\phi - \phi Q) - (l\phi - \phi l) - 4\phi h = -\eta \otimes \phi Q\xi + (\eta \circ Q\phi) \otimes \xi.$$

So far we have not used the fact that we are in dimension three and hence this is the critical point condition for the intergral of the generalized Tanaka-Webster scalar curvature as defined by Tanno [22]. Now in dimension 3 we can combine this condition with (3.6) to get $h = 0$.

## 5. The Abbena-Thurston manifold as a critical point

In 1976 W. Thurston [24] gave an example of a compact symplectic manifold with no Kähler structure. We will begin by discussing this manifold briefly and then turn to a natural Riemannian metric on this manifold introduced by E. Abbena [1]. For details of the topological obstructions to a Kähler structure we refer to [24] or [1] and simply remark here that the first Betti number of this manifold is 3 whereas the odd-dimensional Betti numbers of a compact Kähler are even.

Let $G$ be the closed connected subgroup of $GL(4, \mathbf{C})$ defined by

$$\left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & 0 \\ 0 & 1 & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i a} \end{pmatrix} \middle| \, a_{12}, a_{13}, a_{23}, a \in \mathbf{R} \right\},$$

i.e. $G$ is the product of the Heisenberg group and $S^1$. Let $\Gamma$ be the discrete subgroup of $G$ with integer entries and $M = G/\Gamma$. Denote by $x, y, z, t$ coordinates on $G$, say for $A \in G$, $x(A) = a_{12}$, $y(A) = a_{23}$, $z(A) = a_{13}$, $t(A) = a$. If $L_B$ is left translation by $B \in G$, $L_B^* dx = dx$, $L_B^* dy = dy$, $L_B^*(dz - xdy) = dz - xdy$, $L_B^* dt = dt$. In particular these forms are invariant under the action of $\Gamma$; let $\pi : G \longrightarrow M$, then there exist 1-forms $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ on $M$ such that $dx = \pi^*\alpha_1$, $dy = \pi^*\alpha_2$, $dz - xdy = \pi^*\alpha_3$, $dt = \pi^*\alpha_4$. Setting $\Omega = \alpha_4 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3$ we see that $\Omega \wedge \Omega \neq 0$ and $d\Omega = 0$ on $M$ giving $M$ a symplectic structure.

The vector fields

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \ \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \ \mathbf{e}_3 = \frac{\partial}{\partial z}, \ \mathbf{e}_4 = \frac{\partial}{\partial t}$$

are dual to $dx$, $dy$, $dz - xdy$, $dt$ and are left invariant. Moreover $\{\mathbf{e}_i\}$ is orthonormal with respect to the left invariant metric on $G$ given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2 + dt^2.$$

On $M$ the corresponding metric is $g = \sum \alpha_i \otimes \alpha_i$. The Riemannian manifold $(M, g)$ is refered to as the *Abbena-Thurston manifold.*

Moreover $M$ carries an almost complex structure defined by

$$J\mathbf{e}_1 = \mathbf{e}_4, \ J\mathbf{e}_2 = -\mathbf{e}_3, \ J\mathbf{e}_3 = \mathbf{e}_2, \ J\mathbf{e}_4 = -\mathbf{e}_1.$$

Then noting that $\Omega(X, Y) = g(X, JY)$, we see that $g$ is an associated metric.

The curvature of $g$ was computed by E. Abbena in [1]. With respect to the basis $\{\mathbf{e}_i\}$ the non-zero components of the curvature tensor are

$$R_{1221} = \frac{3}{4}, \ R_{2332} = -\frac{1}{4}, \ R_{1331} = -\frac{1}{4}.$$

Thus the Ricci operator $Q$ is given by the matrix

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we note that $Q^2$ is parallel with respect to the Levi-Civita connection of $g$ but that $Q$ is not parallel.

The following observation stems from conversations between Won-Tae Oh and myself.

**Proposition 5.1 (Blair-Oh).** *The Abbena-Thurston manifold is a critical point of the functional*

$$I(g) = \int_M (\frac{4}{3} trQ^3 - R)\, dV_g$$

*on $\mathcal{M}$.*

**Sketch of the proof.** Computing the critical point condition for $I(g)$ on $\mathcal{M}$ in general we find that it is

$$2(\nabla_m\nabla_i R_{jk}R^{km} + \nabla_m\nabla_j R_{ik}R^{km} - \nabla^m\nabla_m R_{jk}R^k{}_i - g_{ij}\nabla_m\nabla_l R^m{}_k R^{kl}$$

$$-2R_{im}R^m{}_k R^k{}_j + \frac{1}{2}R_{ij}) + \frac{1}{2}(\frac{4}{3}trQ^3 - R)g_{ij} = cg_{ij}.$$

Now since $Q^2$ is parallel and $Q^3 = \frac{1}{4}Q$ on the Abbena-Thurston manifold we see that this metric on the underlying manifold $M = G/\Gamma$ is a critical point of $I(g)$.

From the expression for $Q$ it is clear that $(M,g)$ is not Einstein nor is $QJ = JQ$. Thus this metric is not a critical point for $A(g) = \int_M R\,dV_g$ considered as a functional on $\mathcal{M}$ or on $\mathcal{A}$ or for $K(g) = \int_M R - R^*\,dV_g$ on $\mathcal{A}$. In particular it does not give a negative answer to the question of whether or not an almost Kähler manifold satisfying $QJ = JQ$ is Kählerian. On the other hand $(M,g)$ is a critical point for $K$ in a different context; C. M. Wood [25] showed that the Abbena-Thurston manifold is a critical point of $K$ defined with respect to variations through almost complex structures $J$ which preserve $g$. For this problem the critical point condition is

$$[J, \nabla^*\nabla J] = 0,$$

where $\nabla^*\nabla J$ is the rough Laplacian of the metric in question.

## 6. Problems involving other integrands

Finally we turn to a brief discussion of some related problems. In the Riemannian geometry of contact metric manifolds the tensor fields $l$ and $S$ defined by $lX = R(X,\xi)\xi$ and $S(X,Y) = R(X,Y)\xi$ play important roles. For example on a K-contact manifold $l$ is the identity and on a Sasakian manifold $S(X,Y) = \eta(Y)X - \eta(X)Y$. More generally we have noted (equation (2.3)) that

$$\nabla_\xi h = \phi - \phi h^2 - \phi l.$$

Thus it seems reasonable to consider functionals defined by integrals such as $\int_M |l|^2\,dV_g$ and $\int_M |S|^2\,dV_g$. In the case of the first of these Mr. S. R. Deng computed the critical point condition of $\int_M |l|^2\,dV_g$ as a functional on $\mathcal{A}$ and noted the following.

**Proposition 6.1 (Deng).** *Let $M$ be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then a K-contact metric is a critical point of the functional $\int_M |l|^2\,dV_g$ on $\mathcal{A}$. More generally if for a metric $g$, $\nabla_\xi h = 0$, then $g$ is a critical point if and only if $h^3 - h = 0$.*

The original functionals $A(g)$, $B(g)$, $C(g)$, $D(g)$ on $\mathcal{M}$ have been study further in the context of contact geometry by Muto [17] and Yamaguchi and Chūman [26]; the general thrust of their work is to suppose that a critical point is a Sasakian metric. For example we have the following results.

**Theorem 6.2 (Muto).** *If a critical point $g$ of $A(g)$, $B(g)$, $C(g)$ or $D(g)$ is a Sasakian metric, then its scalar curvature is constant.*

We remarked at the outset that Einstein metrics were critical points of $B(g)$. From Theorem 6.2 and the critical point condition for $B(g)$ we have an immediate converse in the case of a Sasakian metric.

**Theorem 6.3 (Yamaguchi and Chūman).** *In order for a Sasakian metric to be a critical point of $B(g)$ it is necessary and sufficient that it be an Einstein metric.*

The two papers [17] and [26] give many results and focus in particular on Sasakian submersions, discussing relations between $B(g)$, $C(g)$ and $D(g)$ defined relative to the bundle space and the base space of the submersion. There are other contexts where some of these functionals have been discussed, but further discussion would take us beyond the scope of these lectures.

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