APPENDIX

A.1 THE CLASSICAL BOCHNER THEOREM

In this section we recall the classical theorem of Bochner and for the sake of completeness we will give the proof.
First of all we say that a family Λ of probability measures on $(E, \mathcal{B}(E))$ is **tight** if for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset E$ such that

$$\mu(K_\varepsilon) \geq 1 - \varepsilon \quad \text{for all } \mu \in \Lambda.$$

Here E is a separable Banach space and $\mathcal{B}(E)$ its Borel σ-field.
A sequence of measures (μ_p) on $(E, \mathcal{B}(E))$ is said to be **weakly convergent** to a measure μ if for every $\varphi \in C_b(E)$

$$\lim_{p \to \infty} \int_E \varphi(x)\mu_p(dx) = \int_E \varphi(x)\mu(dx).$$

A family Λ of measures on $(E, \mathcal{B}(E))$ is said **relatively compact** if for an arbitrary sequence $(\mu_p) \subset \Lambda$ contains a weakly convergent subsequence (μ_{p_k}) to a measure μ on $(E, \mathcal{B}(E))$.
The following result is due to Prokhorov (cf. [12, Theorem 2.3]).

Theorem A.1.1 *A set Λ of probability measures on $(E, \mathcal{B}(E))$ is tight if and only if is relatively compact.*

For the proof of the Bochner theorem we need the following lemma.

Lemma A.1.2 *Assume that (μ_p) is a sequence of probability measures on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$. If $\varphi_p(z) := \widehat{\mu_p}(z)$ converges to $\varphi(z)$ for all $z \in \mathbb{R}^N$ and if this convergence is uniform in $\{z \in \mathbb{R}^N : |z| \leq a\}$ for a small number a, then $\{\mu_p : p \in \mathbb{N}\}$ is tight.*

Proof: Since φ_p is continuous and (φ_p) converges uniformly in a neighborhood of 0, it follows that φ is continuous at 0 and $\varphi(0) = 1$. Hence, for any $\varepsilon > 0$ there is $\delta \in (0, a)$ such that

$$|\varphi(z) - 1| < \varepsilon \quad \text{for all } |z| < \delta.$$

It follows now from the uniform convergence of (φ_p) to φ in $\{z \in \mathbb{R}^N : |z| < \delta\}$ that there exists $M = M(\varepsilon)$ independent of z such that

$$|\varphi_p(z) - 1| < \frac{\varepsilon}{2}, \quad \forall p \geq M, \forall |z| < \delta.$$

So, by Fubini's theorem we have

$$\begin{aligned} 1 - \frac{\varepsilon}{2} &< \frac{1}{(2\delta)^N} \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} \Re\varphi_p(z)\, dz \\ &= \frac{1}{(2\delta)^N} \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} \int_{\mathbb{R}^N} \cos\langle z, x\rangle \mu_p(dx)\, dz \\ &= \int_{\mathbb{R}^N} \left(\frac{\sin \delta x_1}{\delta x_1}\right) \cdots \left(\frac{\sin \delta x_N}{\delta x_N}\right) \mu_p(dx), \end{aligned}$$

where the last equality can be seen by induction.
Since $\left|\frac{\sin \delta x_j}{\delta x_j}\right|$ is dominated by 1 on $[-R, R]$ and by $\frac{1}{\delta R}$ elsewhere, we obtain

$$\int_{\mathbb{R}^N} \left(\frac{\sin \delta x_1}{\delta x_1}\right) \cdots \left(\frac{\sin \delta x_N}{\delta x_N}\right) \mu_p(dx) \leq \mu_p([-R, R]^N) + \left(\frac{1}{\delta R}\right)^N.$$

Take now $R := \frac{1}{\delta}\left(\frac{2}{\varepsilon}\right)^{\frac{1}{N}}$, it follows that

$$1 - \varepsilon < \mu_p([-R, R]^N) \quad \text{for all } p \geq M.$$

This gives the proof of the lemma. □

We are now ready to show the classical Bochner theorem. The arguments are taking from the proof in one dimensional case (see [20, Theorem 2.6.6]).

Theorem A.1.3 *A functional $\varphi : \mathbb{R}^N \to \mathbb{C}$ is the Fourier transform of a probability measure on $\mathbb{R}^N$ if and only if φ is a continuous positive definite functional satisfying $\varphi(0) = 1$.*

Proof: It suffices to prove the sufficiency. Assume that $\varphi : \mathbb{R}^N \to \mathbb{C}$ is a continuous positive definite functional with $\varphi(0) = 1$. Then, by Lemma 1.1.3, φ is uniformly continuous and bounded. Take now $g : \mathbb{R}^N \to \mathbb{C}$ integrable, bounded and uniformly continuous. If we set $\dot{y} := (y_2, \ldots, y_N) \in \mathbb{R}^{N-1}$ then we have

$$
\begin{aligned}
& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(\xi-\eta)g(\xi)\overline{g(\eta)}\,d\xi d\eta \\
= \; & \int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(\xi_1-\eta_1,\dot{\xi}-\dot{\eta})g(\xi_1,\dot{\xi})\overline{g(\eta_1,\dot{\eta})}\,d\xi_1 d\eta_1\right) d\dot{\xi}d\dot{\eta} \\
= \; & \int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}} \lim_{p\to\infty} \sum_{l,k=-p^2}^{p^2} \varphi\left(\frac{l}{p}-\frac{k}{p},\dot{\xi}-\dot{\eta}\right) g\left(\frac{l}{p},\dot{\xi}\right)\overline{g\left(\frac{k}{p},\dot{\eta}\right)}\left(\frac{1}{p}\right)^2 d\dot{\xi}d\dot{\eta} \\
\geq \; & 0. \qquad (9)
\end{aligned}
$$

Put $g(\xi) := \mathcal{N}(0, \frac{p}{4} Id_{\mathbb{R}^N})(\xi)e^{-i\langle x,\xi\rangle}$, $\xi, x \in \mathbb{R}^N$. Since

$$
\begin{aligned}
\mathcal{N}(0,\frac{p}{4}Id_{\mathbb{R}^N})(\xi) &= \mathcal{N}(0,\frac{p}{4}Id_{\mathbb{R}^N})(-\xi) \text{ and} \\
\mathcal{N}(0,\frac{p}{4}Id_{\mathbb{R}^N})(\xi) * \mathcal{N}(0,\frac{p}{4}Id_{\mathbb{R}^N})(\xi) &= \mathcal{N}(0,\frac{p}{2}Id_{\mathbb{R}^N})(\xi)
\end{aligned}
$$

it follows that, for $x \in \mathbb{R}^N$,

$$
\int_{\mathbb{R}^N} g(\xi+\eta)\overline{g(\eta)}\,d\eta = e^{-i\langle x,\xi\rangle}\frac{1}{(p\pi)^{\frac{N}{2}}}e^{-\frac{|\xi|^2}{p}},
$$

where $\mathcal{N}(0, \frac{p}{4} Id_{\mathbb{R}^N})(\xi) : \frac{1}{(\pi(p/2))^{\frac{N}{2}}}e^{-2\frac{|\xi|^2}{p}}$ for $\xi \in \mathbb{R}^N$ and $Id_{\mathbb{R}^N}$ denotes the identity operator in $\mathbb{R}^N$. So by (9) we obtain

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(\xi-\eta)g(\xi)\overline{g(\eta)}\,d\xi d\eta \\
&= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} g(\xi+\eta)\overline{g(\eta)}\,d\eta\right)\varphi(\xi)\,d\xi \\
&= \frac{1}{(p\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N} \varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\langle x,\xi\rangle}\,d\xi.
\end{aligned}
$$

Thus,

$$
f_p(x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\langle x,\xi\rangle}\,d\xi \geq 0
$$

for $x \in \mathbb{R}^N$. Define the measure $\mu_p(dx) := f_p(x)dx$. We propose to show now that μ_p is a probability measure on $\mathbb{R}^N$. First, by applying Fubini's theorem, observe that

$$
\begin{aligned}
& \mu_p([-a_1,a_1]\times\ldots\times[-a_N,a_N]) \\
= \; & \frac{1}{(2\pi)^N}\int_{-a_1}^{a_1}\ldots\int_{-a_N}^{a_N}\int_{\mathbb{R}^N} \varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\xi_1 x_1}\ldots e^{-i\xi_N x_N}\,d\xi\,dx_1\ldots dx_N \\
= \; & \frac{1}{\pi^N}\int_{\mathbb{R}^N} \varphi(\xi)e^{-\frac{|\xi|^2}{p}}\left(\frac{\sin a_1\xi_1}{\xi_1}\right)\ldots\left(\frac{\sin a_N\xi_N}{\xi_N}\right) d\xi.
\end{aligned}
$$

On the other hand, for $m \in \mathbb{N}$, we have

$$\frac{1}{m^N}\int_0^m \dots \int_0^m \mu_p([-a_1,a_1]\times\dots\times[-a_N,a_N])\,da_1\dots da_N = \\ \int_0^1 \dots \int_0^1 \mu_p([-a_1 m,a_1 m]\times\dots\times[-a_N m,a_N m])\,da_1\dots da_N.$$

Since $\mu_p([-a_1 m,a_1 m]\times\dots\times[-a_N m,a_N m]) \uparrow \mu_p(\mathbb{R}^N)$ as $m\to\infty$, it follows from the monotone convergence theorem that

$$\begin{aligned} &\mu_p(\mathbb{R}^N) \\ = \;& \lim_{m\to\infty}\frac{1}{m^N}\int_0^m \dots \int_0^m \mu_p([-a_1,a_1]\times\dots\times[-a_N,a_N])\,da \\ = \;& \lim_{m\to\infty}\frac{1}{(\pi m)^N}\int_0^m \dots \int_0^m \int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}\left(\frac{\sin a_1\xi_1}{\xi_1}\right)\dots\left(\frac{\sin a_N\xi_N}{\xi_N}\right)d\xi\,da \\ = \;& \lim_{m\to\infty}\frac{1}{\pi^N}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}\left(\frac{1-\cos m\xi_1}{m\xi_1^2}\right)\dots\left(\frac{1-\cos m\xi_N}{m\xi_N^2}\right)d\xi \\ = \;& \lim_{m\to\infty}\frac{1}{\pi^N}\int_{\mathbb{R}^N}\varphi(\frac{\xi_1}{m},\dots,\frac{\xi_N}{m})e^{-\frac{|\xi|^2}{mp}}\left(\frac{1-\cos \xi_1}{\xi_1^2}\right)\dots\left(\frac{1-\cos \xi_N}{\xi_N^2}\right)d\xi, \end{aligned}$$

where $a := (a_1,\dots,a_N)$. Since $\varphi(0)=1$ and

$$\frac{1-\cos\xi_j}{\xi_j^2}\ge 0,\quad \int_{\mathbb{R}}\frac{1-\cos\xi_j}{\xi_j^2} = \pi, \quad \forall j=1,\dots,N, \tag{10}$$

it follows from the dominated convergence theorem that

$$\mu_p(\mathbb{R}^N) = \varphi(0) = 1.$$

Let compute now the Fourier transform of μ_p. For $a_j \ge 0$ and $m\in\mathbb{N}$, observe that

$$\begin{aligned} &\left|\frac{1}{(2\pi)^N}\int_{-a_N m}^{a_N m}\dots\int_{-a_1 m}^{a_1 m} e^{i\langle z,x\rangle}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\langle \xi,x\rangle}\,d\xi dx\right| \\ \le \;& \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\varphi(\xi)e^{-\frac{|\xi|^2}{p}}e^{-i\langle \xi,x\rangle}\,d\xi dx \\ = \;& \mu_p(\mathbb{R}^N) = 1. \end{aligned}$$

So it follows from the dominated convergence theorem that

$$\begin{aligned}
&\widehat{\mu_p}(z)\\
=\;& \lim_{m\to\infty} \frac{1}{(2\pi)^N} \int_0^1 \cdots \int_0^1 \left[\int_{-a_N m}^{a_N m} \cdots \int_{-a_1 m}^{a_1 m} \int_{\mathbb{R}^N} \varphi(\xi) e^{-\frac{|\xi|^2}{p}} e^{-i\langle \xi - z, x\rangle}\, d\xi dx \right] da \\
=\;& \lim_{m\to\infty} \frac{1}{\pi^N} \int_0^1 \cdots \int_0^1 \int_{\mathbb{R}^N} \varphi(\xi) e^{-\frac{|\xi|^2}{p}} \left(\frac{\sin a_1 m(\xi_1 - z_1)}{\xi_1 - z_1} \right) \cdots \\
& \qquad \left(\frac{\sin a_N m(\xi_N - z_N)}{\xi_N - z_N} \right) d\xi da \\
=\;& \lim_{m\to\infty} \frac{1}{\pi^N} \int_{\mathbb{R}^N} \varphi(\xi) e^{-\frac{|\xi|^2}{p}} \left(\frac{1 - \cos m(\xi_1 - z_1)}{m(\xi_1 - z_1)^2} \right) \cdots \left(\frac{1 - \cos m(\xi_N - z_N)}{m(\xi_N - z_N)^2} \right) d\xi \\
=\;& \lim_{m\to\infty} \frac{1}{\pi^N} \int_{\mathbb{R}^N} \varphi(z + \frac{\xi}{m}) e^{-\frac{|z+\frac{\xi}{m}|^2}{p}} \left(\frac{1 - \cos \xi_1}{\xi_1^2} \right) \cdots \left(\frac{1 - \cos \xi_N}{\xi_N^2} \right) d\xi.
\end{aligned}$$

So, again by the dominated convergence theorem and (10), we obtain

$$\widehat{\mu_p}(z) = \varphi(z) e^{-\frac{|z|^2}{p}}, \quad z \in \mathbb{R}^N.$$

Finally,

$$\lim_{p\to\infty} \widehat{\mu_p}(z) = \varphi(z)$$

uniformly in $|z| \le 1$. The theorem follows now from Lemma A.1.2 and Theorem A.1.1. $\square$

A.2 C_0-SEMIGROUPS

In this section we give a general discussion of the abstract Cauchy problem for unbounded linear operators on a Banach space and its relation to the theory of C_0-semigroups. For more details we refer to the recent books of Engel-Nagel [16] and Arendt-Batty-Hieber-Neubrander [1]. A particular attention will be dedicated to the class of eventually norm continuous C_0-semigroups.

We consider the abstract Cauchy problem

$$(ACP) \qquad \begin{cases} \frac{du}{dt}(t) = Au(t), & t \ge 0, \\ u(0) = x, \end{cases}$$

where A is a possibly unbounded linear operator with domain $D(A)$ on a Banach space X and $x \in X$. A *classical solution* of (ACP) is a function $u \in C^1(\mathbb{R}_+, X)$ such that $u(t) \in D(A)$ for all $t \ge 0$ and u satisfies (ACP).

Now we introduce C_0-semigroups.

Definition A.2.1 *A family $T(\cdot) := (T(t))_{t\ge0}$ of bounded linear operators on X is called a C_0-semigroup if*

(i) $\lim_{t\downarrow 0} \|T(t)x - x\| = 0, \quad \forall x \in X,$

(ii) $T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$ *and* $T(0) = Id$.

The generator of $T(\cdot)$ is the linear operator A defined by

$$\begin{aligned} D(A) &= \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists } \}, \\ Ax &= \lim_{t\downarrow 0} \frac{T(t)x - x}{t}, \quad x \in D(A). \end{aligned}$$

One can prove that the generator is always a closed and densely defined operator. The domain $D(A)$ satisfies

$$T(t)D(A) \subseteq D(A) \text{ and } AT(t)x = T(t)Ax, \quad \forall t \geq 0.$$

Moreover, for $x \in D(A)$,

$$\frac{d}{dt} T(t)x = AT(t)x, \quad t \geq 0.$$

This shows that for $x \in D(A)$ the problem (ACP) has a classical solution $u(\cdot) := T(\cdot)x$. We say that (ACP) is *well-posed* if for each initial value $x \in D(A)$ there is a unique classical solution $u(\cdot, x)$ satisfying

> for any sequence $(x_n) \subset D(A)$ with $\lim_{n\to\infty} \|x_n - x\| = 0$ for $x \in D(A)$, the corresponding classical solutions $u(\cdot, x_n)$ converges to $u(\cdot, x)$ uniformly on compact subsets of $\mathbb{R}_+$.

The following theorem shows that wellposedness is equivalent to generation of C_0-semigroups.

Theorem A.2.2 *Let A be a linear operator with domain $D(A)$ on a Banach space X. Then the following assertion are equivalent:*

(a) *A is the generator of a C_0-semigroup on X.*

(b) *The abstract Cauchy problem (ACP) associated with A is well-posed.*

On the other hand, for a C_0-semigroup $T(\cdot)$, one has

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0,$$

for some constants $\omega \in \mathbb{R}$ and $M \geq 1$. If we denote by

$$\omega_0(A) := \inf\{\omega \in \mathbb{R} : \text{ there is } M_\omega \geq 1 \text{ with } \|T(t)\| \leq M_\omega e^{\omega t}, \forall t \geq 0\}$$

the growth bound of the C_0-semigroup $T(\cdot)$ with generator A, then $(\omega_0(A), \infty) \subset \rho(A)$, the resolvent set of A, and the resolvent $R(\lambda, A)$ of A is given by

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt, \quad x \in X, \, \lambda > \omega_0(A).$$

In the following proposition we collect some properties of C_0-semigroups and their generators.

Proposition A.2.3 *Let $T(\cdot)$ be a C_0-semigroup on a Banach space X. If $(A, D(A))$ denotes its generator then the following assertions hold:*

(i) $\int_0^t T(s)x\,ds \in D(A)$ *and* $A\int_0^t T(s)x\,ds = T(t)x - x$ *for all* $x \in X$ *and* $t \geq 0$.

(ii) $A\int_0^t T(s)x\,ds = \int_0^t T(s)Ax\,ds = T(t)x - x$ *for all* $x \in D(A)$ *and* $t \geq 0$.

(iii) $\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x$ *for all* $x \in X$.

(iv) $R(\lambda, A)T(t) = T(t)R(\lambda, A)$ *for all* $\lambda \in \rho(A)$ *and* $t \geq 0$.

In many applications it is difficult to identify the domain of the generator of a C_0-semigroup. It is often the case that one can find a "large" subspace of $D(A)$ as defined now.

Definition A.2.4 *A subspace D of $D(A)$, the domain of a linear operator A on a Banach space X is called a core for A if D is dense in $D(A)$ for the graph norm*

$$\|x\|_A := \|x\| + \|Ax\|, \quad x \in D(A).$$

A useful criterion for subspaces to be a core for the generator of a C_0-semigroup is given by the following proposition.

Proposition A.2.5 *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and D be a subspace of $D(A)$. If D is dense in X and invariant under $(T(t))_{t\geq 0}$, then D is a core for A.*

We propose now to introduce different classes of semigroups. In the sequel we denote the sector in $\mathbb{C}$ of angle δ by

$$\Sigma_\delta := \{\lambda \in \mathbb{C} : |\arg\lambda| < \delta\} \setminus \{0\}.$$

Definition A.2.6 *A family $(T(z))_{z\in\Sigma_\theta\cup\{0\}} \subset \mathcal{L}(X)$ on a Banach space X is called an analytic semigroup (of angle $\theta \in (0, \frac{\pi}{2}]$) if*

(a1) $T(0) = Id$ *and* $T(z_1 + z_2) = T(z_1)T(z_2)$ *for all* $z_1, z_2 \in \Sigma_\theta$.

(a2) *The map $z \mapsto T(z)$ is analytic in Σ_θ.*

(a3) $\lim_{\Sigma_{\theta'}\ni z\to 0} T(z)x = x$ *for all* $x \in X$ *and* $0 < \theta' < \theta$.

If, in addition

(a4) $\|T(z)\|$ *is bounded in* $\Sigma_{\theta'}$ *for every* $0 < \theta' < \theta$,

we call $(T(z))_{z\in\Sigma_\theta\cup\{0\}}$ a bounded analytic semigroup.

The following theorem gives useful characterization of generators of bounded analytic semigroups.

Theorem A.2.7 *Let $(A, D(A)$ be an operator on a Banach space X. Then the following assertions are equivalent:*

(i) *A generates a bounded analytic semigroup $(T(z))_{z\in\Sigma_\theta\cup\{0\}}$ on X.*

(ii) *A generates a bounded C_0-semigroup $T(\cdot)$ on X with $\mathrm{rg}(T(t)) \subset D(A)$ for all $t > 0$, and*

$$\|AT(t)\| \leq \frac{M}{t}$$

for some positive constant M.

(iii) *There is $\delta \in (0, \frac{\pi}{2})$ such that $e^{\pm i\delta}A$ generate bounded C_0-semigroups on X.*

(iv) *$\Sigma_{\theta+\frac{\pi}{2}} \subset \rho(A)$ and for each $\varepsilon \in (0, \theta)$ there is $M_\varepsilon \geq 1$ such that*

$$\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|} \quad \textit{for all } 0 \neq \lambda \in \overline{\Sigma}_{\theta+\frac{\pi}{2}-\varepsilon}.$$

From (ii) above we see that if $T(\cdot)$ is an analytic semigroup, then the maps $0 < t \mapsto T(t)x$ are differentiable for every $x \in X$. This motivate the following definition.

Definition A.2.8 *A C_0-semigroup $T(\cdot)$ on a Banach space X is called eventually (resp. immediately) differentiable if there is $t_0 \geq 0$ such that the maps $(t_0, \infty) \ni t \mapsto T(t)x$ (resp. $(0, \infty) \ni t \mapsto T(t)x$) are differentiable for every $x \in X$.*

A characterization of differentiable semigroups in terms of the spectrum and the growth of the resolvent can be proved (cf. [16, Theorem II.4.14]).

Finally we recall the class of eventually norm continuous C_0-semigroups.

Definition A.2.9 *A C_0-semigroup $T(\cdot)$ on a Banach space X is called eventually (resp. immediately) norm continuous if there is $t_0 \geq 0$ such that the mapping $(t_0, \infty) \ni t \mapsto T(t) \in \mathcal{L}(X)$ (resp. $(0, \infty) \ni t \mapsto T(t) \in \mathcal{L}(X)$) is norm continuous.*

It is an easy exercise to see that the following implications between the three classes of semigroups hold:

$$\begin{aligned}&\text{analytic} \Longrightarrow \text{immediately differentiable} \Longrightarrow \text{immediately norm continuous,}\\&\text{analytic} \Longrightarrow \text{eventually differentiable} \Longrightarrow \text{eventually norm continuous.}\end{aligned} \tag{11}$$

On Hilbert spaces eventually norm C_0-semigroups are completely characterized (cf. [16, Theorem II.4.20]). But in general Banach spaces such a characterization remain open. However a necessary condition can be obtained as the following theorem shows.

Theorem A.2.10 *If $(A, D(A))$ is the generator of an eventually norm continuous C_0-semigroup $T(\cdot)$ on a Banach space X, then, for every $a \in \mathbb{R}$, the set*

$$\{\lambda \in \sigma(A) : \Re\lambda \geq a\}$$

is bounded.