CHAPTER 3

THE ORNSTEIN-UHLENBECK SEMIGROUP

In this chapter we are concerned with the Ornstein-Uhlenbeck semigroup, first on $C_b(H)$, and finally on L^p–spaces with invariant measure. The Ornstein-Uhlenbeck semigroup is related to the solution of the following linear stochastic differential equation

$$(SDE) \begin{cases} dX(t,x) = AX(t,x)dt + Q^{\frac{1}{2}}dW(t), \quad t \geq 0 \\ X(0,x) = x \in H, \end{cases}$$

where $Q \in \mathcal{L}(H)$ is selfadjoint and nonnegative and A generates a C_0–semigroup $(e^{tA})_{t\geq 0}$ on H. The process W is a standard cylindrical Wiener process on H. Under appropriate assumptions (see [12]) the solution to (SDE) is a Gaussian and Markov process in H, called the Ornstein-Uhlenbeck process. The associated Ornstein-Uhlenbeck semigroup on $B_b(H)$, the space of bounded and Borel functions from H into $\mathbb{R}$, is given by

$$R_t\varphi(x) := \mathbb{E}\left(\varphi(X(t,x))\right), \quad t \geq 0,\, x \in H,\, \varphi \in B_b(H).$$

This is the semigroup solution of the associated Kolmogorov equation

$$(KE) \begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\mathrm{Tr}(QD^2u(t,x) + \langle x, A^*Du(t,x)\rangle, \quad t > 0,\, x \in H, \\ u(0,x) = \varphi(x), \quad x \in H. \end{cases}$$

The basic assumption in this chapter is

$$\text{(H1)} \qquad Q_t := \int_0^t e^{sA}Qe^{sA^*}\,ds \in \mathcal{L}_1^+(H), \quad t > 0.$$

Under (H1) and by the change of variables

$$v(t, e^{tA}x) := u(t,x), \quad t \geq 0,\, x \in H,$$

one can see (cf. [8], [4]) that v is the unique solution of the parabolic equation

$$(PE)\quad \begin{cases} \frac{\partial}{\partial t}v(t,x) = \frac{1}{2}\mathrm{Tr}\left(e^{tA}Qe^{tA^*}D^2v(t,x)\right), & t>0,\, x\in H,\\ v(0,x)=\varphi(x), \quad x\in H, \end{cases}$$

and is given by

$$v(t,x)=\int_H \varphi(x+y)\mathcal{N}(0,Q_t)(dy), \quad x\in H,\, t\ge 0,$$

where $\varphi \in BUC^2(H)$. Therefore, if we suppose (H1) then the Ornstein-Uhlenbeck semigroup is given by

$$R_t\varphi(x)=\int_H \varphi(e^{tA}x+y)\mathcal{N}(0,Q_t)(dy), \quad x\in H,\, t\ge 0,$$

for $\varphi \in B_b(H)$. Now, by Lemma 1.2.7, we have, for $\varphi \in B_b(H)$,

$$R_t\varphi(x)=\int_H \varphi(y)\mathcal{N}(e^{tA}x,Q_t)(dy), \quad x\in H,\, t\ge 0.$$

3.1 The Ornstein-Uhlenbeck semigroup on $C_b(H)$

The aim of this section is to study the global regularity of the Ornstein-Uhlenbeck semigroup $(R_t)_{t\ge 0}$ on $C_b(H)$. Existence and uniqueness of a classical solution for (KE) will be also considered.

In this section we assume the *controllability condition* (see [31])

$$\text{(H2)} \qquad e^{tA}(H)\subseteq Q_t^{\frac{1}{2}}(H) \quad \text{for all } t>0.$$

If we suppose in addition that $(e^{tA})_{t\ge 0}$ is exponentially stable, that is, there are constants $M\ge 1$ and $\omega>0$ such that $\|e^{tA}\|\le Me^{-t\omega}$ for all $t\ge 0$, then it follows from the strong continuity of the semigroup $(e^{tA})_{t\ge 0}$ and Exercise 3.3.22 that, for any $t>0$, the subspace $Q_t^{\frac{1}{2}}(H)$ is dense in H and so, by Remark 1.3.2,

$$\ker Q_t=\{0\} \quad \text{for all } t>0.$$

This will be needed for the application of the Cameron-Martin formula. Regularity properties of the semigroup $(R_t)_{t\ge 0}$ are given by the following result.

Theorem 3.1.1 *Suppose that (H1) and (H2) are satisfied and* $\ker Q_t=\{0\}$ *for all* $t>0$. *Then, for any* $\varphi\in B_b(H)$ *and* $t>0$, *we have* $R_t\varphi \in BUC^\infty(H)$

and in particular, for $x, y, z \in H$,

$$\begin{aligned}
\langle DR_t\varphi(x), y\rangle &= \int_H \langle \Lambda_t y, Q_t^{-\frac{1}{2}} h\rangle \varphi(e^{tA}x + h)\mathcal{N}(0, Q_t)(dh), \\
\langle D^2 R_t\varphi(x)y, z\rangle &= \int_H \left[\langle \Lambda_t y, Q_t^{-\frac{1}{2}} v\rangle\langle \Lambda_t z, Q_t^{-\frac{1}{2}} v\rangle - \langle \Lambda_t y, \Lambda_t z\rangle\right] \cdot \\
&\qquad \varphi(e^{tA}x + v)\mathcal{N}(0, Q_t)(dv),
\end{aligned}$$

where $\Lambda_t := Q_t^{-\frac{1}{2}} e^{tA}$, $t > 0$. Moreover,

$$\begin{aligned}
|DR_t\varphi(x)| &\le \|\Lambda_t\|\|\varphi\|_\infty, \\
\|D^2 R_t\varphi(x)\| &\le \sqrt{2}\|\Lambda_t\|^2\|\varphi\|_\infty.
\end{aligned}$$

Furthermore, if for any $t > 0$, $R_t B_b(H) \subset C_b(H)$, then (H2) holds.

Proof: Let $t > 0$, $\varphi \in B_b(H)$ and $x \in H$. Since, by (H2), $e^{tA}x \in Q_t^{\frac{1}{2}}(H)$, it follows from the Cameron-Martin formula (see Corollary 1.3.5) that $\mathcal{N}(e^{tA}x, Q_t) \sim \mathcal{N}(0, Q_t)$ and

$$\frac{d\mathcal{N}(e^{tA}x, Q_t)}{d\mathcal{N}(0, Q_t)}(y) = \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} y\rangle\right).$$

Thus,

$$R_t\varphi(x) = \int_H \varphi(y)\exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} y\rangle\right)\mathcal{N}(0, Q_t)(dy).$$

Therefore, by a change of variables (see Lemma 1.2.7), we obtain

$$\begin{aligned}
\langle DR_t\varphi(x), y\rangle &= \int_H \langle \Lambda_t y, Q_t^{-\frac{1}{2}}(h - e^{tA}x)\rangle\varphi(h)\mathcal{N}(e^{tA}x, Q_t)(dh) \\
&= \int_H \langle \Lambda_t y, Q_t^{-\frac{1}{2}} h\rangle\varphi(e^{tA}x + h)\mathcal{N}(0, Q_t)(dh).
\end{aligned}$$

So by Proposition 1.3.1 we have

$$\begin{aligned}
|\langle DR_t\varphi(x), y\rangle|^2 &\le \|\varphi\|_\infty \int_H |\langle \Lambda_t y, Q_t^{\frac{1}{2}} h\rangle|^2\mathcal{N}(0, Q_t)(dh) \\
&= \|\varphi\|_\infty|\Lambda_t y|^2
\end{aligned}$$

for all $y \in H$. Similarly one obtains the second derivative of $R_t\varphi$ and the estimate follows by a simple computation. Let now prove the last assertion. Suppose that for any $\varphi \in B_b(H)$, the function $R_t\varphi(\cdot)$ is continuous and there is $x_0 \in H$ such that $e^{tA}x_0 \notin Q_t^{\frac{1}{2}}(H)$. It follows from the Cameron-Martin formula (Corollary 1.3.5) that, for any $n \in \mathbb{N}$, $\mathcal{N}(\frac{1}{n}e^{tA}x_0, Q_t) \perp \mathcal{N}(0, Q_t)$. This means that , for any $n \in \mathbb{N}$, there is $\Gamma_n \in \mathcal{B}(H)$ with

$$\mathcal{N}\left(\frac{1}{n}e^{tA}x_0, Q_t\right)(\Gamma_n) = 0 \text{ and } \mathcal{N}(0, Q_t)(\Gamma_n) = 1.$$

If we set $\Gamma := \cap_{n\in\mathbb{N}}\Gamma_n$, then

$$\mathcal{N}\left(\frac{1}{n}e^{tA}x_0, Q_t\right)(\Gamma) = 0 \text{ and } \mathcal{N}(0, Q_t)(\Gamma) = 1.$$

Now, we consider the characteristic function $\varphi := \chi_\Gamma$. Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} R_t\varphi\left(\frac{x_0}{n}\right) &= \mathcal{N}\left(\frac{1}{n}e^{tA}x_0, Q_t\right)(\Gamma) = 0 \text{ and} \\ R_t\varphi(0) &= \mathcal{N}(0, Q_t)(\Gamma) = 1. \end{aligned}$$

Hence, the function $R_t\varphi(\cdot)$ is not continuous at zero. This end the proof of the theorem. □

We show now that the Ornstein-Uhlenbeck semigroup $(R_t)_{t\geq 0}$ solves the Kolmogorov equation (KE) in the following sense.
We say that a function $u(t, x)$, $t \geq 0, x \in H$, is a *classical solution* of (KE) if

(a) $u : [0,\infty) \times H \to \mathbb{R}$ is continuous and $u(0, \cdot) = \varphi$,

(b) $u(t, \cdot) \in BUC^2(H)$ for all $t > 0$, and $QD^2u(t, x)$ is a trace class operator on H for all $x \in H$ and $t > 0$,

(c) $Du(t, x) \in D(A^*)$ for all $x \in H$ and $t > 0$,

(d) for any $x \in H$, $u(\cdot, x)$ is continuously differentiable on $(0,\infty)$ and fulfills (KE)

Under appropriate conditions we show now the existence and the uniqueness of a classical solution for (KE) (cf. [13, Theorem 6.2.4]).

Theorem 3.1.2 *Suppose(H1), (H2) and* $\ker Q_t = \{0\}$ *for all* $t > 0$. *If* $\Lambda_t A$ *has a continuous extension* $\overline{\Lambda_t A}$ *on* H *and* $\Lambda_t Q^{\frac{1}{2}}$ *is a Hilbert-Schmidt operator on* H *for every* $t > 0$*, then (KE) has a unique classical solution.*

Proof: For $\varphi \in B_b(H)$ we know, from Theorem 3.1.1, that, for any $t > 0$, $R_t\varphi \in BUC^\infty(H)$ and

$$\langle DR_t\varphi(x), Ay\rangle = \int_H \langle \Lambda_t Ay, Q_t^{-\frac{1}{2}}h\rangle\varphi(e^{tA}x + h)\mathcal{N}(0, Q_t)(dh)$$

for $y \in D(A)$, $t > 0$ and $x \in H$. So by Proposition 1.3.1, we obtain

$$|\langle DR_t\varphi(x), Ay\rangle| \leq \|\varphi\|_\infty\|\overline{\Lambda_t A}\||y|, \quad \forall y \in D(A),$$

for $t > 0$ and $x \in H$. Hence, $DR_t\varphi(x) \in D(A^*)$ for all $x \in H$ and $t > 0$.
Again from Theorem 3.1.1 we deduce that

$$\begin{aligned} &\langle D^2R_t\varphi(x)Q^{\frac{1}{2}}e_j, Q^{\frac{1}{2}}e_j\rangle = \\ &= \int_H \left(\langle \Lambda_t Q^{\frac{1}{2}}e_j, Q_t^{-\frac{1}{2}}y\rangle^2 - |\Lambda_t Q^{\frac{1}{2}}e_j|^2\right)\varphi(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy) \end{aligned}$$

for $x \in H$, $t > 0$ and $j \in \mathbb{N}$. It follows from Proposition 1.3.1 that

$$\left|\langle D^2 R_t\varphi(x) Q^{\frac{1}{2}} e_j, Q^{\frac{1}{2}} e_j\rangle\right| \leq 2|\Lambda_t Q^{\frac{1}{2}} e_j|^2 \|\varphi\|_\infty$$

for $x \in H$ and $t > 0$. This implies that $QD^2 R_t\varphi(x)$ is a trace class operator on H for all $x \in H$ and $t > 0$.
For any $x \in H$, the function $t \mapsto R_t\varphi(x)$ fulfills (KE) follows from a straightforward computation and is left to the reader. The uniqueness follows from the fact that Equation (PE) has a unique solution for an initial data $\varphi \in BUC^2(H)$. □

If the semigroup $(e^{tA})_{t\geq 0}$ is exponentially stable then the assumption "$\Lambda_t Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on H" is automatically satisfied as the following corollary shows.

Corollary 3.1.3 *Assume (H1) and (H2). If $\Lambda_t A$ has a continuous extension $\overline{\Lambda_t A}$ on H for every $t > 0$ and $(e^{tA})_{t\geq 0}$ is exponentially stable then (KE) has a unique classical solution.*

Proof: It suffices to prove that the assumptions of Theorem 3.1.2 are satisfied. Since

$$\Lambda_t = Q_t^{-\frac{1}{2}} e^{tA} = (Q_t^{-\frac{1}{2}} Q_\infty^{\frac{1}{2}})(Q_\infty^{-\frac{1}{2}} e^{\frac{t}{2}A}) e^{\frac{t}{2}A}, \quad t > 0,$$

it follows from Exercise 3.3.22 that Λ_t is a trace class operator and hence $\Lambda_t Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on H for every $t > 0$. □

3.2 Sobolev spaces with respect to Gaussian measures on H

In this section we propose to define and study the Sobolev spaces $W^{1,2}(H,\mu)$, $W_B^{1,2}(H,\mu)$ and $W^{2,2}(H,\mu)$, where $\mu := \mathcal{N}(0,B)$ and $B \in \mathcal{L}_1^+(H)$. Without loss of generality we suppose that $\ker B = \{0\}$ and consider an orthonormal system (e_k) and positive numbers λ_k with $Be_k = \lambda_k e_k$ for $k \in \mathbb{N}$.
Define the subspaces $\mathcal{E}(H)$ and $\mathcal{E}_A(H)$ of $BUC(H)$ by

$$\begin{aligned} \mathcal{E}(H) &:= \operatorname{Span}\{e^{i\langle x,h\rangle}; h \in H\} \\ \mathcal{E}_A(H) &:= \operatorname{Span}\{e^{i\langle x,h\rangle}; h \in D(A^*)\}. \end{aligned}$$

In the sequel the following lemma will play a crucial role.

Lemma 3.2.1 *For any $\varphi \in BUC(H)$, there is a sequence $(\varphi_{n,k})_{n,k\in\mathbb{N}} \subset \mathcal{E}(H)$ with*

(a) $\lim_{k\to\infty}\lim_{n\to\infty}\varphi_{n,k}(x)=\varphi(x), \quad \forall x\in H,$

(b) $\|\varphi_{n,k}\|_\infty \le \|\varphi\|_\infty, \quad \forall n,k\in\mathbb{N}.$

Thus, $\mathcal{E}(H)$ (resp. $\mathcal{E}_A(H)$) is dense in $L^2(H,\mu)$.

Proof: Since $D(A^*)$ is dense in H and $BUC(H)$ is dense in $L^2(H,\mu)$, and by the dominated convergence theorem, it suffices to show the existence of such a sequence.
To this purpose we assume first that $\dim H := d < \infty$ and consider the function φ_n satisfying

(i) φ_n is periodic with period n in all coordinate x_k, $k=1,\dots,d$,

(ii) $\varphi_n(x)=\varphi(x), \quad \forall x\in[-n-\frac{1}{2}, n-\frac{1}{2}]^d$,

(iii) $\|\varphi_n\|_\infty \le \|\varphi\|_\infty$.

Hence,

$$\lim_{n\to\infty}\varphi_n(x)=\varphi(x), \quad \forall x\in H.$$

On the other hand, any function φ_n, $n\in\mathbb{N}$, can be approximate, by using Fourier series, by functions in $\mathcal{E}(H)$. This proves the lemma for finite dimensional Hilbert spaces.
In the general case, let $\varphi\in BUC(H)$. Take

$$\psi_k(x):=\varphi(x_1,x_2,\dots,x_k,0,\dots), \quad x\in H,\ k\in\mathbb{N}.$$

Then it follows from the first step that there is $(\varphi_{n,k})_{n,k\in\mathbb{N}}\subset\mathcal{E}(H)$ with

$$\begin{aligned}\lim_{n\to\infty}\varphi_{n,k}(x) &= \psi_k(x), \quad \forall x\in H,\\ \|\varphi_{n,k}\|_\infty &\le \|\psi_k\|_\infty \le \|\varphi\|_\infty.\end{aligned}$$

Therefore, for any $x\in H$,

$$\lim_{k\to\infty}\lim_{n\to\infty}\varphi_{n,k}(x)=\varphi(x), \quad \forall x\in H.$$

□

For any $k\in\mathbb{N}$ we define the partial derivative in the direction e_k by

$$D_k\varphi(x):=\lim_{t\to 0}\frac{1}{t}(\varphi(x+te_k)-\varphi(x)), \quad x\in H$$

for $\varphi\in\mathcal{E}_A(H)$ (or $\varphi\in\mathcal{E}(H)$). We note that for $\varphi(x):=e^{i\langle x,h\rangle}$, we have $D_k\varphi(x)=ihe^{i\langle x,h\rangle}$ for $x,h\in H$.
The following proposition gives an integration by part formula.

Proposition 3.2.2 *For $\varphi,\tilde\varphi\in\mathcal{E}(H)$ and $k\in\mathbb{N}$ the following holds*

$$\int_H D_k\varphi(x)\tilde\varphi(x)\mu(dx) = -\int_H \varphi(x)D_h\tilde\varphi(x)\mu(dx)+\frac{1}{\lambda_k}\int_H x_k\varphi(x)\tilde\varphi(x)\mu(dx).$$

Proof: For $\varphi, \tilde{\varphi} \in \mathcal{E}(H)$ we have

$$\begin{aligned}\int_H D_k\varphi(x)\tilde{\varphi}(x)\mu(dx) &= \int_H ih_k e^{i\langle x,h\rangle} e^{i\langle x,\tilde{h}\rangle}\mu(dx) \\ &= ih_k \int_H e^{i\langle x,h+\tilde{h}\rangle}\mu(dx) \\ &= ih_k e^{-\frac{1}{2}\langle B(h+\tilde{h}),h+\tilde{h}\rangle} \text{ and} \\ \int_H \varphi(x)D_k\tilde{\varphi}(x)\mu(dx) &= i\tilde{h}_k e^{-\frac{1}{2}\langle B(h+\tilde{h}),h+\tilde{h}\rangle}.\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}&\frac{1}{\lambda_k}\int_H x_k\varphi(x)\tilde{\varphi}(x)\mu(dx) = \\ &= \frac{1}{\lambda_k}\int_H x_k e^{i\langle x,h+\tilde{h}\rangle}\mu(dx) \\ &= \frac{1}{i\lambda_k}\frac{d}{dt}\left(\int_H e^{it\langle x,e_k\rangle}e^{i\langle x,h+\tilde{h}\rangle}\mu(dx)\right)_{|t=0} \\ &= \frac{1}{i\lambda_k}\frac{d}{dt}\left(\int_H e^{i\langle x,te_k+h+\tilde{h}\rangle}\mu(dx)\right)_{|t=0} \\ &= \frac{1}{i\lambda_k}\frac{d}{dt}\left[\exp\left(-\frac{1}{2}\langle B(te_k+h+\tilde{h}),te_k+h+\tilde{h}\rangle\right)\right]_{|t=0} \\ &= \frac{1}{i\lambda_k}\left[-\lambda_k(h_k+\tilde{h}_k)e^{-\frac{1}{2}\langle B(h+\tilde{h}),h+\tilde{h}\rangle}\right] \\ &= i(h_k+\tilde{h}_k)e^{-\frac{1}{2}\langle B(h+\tilde{h}),h+\tilde{h}\rangle}.\end{aligned}$$

This proves the integration by part formula. □

The following proposition permits us to define the first Sobolev space with respect to the Gaussian measure μ.

Proposition 3.2.3 *For any $k \in \mathbb{N}$, the operator D_k with domain $\mathcal{E}(H)$ is closable on $L^2(H,\mu)$.*

Proof: Let $(\varphi_n) \subset \mathcal{E}(H)$ be such that $\lim_{n\to\infty}\varphi_n = 0$ and $\lim_{n\to\infty} D_k\varphi_n = \psi$ in $L^2(H,\mu)$. By Proposition 3.2.2 we have

$$\int_H D_k\varphi_n(x)\varphi(x)\mu(dx)+\int_H \varphi_n(x)D_k\varphi(x)\mu(dx) = \frac{1}{\lambda_k}\int_H x_k\varphi_n(x)\varphi(x)\mu(dx).$$

By Hölder's inequality, one can estimate the right hand side of the above equation and obtains

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_H x_k\varphi_n(x)\varphi(x)\mu(dx)\right|^2 \le \\ &\le \lim_{n\to\infty}\left(\int_H \varphi_n(x)^2\mu(dx)\cdot\int_H x_k^2\varphi(x)^2\mu(dx)\right) = 0\end{aligned}$$

for $\varphi \in \mathcal{E}(H)$. Hence,

$$\int_H \psi(x)\varphi(x)\mu(dx) = 0, \quad \forall \varphi \in \mathcal{E}(H).$$

Since $\mathcal{E}(H)$ is dense in $L^2(H,\mu)$, it follows that $\psi \equiv 0$. □

In the sequel we use the notation $D_k := \overline{D_k}$ for $k \in \mathbb{N}$.

Definition 3.2.4 *The first order Sobolev space $W^{1,2}(H,\mu)$ is defined by*

$$W^{1,2}(H,\mu) :=$$
$$\{\varphi \in L^2(H,\mu) : \varphi \in D(D_k),\ \forall k \in \mathbb{N},\ \text{and}\ \sum_{k=1}^{\infty} \int_H |D_k\varphi(x)|^2 \mu(dx) < \infty\}.$$

For $\varphi \in W^{1,2}(H,\mu)$, we denote by

$$D\varphi(x) := \sum_{k=1}^{\infty} D_k\varphi(x) e_k, \quad x \in H,$$

the gradient of φ at x, which exists as a $L^2(H,\mu)$–function and hence for almost every $x \in H$. It is clear that $W^{1,2}(H,\mu)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_{W^{1,2}(H,\mu)} :=$$
$$\langle \varphi, \psi \rangle_{L^2(H,\mu)} + \int_H \langle D\varphi(x), D\psi(x) \rangle \mu(dx), \quad \varphi, \psi \in W^{1,2}(H,\mu),$$

is a Hilbert space.

Now, we show that Proposition 3.2.2 remains valid in $W^{1,2}(H,\mu)$. To this purpose we need the following lemma.

Lemma 3.2.5 *If $\varphi \in W^{1,2}(H,\mu)$, then, for any $k \in \mathbb{N}$, $x_k\varphi \in L^2(H,\mu)$.*

Proof: It is easy to see that Proposition 3.2.2 holds for all $\varphi \in W^{1,2}(H,\mu)$ and $\tilde{\varphi} \in \mathcal{E}(H)$. So if we apply Proposition 3.2.2 with $\varphi = x_k g$ and $\tilde{\varphi} = g$ for $k \in \mathbb{N}$ and $g \in \mathcal{E}(H)$, then

$$\begin{aligned}
&\int_H x_k^2 g(x)^2 \mu(dx) = \\
&= \lambda_k \int_H (g(x) + x_k D_k g(x)) g(x) \mu(dx) + \lambda_k \int_H x_k g(x) D_k g(x) \mu(dx) \\
&= \lambda_k \int_H g(x)^2 \mu(dx) + 2\lambda_k \int_H x_k g(x) D_k g(x) \mu(dx).
\end{aligned}$$

So by Young's inequality we obtain

$$\begin{aligned}
&\int_H x_k^2 g(x)^2 \mu(dx) \le \\
&\le \lambda_k \int_H g(x)^2 \mu(dx) + \frac{1}{2} \int_H x_k^2 g(x)^2 \mu(dx) + 2\lambda_k^2 \int_H D_k g(x)^2 \mu(dx).
\end{aligned}$$

Thus,

$$\int_H x_k^2 g(x)^2 \mu(dx) \le 2\lambda_k \int_H g(x)^2 \mu(dx) + 4\lambda_k^2 \int_H D_k g(x)^2 \mu(dx).$$

This end the proof of the lemma. □

From the above lemma we obtain the following corollaries.

Corollary 3.2.6 *If $\varphi \in W^{1,2}(H,\mu)$, then $|x|\varphi \in L^2(H,\mu)$ and the following holds*

$$\int_H |x|^2 \varphi(x)^2 \mu(dx) \le 2\mathrm{Tr}B \int_H \varphi(x)^2 \mu(dx) + 4\|B\|^2 \int_H |D\varphi(x)|^2 \mu(dx).$$

Corollary 3.2.7 *For $\varphi, \psi \in W^{1,2}(H,\mu)$ the following holds*

$$\int_H D_k\varphi(x)\psi(x)\mu(dx) + \int_H \varphi(x) D_k\psi(x)\mu(dx) = \frac{1}{\lambda_k}\int_H x_k \varphi(x)\psi(x)\mu(dx).$$

By the same proof as for the first derivative one can see that, for any $h,k \in \mathbb{N}$ the operator $D_hD_k : \mathcal{E}(H) \to L^2(H,\mu)$ is closable on $L^2(H,\mu)$ and as before we use the notation $D_hD_k := \overline{D_hD_k}$.

Definition 3.2.8 *The second order Sobolev space $W^{2,2}(H,\mu)$ is defined by*

$$W^{2,2}(H,\mu) :=$$
$$\{\varphi \in L^2(H,\mu) : \varphi \in \bigcap_{h,k\in\mathbb{N}} D(D_hD_k) \text{ and } \sum_{h,k=1}^{\infty} \int_H |D_hD_k\varphi(x)|^2 \mu(dx) < \infty\}.$$

If $\varphi \in W^{2,2}(H,\mu)$, then, for a.e. $x \in H$ one can define a Hilbert-Schmidt operator $D^2\varphi(x)$ (since $\sum_{h,k\in\mathbb{N}} |D_hD_k\varphi(x)|^2 < \infty$ for a.e. $x \in H$) by

$$\langle D^2\varphi(x)y, z\rangle := \sum_{h,k=1}^{\infty} D_hD_k\varphi(x) y_h z_k, \quad y,z \in H, \text{ a.e. } x \in H.$$

It is easy to see that $W^{2,2}(H,\mu)$ endowed with the inner product

$$\langle \varphi, \psi\rangle_{W^{2,2}(H,\mu)} := \langle \varphi, \psi\rangle_{W^{1,2}(H,\mu)} + \sum_{h,k=1}^{\infty} \int_H \langle D_hD_k\varphi(x), D_hD_k\psi(x)\rangle \mu(dx)$$

is a Hilbert space.

In a similar way one can obtain the following useful result.

Proposition 3.2.9 *If $\varphi \in W^{2,2}(H,\mu)$, then $|x|\varphi \in W^{1,2}(H,\mu)$, $|x|^2\varphi \in L^2(H,\mu)$ and the following estimates hold*

$$\begin{aligned}\int_H |x|^2|D\varphi(x)|^2\mu(dx) &\leq 2\int_H \varphi(x)^2\mu(dx) + 4\mathrm{Tr}B\int_H |D\varphi(x)|^2\mu(x) + \\ &\quad 8\|B\|^2\int_H \mathrm{Tr}(D^2\varphi(x))^2\mu(dx), \\ \int_H |x|^4\varphi(x)^2\mu(dx) &\leq c\left(\int_H \varphi(x)^2\mu(dx) + \int_H |D\varphi(x)|^2\mu(dx) + \right. \\ &\quad \left. \int_H \mathrm{Tr}(D^2\varphi(x))^2\mu(dx)\right).\end{aligned}$$

For the characterization of the generator of the Ornstein-Uhlenbeck semigroup on $L^2(H,\mu)$ we need the notion of Malliavin derivatives.

We consider the operator $D_B : \mathcal{E}(H) \to L^2(H,\mu;H)$ defined by

$$D_B\varphi := B^{\frac{1}{2}}D\varphi \quad \text{for } \varphi \in \mathcal{E}(H).$$

Here $L^2(H,\mu;H)$ denotes the space of all strongly measurable functions $\Phi : H \to H$ satisfying $\int_H |\Phi(x)|^2\mu(dx) < \infty$.

Proposition 3.2.10 *The operator D_B with domain $\mathcal{E}(H)$ is closable in $L^2(H,\mu;H)$.*

Proof: Let $(\varphi_n) \subset \mathcal{E}(H)$ and $F \in L^2(H,\mu;H)$ are such that $\lim_{n\to\infty}\varphi_n = 0$ in $L^2(H,\mu)$ and $\lim_{n\to\infty} D_B\varphi_n = F$ in $L^2(H,\mu;H)$. This means that

$$\begin{aligned}&\lim_{n\to\infty}\int_H |D_B\varphi_n(x) - F(x)|^2\mu(dx) = \\ &= \lim_{n\to\infty}\int_H \sum_{k=1}^{\infty} |\sqrt{\lambda_k}D_k\varphi_n(x) - F_k(x)|^2\mu(dx) = 0.\end{aligned}$$

Since we have supposed that $\ker B = \{0\}$, it follows that, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty} D_k\varphi_n = \frac{1}{\sqrt{\lambda_k}}F_k \quad \text{in } L^2(H,\mu).$$

So by Proposition 3.2.3 we have, for any $k \in \mathbb{N}$, $F_k \equiv 0$, which proves the claim. □

As before we use the notation $D_B := \overline{D_B}$ and this will be called the *Malliavin derivative*. In a similar way we define the following spaces

$$\begin{aligned}W_B^{1,2}(H,\mu) &:= \{\varphi \in L^2(H,\mu) : D_B\varphi \in L^2(H,\mu;H)\}, \\ W_B^{2,2}(H,\mu) &:= \{\varphi \in L^2(H,\mu) : \varphi \in \bigcap_{h,k\in\mathbb{N}} D(D_hD_k) \text{ and} \\ &\quad \sum_{h,k=1}^{\infty}\int_H \lambda_h\lambda_k|D_hD_k\varphi(x)|^2\mu(dx) < \infty\}.\end{aligned}$$

3.3 The Ornstein-Uhlenbeck semigroup on L^p-spaces with invariant measure

The aim of this section is to study the Ornstein-Uhlenbeck semigroup on L^p–spaces with respect to an invariant measure.
Under appropriate assumptions we prove the existence and uniqueness of an invariant measure μ for the Ornstein-Uhlenbeck semigroup (R_t). This allows us to extend (R_t) to a C_0–semigroup on $L^p(H,\mu)$, $1 \le p < \infty$. We find sufficient conditions for the existence and uniqueness of a classical solution for (KE) on $L^p(H,\mu)$, $1 < p < \infty$ and finally we characterize the domain of the generator of the symmetric Ornstein-Uhlenbeck semigroup on $L^2(H,\mu)$.

In order to have an invariant measure for the Ornstein-Uhlenbeck semigroup we suppose in this section the following assumptions

(H3) $A : D(A) \to H$ generates a C_0 – semigoup $(e^{tA})_{t\ge 0}$ satisfying $\|e^{tA}\| \le Me^{-\omega t}$ for some constants $M \ge 1$, $\omega > 0$.

(H4) $Q \in \mathcal{L}(H)$ is a symmetric and positive operator and

$$Q_t := \int_0^t e^{sA} Q e^{sA^*}\, ds \in \mathcal{L}_1^+(H), \quad t \ge 0.$$

If we set $Q_\infty x := \int_0^\infty e^{sA} Q e^{sA^*}\, ds$, $x \in H$, then

$$Q_\infty x = \sum_{n=0}^\infty \int_n^{n+1} e^{sA} Q e^{sA^*}\, ds = \sum_{n=0}^\infty e^{nA} Q_1 e^{nA^*} x, \quad x \in H.$$

Hence,

$$\mathrm{Tr} Q_\infty \le M^2 \mathrm{Tr} Q_1 \sum_{n=0}^\infty e^{-2\omega n} < \infty,$$

which implies that $Q_\infty \in \mathcal{L}_1^+(H)$.
The following result shows the existence and uniqueness of invariant measure for the Ornstein-Uhlenbeck semigroup.

Proposition 3.3.1 *Assume that (H3) and (H4) hold. Then the Gaussian measure $\mu := \mathcal{N}(0, Q_\infty)$ is the unique invariant measure for the Ornstein-Uhlenbeck semigroup $(R_t)_{t\ge 0}$. This means that, for all $\varphi \in BUC(H)$,*

$$\int_H R_t\varphi(x)\mu(dx) = \int_H \varphi(x)\mu(dx).$$

Moreover, for all $\varphi \in BUC(H)$ and $x \in H$,

$$\lim_{t\to\infty} R_t\varphi(x) = \int_H \varphi(x)\mu(dx).$$

Proof: It follows from Lemma 3.2.1 that it suffices to show the proposition for $\varphi \in \mathcal{E}_A(H)$. For $\varphi_h(x) := e^{i\langle h,x\rangle}$, $x, h \in H$, we have

$$\begin{aligned}
\int_H R_t\varphi_h(x)\mu(dx) &= \int_H \int_H e^{i\langle h, e^{tA}x+y\rangle}\mathcal{N}(0,Q_t)(dy)\mu(dx) \\
&= \int_H e^{i\langle e^{tA}x,h\rangle - \frac{1}{2}\langle Q_t h,h\rangle}\mu(dx) \\
&= e^{-\frac{1}{2}\langle Q_t h,h\rangle - \frac{1}{2}\langle Q_\infty e^{tA^*}h, e^{tA^*}h\rangle} \\
&= e^{-\frac{1}{2}\langle (Q_t + e^{tA}Q_\infty e^{tA^*})h,h\rangle} \\
&= \int_H \varphi_h(x)\mu(dx),
\end{aligned}$$

where the last equality follows from the equation

$$Q_t + e^{tA}Q_\infty e^{tA^*} = Q_\infty, \quad t \geq 0. \tag{3.1}$$

On the other hand, we obtain

$$\begin{aligned}
\lim_{t\to\infty} R_t\varphi_h(x) &= \lim_{t\to\infty} e^{i\langle e^{tA}h,x\rangle - \frac{1}{2}\langle Q_t h,h\rangle} \\
&= e^{-\frac{1}{2}\langle Q_\infty h,h\rangle} \\
&= \int_H \varphi_h(x)\mu(dx).
\end{aligned}$$

For the uniqueness, we suppose that there is an invariant measure ν for (R_t). In particular ν satisfies

$$\int_H R_t\varphi_h(x)\nu(dx) = \int_H \varphi_h(x)\nu(dx)$$

for $\varphi_h(x) := e^{i\langle h,x\rangle}$, $x, h \in H$. This implies that

$$e^{-\frac{1}{2}\langle Q_t h,h\rangle}\widehat{\nu}(e^{tA^*}h) = \widehat{\nu}(h).$$

So by letting $t \to \infty$ we obtain

$$\widehat{\nu}(h) = e^{-\frac{1}{2}\langle Q_\infty h,h\rangle} = \widehat{\mu}(h)$$

and the uniqueness follows now from the characterization of Gaussian measures (see Theorem 1.2.5). □

Now, one can extend the semigroup $(R_t)_{t\geq 0}$ to a C_0–semigroup on $L^p(H,\mu)$,
$1 \leq p < \infty$.

Theorem 3.3.2 *Assume that (H3) and (H4) are satisfied. Then, for all $t \geq 0$, R_t can be extended to a bounded linear operator on $L^p(H,\mu)$ and $(R_t)_{t\geq 0}$ defines a C_0–semigroup of contractions on $L^p(H,\mu)$ for $1 \leq p < \infty$.*

Proof: Let $t \geq 0$ and $\varphi \in BUC(H)$. By Hölder's inequality we have

$$|R_t\varphi(x)|^p \leq (R_t|\varphi|^p)(x), \quad x \in H.$$

Hence,

$$\begin{aligned}\int_H |R_t\varphi(x)|^p \mu(dx) &\leq \int_H R_t|\varphi|^p(x)\mu(dx) \\ &= \int_H |\varphi(x)|^p \mu(dx).\end{aligned}$$

So, the first assertion follows from the density of $BUC(H)$ in $L^p(H,\mu)$ for $1 \leq p < \infty$ and we have

$$\|R_t\varphi\|_{L^p(H,\mu)} \leq \|\varphi\|_{L^p(H,\mu)}, \quad t \geq 0, \varphi \in L^p(H,\mu).$$

Finally, the strong continuity follows from the dominated convergence theorem. □

As in Section 3.1 we show that $u(t,x) := (R_t\varphi)(x)$, $t \geq 0, x \in H$, and $\varphi \in L^p(H,\mu)$ is the unique classical solution of (KE), which means that

(a) u is continuous on $[0,\infty) \times H$, $u(t,\cdot) \in C^2(H)$ for all $t > 0$,

(b) $QD^2u(t,x)$ is a trace class operator on H and $Du(t,x) \in D(A^*)$ for every $t > 0$ and $x \in H$,

(c) A^*Du and $\mathrm{Tr}(QD^2u)$ are two continuous functions on $(0,\infty) \times H$ and u satisfies (KE) for all $t > 0$ and $x \in D(A)$.

This result can be found in [6, Theorem 5].
To this purpose we need the following lemmas (see [6, Proposition 2] and [5, Proposition 1] or [13, Theorem 10.3.5]).

Lemma 3.3.3 *Suppose (H2), (H3) and (H4). Then the following hold.*

(i) *The family $S_0(t) := Q_\infty^{-\frac{1}{2}} e^{tA} Q_\infty^{\frac{1}{2}}$, $t \geq 0$, defines a C_0–semigroup of contractions on H.*

(ii) *The operators $S_0(t)S_0^*(t)$, $t > 0$, satisfy*

$$\begin{aligned}&\|S_0(t)S_0^*(t)\| < 1 \text{ and} \\ &\Lambda_t\Lambda_t^*(Q_\infty^{-\frac{1}{2}}e^{tA})^*(I - S_0(t)S_0^*(t))^{-1}(Q_\infty^{-\frac{1}{2}}e^{tA}).\end{aligned}$$

(iii) *For $0 < t_0 < t_1$, the function $[t_0,t_1] \ni t \mapsto \Lambda_t \in \mathcal{L}(H)$ is bounded.*

Lemma 3.3.4 *Assume (H2), (H3) and (H4) and let* $\varphi \in L^p(H,\mu)$, $1 < p < \infty$. *Then, for any* $t > 0$, $(R_t\varphi)(\cdot) \in C^\infty(H)$ *and*

$$|D^n R_t \varphi(x)| \le c(t,n,p,\varphi) < \infty$$

uniformly on bounded subsets of H *for* $n = 0, 1, \dots$ *and some constant* $c(t,n,p,\varphi) > 0$.

Proof of Lemma 3.3.3: (i) It follows from (H2) and Exercise 3.3.22 that $S_0(t)$, $t \ge 0$, are bounded linear operators on H and

$$S_0^*(t) = \overline{Q_\infty^{\frac{1}{2}} e^{tA^*} Q_\infty^{-\frac{1}{2}}}, \quad t \ge 0,$$

which can be defined on H, since $\ker Q_\infty = \{0\}$ and hence, $\overline{Q_\infty^{\frac{1}{2}}(H)} = H$ by Remark 1.3.2. Now, from (3.1), we obtain

$$0 \le \langle Q_t x, x\rangle = \langle (I - S_0(t)S_0^*(t)) Q_\infty^{\frac{1}{2}} x, Q_\infty^{\frac{1}{2}} x\rangle, \quad t \ge 0, x \in H.$$

Hence, $\|S_0^*(t) Q_\infty^{\frac{1}{2}} x\| \le \|Q_\infty^{\frac{1}{2}} x\|$, $t \ge 0$, $x \in H$. Since $\overline{Q_\infty^{\frac{1}{2}}(H)} = H$, we deduce that

$$\|S_0(t)\| \le 1, \quad t \ge 0. \tag{3.2}$$

The semigroup property can be easily verified. It suffices now to show that $S_0(\cdot)$ is weakly continuous at zero. Let $x, y \in H$. Then,

$$\lim_{t \to 0^+} \langle S_0(t)x, Q_\infty^{\frac{1}{2}} y\rangle = \langle x, Q_\infty^{\frac{1}{2}} y\rangle,$$

and the weak continuity follows from (3.2) and the density of $Q_\infty^{\frac{1}{2}}(H)$ in H.
(ii) From (3.1) and Exercise 3.3.22 it follows that

$$I - S_0(t)S_0^*(t) = (Q_\infty^{-\frac{1}{2}} Q_t^{\frac{1}{2}})(\overline{Q_t^{\frac{1}{2}} Q_\infty^{-\frac{1}{2}}}), \quad t > 0.$$

By Exercise 3.3.22 we have that $Q_\infty^{-\frac{1}{2}} Q_t^{\frac{1}{2}}$ has a bounded inverse and so does $I - S_0(t)S_0^*(t)$ for $t > 0$. Since $I - S_0(t)S_0^*(t)$ is selfadjoint and positive, we deduce that

$$\|S_0(t)S_0^*(t)\| < 1 \quad \text{for all } t > 0.$$

On the other hand, by Exercise 3.3.22, we have

$$\begin{aligned}
\Lambda_t^*\Lambda_t &= (Q_t^{-\frac{1}{2}} e^{tA})^* (Q_t^{-\frac{1}{2}} e^{tA}) \\
&= (Q_\infty^{-\frac{1}{2}} e^{tA})^* (Q_t^{-\frac{1}{2}} Q_\infty^{\frac{1}{2}})^* (Q_t^{-\frac{1}{2}} Q_\infty^{\frac{1}{2}}) (Q_\infty^{-\frac{1}{2}} e^{tA}) \\
&= (Q_\infty^{-\frac{1}{2}} e^{tA})^* (I - S_0(t)S_0^*(t))^{-1} (Q_\infty^{-\frac{1}{2}} e^{tA})
\end{aligned}$$

for every $t > 0$.
(iii) Take $a > 0$ such that

$$\|S_0(t_0)S_0^*(t_0)\| < a < 1.$$

Then,

$$\begin{aligned} \|S_0(t)S_0^*(t)\| &= \|S_0(t-t_0)S_0(t_0)S_0^*(t_0)S_0^*(t-t_0)\| \\ &\le \|S_0(t_0)S_0^*(t_0)\| < a \end{aligned}$$

for $t \in [t_0, t_1]$. Now, (iii) follows from the identity

$$Q_\infty^{-\frac{1}{2}} e^{tA} = (Q_\infty^{-\frac{1}{2}} e^{t_0 A}) e^{(t-t_0)A}$$

for $t \in [t_0, t_1]$. □

Proof of Lemma 3.3.4: We fix $t > 0$ and $\varphi \in L^p(H, \mu)$. Suppose without loss of generality that

$$\int_H |\varphi(e^{tA}x + y)|^p \mathcal{N}(0, Q_t)(dy) < \infty \quad \text{for } x = 0. \tag{3.3}$$

Let consider a sequence $(\varphi_n) \subset B_b(H)$ with $|\varphi_n(x)| \le |\varphi(x)|$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for μ–a.a. x and hence, by Exercise 3.3.20, for $\mathcal{N}(0, Q_t)$–a.a. x. So, by (3.3), φ_n converges also to φ in $L^p(H, \mathcal{N}(0, Q_t))$. On the other hand, we know from Theorem 3.1.1 that $R_t\varphi_n \in BUC^\infty(H)$. So, by the Cameron-Martin formula and Hölder's inequality, we obtain

$$\begin{aligned} & |R_t\varphi(x) - R_t\varphi_n(x)| \\ \le & \int_H |\varphi(e^{tA}x + y) - \varphi_n(e^{tA}x + y)| \mathcal{N}(0, Q_t)(dy) \\ = & \int_H \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} y\rangle\right) |\varphi(y) - \varphi_n(y)| \mathcal{N}(0, Q_t)(dy) \\ \le & \left(\int_H \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} y\rangle\right)^q \mathcal{N}(0, Q_t)(dy)\right)^{\frac{1}{q}} \\ & \left(\int_H |\varphi(y) - \varphi_n(y)|^p \mathcal{N}(0, Q_t)(dy)\right)^{\frac{1}{p}} \end{aligned}$$

for $\frac{1}{p} + \frac{1}{q} = 1$. Thus, it follows from Proposition 1.3.3 that

$$\sup_{\|x\| \le K} |R_t\varphi(x) - R_t\varphi_n(x)| \le \sup_{\|x\| \le K} \exp\left(\frac{q-1}{2}|\Lambda_t x|^2\right) \|\varphi - \varphi_n\|_{L^p(H, \mathcal{N}(0, Q_t))}$$

for $t > 0$ and any constant $K > 0$. This implies that $R_t\varphi \in C(H)$.
On the other hand, from Exercise 3.3.21 and the Cameron-Martin formula,

we have

$$
\begin{aligned}
&|\langle DR_t\varphi_n(x) - DR_t\varphi_m(x), y\rangle| \\
\leq\; & \int_H |\langle \Lambda_t y, Q_t^{-\frac{1}{2}}h\rangle(\varphi_n(e^{tA}x+h) - \varphi_m(e^{tA}x+h))|\mathcal{N}(0,Q_t)(dh) \\
\leq\; & \left(\int_H |\langle \Lambda_t y, Q_t^{-\frac{1}{2}}h\rangle|^{r'}\mathcal{N}(0,Q_t)(dh)\right)^{\frac{1}{r'}} \\
& \left(\int_H |\varphi_n(e^{tA}x+h) - \varphi_m(e^{tA}x+h)|^{r}\mathcal{N}(0,Q_t)(dh)\right)^{\frac{1}{r}} \\
=\; & c_r|\Lambda_t y|\Big(\int_H \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}}h\rangle\right) \\
& |\varphi_n(h) - \varphi_m(h)|^r \mathcal{N}(0,Q_t)(dh)\Big)^{\frac{1}{r}} \\
\leq\; & c_r|\Lambda_t y|\left(\int_H \exp\left(-\frac{b}{2}|\Lambda_t x|^2 + b\langle \Lambda_t x, Q_t^{-\frac{1}{2}}h\rangle\right)\mathcal{N}(0,Q_t)(dh)\right)^{\frac{1}{rb}} \\
& \left(\int_H |\varphi_n(h) - \varphi_m(h)|^p \mathcal{N}(0,Q_t)(dh)\right)^{\frac{1}{p}},
\end{aligned}
$$

where $\frac{1}{r} + \frac{1}{r'} = 1$, $r > 1$, and $\frac{1}{b} + \frac{r}{p} = 1$. So, by Proposition 1.3.3, it follows that

$$
|DR_t\varphi_n(x) - DR_t\varphi_m(x)| \leq c(t,p)\exp\left(\frac{b-1}{2r}|\Lambda_t x|^2\right)\|\varphi_n - \varphi_m\|_{L^p(H,\mathcal{N}(0,Q_t))}
$$

for $x \in H$. Thus, $DR_t\varphi_n$ converges uniformly on bounded subsets of H to a continuous function. Using Theorem 3.1.1 and by the same argument one can show the result for arbitrary n. □

The following result shows the existence and uniqueness of the classical solution for (KE), for any $\varphi \in L^p(H,\mu)$, $1 < p < \infty$.

Theorem 3.3.5 *Let (H2), (H3) and (H4) hold. If the operator $\Lambda_t A$ has a continuous extension $\overline{\Lambda_t A}$ on H then the function $(t,x) \mapsto (R_t\varphi)(x)$ is the unique classical solution for (KE) for any $\varphi \in L^p(H,\mu)$, $1 < p < \infty$.*

Proof: As in Theorem 3.1.2 we prove first that, for every $\varphi \in L^p(H,\mu)$, and $x \in H$,

$$
DR_t\varphi(x) \in D(A^*) \quad \text{for all } t > 0.
$$

Let $t > 0$ and $\varphi \in L^p(H,\mu)$ be fixed. We know from Theorem 3.1.1 and Lemma 3.3.4 that, for $y \in D(A)$,

$$
\langle DR_t\varphi(x), Ay\rangle = \int_H \langle \Lambda_t Ay, Q_t^{-\frac{1}{2}}h\rangle\varphi(e^{tA}x+h)\mathcal{N}(0,Q_t)(dh).
$$

Thus, by Hölder's inequality and Exercise 3.3.21, we obtain

$$\begin{aligned} |\langle DR_t\varphi(x), Ay\rangle| &\leq \left(\int_H |\langle \Lambda_t Ay, Q_t^{-\frac{1}{2}} h\rangle|^{r'} \mathcal{N}(0,Q_t)(dh)\right)^{\frac{1}{r'}} \\ &\quad \left(\int_H |\varphi(e^{tA}x+h)|^r \mathcal{N}(0,Q_t)(dh)\right)^{\frac{1}{r}} \\ &\leq c_r |\Lambda_t Ay| \left(R_t|\varphi|^r(x)\right)^{\frac{1}{r}} \qquad (3.4) \\ &\leq c_r \|\overline{\Lambda_t A}\| |y| \left(R_t|\varphi|^r(x)\right)^{\frac{1}{r}} \end{aligned}$$

for $x \in H$, $\frac{1}{r'}+\frac{1}{r}=1$, $1<r<p$, and all $y \in D(A)$. Since $|\varphi|^r \in L^{\frac{p}{r}}(H,\mu)$, it follows from Lemma 3.3.4 that

$$c(r,\varphi,x) := c_r \left(R_t|\varphi|^r(x)\right)^{\frac{1}{r}} < \infty.$$

Hence, $DR_t\varphi(x) \in D(A^*)$ for $t>0$ and $x \in H$.
On the other hand, by Theorem 3.1.1 and Lemma 3.3.4, we have $D^2R_t\varphi(x)$ exists for all $x \in H$ and

$$\langle D^2R_t\varphi(x)e_j, e_j\rangle = \int_H \left[|\langle \Lambda_t e_j, Q_t^{-\frac{1}{2}} y\rangle|^2 - |\Lambda_t e_j|^2\right] \varphi(e^{tA}x+y)\mathcal{N}(0,Q_t)(dy).$$

Take $1<r<p$. Then, it follows from Hölder's inequality and Exercise 3.3.21 that

$$\begin{aligned} |\langle D^2R_t\varphi(x)e_j, e_j\rangle| &\leq \left(\int_H \left[|\langle \Lambda_t e_j, Q_t^{-\frac{1}{2}} y\rangle|^2 - |\Lambda_t e_j|^2\right]^{r'} \mathcal{N}(0,Q_t)(dy)\right)^{\frac{1}{r'}} \\ &\quad \left(\int_H |\varphi(e^{tA}x+y)|^r \mathcal{N}(0,Q_t)(dy)\right)^{\frac{1}{r}} \\ &\leq c_r |\Lambda_t e_j|^2 \left(R_t|\varphi|^r(x)\right)^{\frac{1}{r}} \qquad (3.5) \end{aligned}$$

for $x \in H$, and $\frac{1}{r'}+\frac{1}{r}=1$, $1<r<p$. By the same argument as above and Corollary 3.1.3 we have $c(r,\varphi,x) := c_r \left(R_t|\varphi|^r(x)\right)^{\frac{1}{r}} < \infty$ and

$$\sum_{j=1}^{\infty} |\langle D^2R_t\varphi(x)e_j, e_j\rangle| \leq c(r,\varphi,x) \sum_{j=1}^{\infty} |\Lambda_j e_j|^2 < \infty.$$

This shows that $D^2R_t\varphi(x)$ is a trace class operator on H for $x \in H$, $t>0$ and $\varphi \in L^p(H,\mu)$. From Corollary 3.1.3 we know that (KE) has a unique classical solution $u(t,x) := R_t\varphi(x)$ for $\varphi \in B_b(H)$. Now, for $\varphi \in L^p(H,\mu)$, there is a sequence $(\varphi_n) \subset B_b(H)$ with $|\varphi_n(x)| \leq |\varphi(x)|$ and $\lim_{n\to\infty} \varphi_n(x) =$

$\varphi(x)$ for μ–a.a. $x \in H$. It follows from Exercise 3.3.23 that

$$
\begin{aligned}
&|R_t\varphi_n(x) - R_t\varphi(x)| \leq \\
&\leq \left(\int_H k(t,x,y)^q \mu(dy)\right)^{\frac{1}{q}} \|\varphi_n - \varphi\|_{L^p(H,\mu)} \\
&= \det(I - S_0(t)S_0^*(t))^{\frac{1-q}{2q}} \det(I + (q-1)S_0(t)S_0^*(t))^{-\frac{1}{2q}} \\
&\quad \exp\left(\frac{q-1}{2}\langle (I + (q-1)S_0(t)S_0^*(t))^{-1} Q_\infty^{-\frac{1}{2}} e^{tA}x, Q_\infty^{-\frac{1}{2}} e^{tA}x\rangle\right)
\end{aligned}
$$

for $t > 0$, $x \in H$ and $\frac{1}{q} + \frac{1}{p} = 1$. So, by Lemma 3.3.3(iii), $R_t\varphi_n(x) \to R_t\varphi(x)$ uniformly in $(t,x) \in [t_0, t_1] \times \{x \in H : |x| \leq K\}$ for $0 < t_0 < t_1$ and any constant $K > 0$. Again by Exercise 3.3.23, we obtain

$$
\begin{aligned}
&R_t|\varphi|^r(x) \leq \\
&\leq \left(\int_H k(t,x,y)^{\frac{p}{r}} \mu(dy)\right)^{\frac{r}{p}} \|\varphi\|_{L^p(H,\mu)}^r \\
&= \det(I - S_0(t)S_0^*(t))^{\frac{r-p}{2p}} \det(I + (\frac{p}{r} - 1)S_0(t)S_0^*(t))^{-\frac{r}{2p}} \\
&\quad \exp\left(\frac{p-r}{2r}\langle (I + (\frac{p}{r} - 1)S_0(t)S_0^*(t))^{-1} Q_\infty^{-\frac{1}{2}} e^{tA}x, Q_\infty^{-\frac{1}{2}} e^{tA}x\rangle\right)
\end{aligned}
$$

for $t > 0$, $x \in H$ and $1 < r < p$. So, by Lemma 3.3.3(iii), (3.4) and (3.5), it follows that $\frac{\partial}{\partial t}R_t\varphi_n(x)$ converges uniformly in $(t,x) \in [t_0, t_1] \times \{x \in H : |x| \leq K\}$. Hence the function $(t,x) \mapsto R_t\varphi(x)$ is a classical solution for (KE). The uniqueness follows from Theorem 3.1.2. □

We propose now to characterize symmetric Ornstein-Uhlenbeck semigroups on $L^2(H,\mu)$. To this purpose we need the following lemma.

Lemma 3.3.6 *Assume that (H3) and (H4) hold. Then the operator Q_∞ is the only positive and symmetric solution of the following Lyapunov equation*

$$\langle Q_\infty x, A^*y\rangle + \langle Q_\infty A^*x, y\rangle = -\langle Qx, y\rangle, \quad x, y \in D(A^*). \tag{3.6}$$

Proof: For $x, y \in D(A^*)$, by using integration by part, we have

$$
\begin{aligned}
\langle Q_\infty x, A^*y\rangle &= \int_0^\infty \langle e^{sA}Qe^{sA^*}x, A^*y\rangle\, ds \\
&= \int_0^\infty \langle Qe^{sA^*}x, \frac{d}{ds}e^{sA^*}y\rangle\, ds \\
&= -\langle Qx, y\rangle - \langle Q_\infty A^*x, y\rangle.
\end{aligned}
$$

Suppose now that there is a positive and symmetric opertor $R \in \mathcal{L}(H)$ solution of the Lyapunov equation (3.6). Then we obtain

$$\frac{d}{dt}\langle Re^{tA^*}x, e^{tA^*}x\rangle = -\langle Qe^{tA^*}x, e^{tA^*}x\rangle, \quad x \in D(A^*).$$

So by integrating between 0 and t we obtain

$$\langle Re^{tA^*}x, e^{tA^*}x\rangle - \langle Rx, x\rangle = -\langle Q_t x, x\rangle, \quad x \in D(A^*).$$

Now, by letting $t \to \infty$ we get

$$\langle Rx, x\rangle = \langle Q_\infty x, x\rangle \quad \text{for all } x \in D(A^*).$$

This implies that $R = Q_\infty$. □

Symmetric Ornstein-Uhlenbeck semigroups on $L^2(H,\mu)$ are characterized by the following result.

Proposition 3.3.7 *Suppose (H3) and (H4) hold. Then the following assertion are equivalent*

(i) $(R_t)_{t\geq 0}$ *is symmetric in* $L^2(H,\mu)$.

(ii) $Q_\infty e^{tA^*} = e^{tA}Q_\infty$ *for all* $t \geq 0$.

(iii) $Qe^{tA^*} = e^{tA}Q$ *for all* $t \geq 0$.

If $(R_t)_{t\geq 0}$ *is symmetric then* $Q_\infty = -\frac{1}{2}A^{-1}Q$.

Proof: For $\varphi(x) := e^{i\langle x,h\rangle}$ and $\tilde{\varphi}(x) := e^{i\langle x,\tilde{h}\rangle}$, $x, h \in H$, we have

$$\begin{aligned} R_t\varphi(x) &= e^{i\langle e^{tA}x,h\rangle - \frac{1}{2}\langle Q_t h,h\rangle} \text{ and} \\ R_t\tilde{\varphi}(x) &= e^{i\langle e^{tA}x,\tilde{h}\rangle - \frac{1}{2}\langle Q_t \tilde{h},\tilde{h}\rangle}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_H R_t\varphi(x)\tilde{\varphi}(x)\mu(dx) &= e^{-\frac{1}{2}\langle Q_t h,h\rangle}\int_H e^{i\langle x,\tilde{h}+e^{tA^*}h\rangle}\mu(dx) \\ &= e^{-\frac{1}{2}\langle Q_t h,h\rangle} e^{i\langle Q_\infty(\tilde{h}+e^{tA^*}h),\tilde{h}+e^{tA^*}h\rangle} \\ &= e^{-\frac{1}{2}\langle (Q_t+e^{tA}Q_\infty e^{tA^*})h,h\rangle} e^{-\frac{1}{2}\langle Q_\infty\tilde{h},\tilde{h}\rangle} e^{-\langle Q_\infty e^{tA^*}h,\tilde{h}\rangle}. \end{aligned}$$

So by (3.1) we obtain

$$\int_H R_t\varphi(x)\tilde{\varphi}(x)\mu(dx) e^{-\frac{1}{2}\langle Q_\infty h,h\rangle - \frac{1}{2}\langle Q_\infty\tilde{h},\tilde{h}\rangle - \langle Q_\infty e^{tA^*}h,\tilde{h}\rangle}.$$

By the same computation we have

$$\int_H R_t\tilde{\varphi}(x)\varphi(x)\mu(dx) e^{-\frac{1}{2}\langle Q_\infty h,h\rangle - \frac{1}{2}\langle Q_\infty\tilde{h},\tilde{h}\rangle - \langle Q_\infty e^{tA^*}\tilde{h},h\rangle}.$$

Therefore,

$$\begin{aligned} \int_H R_t\varphi(x)\tilde{\varphi}(x)\mu(dx) &= \int_H R_t\tilde{\varphi}(x)\varphi(x)\mu(dx) \text{ if and only if} \\ e^{-\langle Q_\infty e^{tA^*}h,\tilde{h}\rangle} &= e^{-\langle Q_\infty e^{tA^*}\tilde{h},h\rangle} \text{ if and only if} \\ Q_\infty e^{tA^*} &= e^{tA}Q_\infty. \end{aligned}$$

Hence the equivalence (i) $\Leftrightarrow$ (ii) follows from the density of $\mathcal{E}_A(H)$ in $L^2(H,\mu)$ (see Lemma 3.2.1).
The implication (iii) $\Rightarrow$ (ii) is trivial. It remains to prove (ii) $\Rightarrow$ (iii). To this purpose we consider $x \in D(A^*)$. It follows from (ii) that $Q_\infty x \in D(A)$ and

$$Q_\infty A^* x = AQ_\infty x.$$

So by Lemma 3.3.6 it follows that $2AQ_\infty = -Q$ and hence

$$Q_\infty = -\frac{1}{2}A^{-1}Q,$$

which proves the last assertion of the theorem. Again by Lemma 3.3.6 we have

$$\begin{aligned}\langle Qe^{tA^*}x, y\rangle &= -\langle Q_\infty e^{tA^*}x, A^*y\rangle - \langle Q_\infty A^* e^{tA^*}x, y\rangle \\ &= -\langle Q_\infty x, A^* e^{tA^*}y\rangle - \langle Q_\infty A^* x, e^{tA^*}y\rangle.\end{aligned}$$

On the other hand, it follows from Lemma 3.3.6 that

$$\begin{aligned}\langle e^{tA}Qx, y\rangle &= \langle Qx, e^{tA^*}y\rangle \\ &= -\langle Q_\infty x, A^* e^{tA^*}y\rangle - \langle Q_\infty A^* x, e^{tA^*}y\rangle.\end{aligned}$$

This implies that

$$\langle Qe^{tA^*}x, y\rangle = \langle e^{tA}Qx, y\rangle, \quad x, y \in D(A^*),\ t \geq 0,$$

which is equivalent to $Qe^{tA^*} = e^{tA}Q$ for all $t \geq 0$. □

In the particular case where A is selfadjoint we have the following result.

Corollary 3.3.8 *If the following assumptions are satisfied*

1. $A : D(A) \to H$ *is selfadjoint and there is* $\omega > 0$ *such that* $\langle Ax, x\rangle \leq -\omega|x|^2$ *for all* $x \in D(A)$,
2. $Qe^{tA} = e^{tA}Q$ *for all* $t \geq 0$,
3. $QA^{-1} \in \mathcal{L}(H)$ *is a trace class operator,*

then $(R_t)_{t\geq 0}$ *is symmetric on* $L^2(H,\mu)$.

Proof: In this particular case we have

$$Q_t = Q\int_0^t e^{2sA}\,ds = \frac{1}{2}QA^{-1}(e^{2tA} - I), \quad t \geq 0.$$

From the third assumption we have $\mathrm{Tr}Q < \infty$ and the second assumption is exactly the third assertion in Proposition 3.3.7. This end the proof of the corollary. □

In the special case $Q = I$ we obtain

Corollary 3.3.9 *Assume that $A : D(A) \to H$ is selfadjoint, there is $\omega > 0$ such that $\langle Ax, x\rangle \le -\omega|x|^2$ for all $x \in D(A)$, A^{-1} is a trace class operator and $Q = I$. Then $(R_t)_{t\ge 0}$ is symmetric on $L^2(H, \mu)$.*

We propose now to describe the generator L_p of the Ornstein-Uhlenbeck semigroup $(R_t)_{t\ge 0}$ on $L^p(H, \mu)$ $1 \le p < \infty$.
We set

$$L_0\varphi(x) := \frac{1}{2}\mathrm{Tr}(QD^2\varphi(x)) + \langle x, A^*D\varphi(x)\rangle, \quad x \in H,\ \varphi \in \mathcal{E}_A(H).$$

Proposition 3.3.10 *If the assumptions (H3) and (H4) are satisfied, then $\mathcal{E}_A(H)$ is a core for L_p.*

Proof: For $\varphi(x) := e^{i\langle h,x\rangle}$, $h \in D(A^*)$, $x \in H$, we have

$$\begin{aligned} R_t\varphi(x) &= \int_H e^{i\langle h, e^{tA}x+y\rangle}\mathcal{N}(0, Q_t)(dy) \\ &= e^{i\langle e^{tA^*}h,x\rangle - \frac{1}{2}\langle Q_t h,h\rangle} \in \mathcal{E}_A(H). \end{aligned}$$

Hence,

$$R_t\mathcal{E}_A(H) \subseteq \mathcal{E}_A(H), \quad \forall t \ge 0.$$

On the other hand we know that

$$\begin{aligned} \lim_{t\to 0^+} \frac{1}{t}(R_t\varphi - \varphi)(x) &= e^{i\langle h,x\rangle}\left(i\langle A^*h, x\rangle - \frac{1}{2}\langle Qh, h\rangle\right) \\ &= L_0\varphi(x), \quad x \in H. \end{aligned}$$

So by the dominated convergence theorem we obtain

$$\lim_{t\to 0^+} \left\| \frac{1}{t}(R_t\varphi - \varphi) - L_0\varphi \right\|_{L^p(H,\mu)} = 0.$$

Thus, $\mathcal{E}_A(H) \subset D(L_p)$ and the assertion follows from the density of $\mathcal{E}_A(H)$ in $L^p(H, \mu)$ (see Lemma 3.2.1) and Proposition A.2.5. □

In the remaining part of this section we propose to describe exactly the domain $D(L_2)$ of the generator of the symmetric Ornstein-Uhlenbeck semigroup on $L^2(H, \mu)$. To this purpose we need some auxiliary results. The following result was proved independently in [3] and [17].

Proposition 3.3.11 *Assume (H3) and (H4). Then the following hold*

$$\begin{aligned} \int_H L_0\varphi(x)\tilde{\varphi}(x)\mu(dx) &= \int_H \langle Q_\infty D\tilde{\varphi}(x), A^*D\varphi(x)\rangle\mu(dx) \\ \int_H L_0\varphi(x)\varphi(x)\mu(dx) &= -\frac{1}{2}\int_H \langle Q^{\frac{1}{2}}D\varphi(x), Q^{\frac{1}{2}}D\varphi(x)\rangle\mu(dx) \end{aligned}$$

for φ, $\tilde{\varphi} \in \mathcal{E}_A(H)$.

Proof: For $\varphi(x) := e^{i\langle h,x\rangle}$, $\tilde\varphi(x) := e^{i\langle \tilde h,x\rangle}$, $h,\tilde h \in D(A^*)$, $x \in H$, we have

$$\begin{aligned}
&\int_H L_0\varphi(x)\tilde\varphi(x)\mu(dx)\\
= &\int_H e^{i\langle h,x\rangle}\left(i\langle A^*h,x\rangle - \frac12\langle Qh,h\rangle\right)e^{i\langle\tilde h,x\rangle}\mu(dx)\\
= &\; i\int_H \langle A^*h,x\rangle e^{i\langle h+\tilde h,x\rangle}\mu(dx) - \frac12\langle Qh,h\rangle e^{-\frac12\langle Q_\infty(h+\tilde h),h+\tilde h\rangle}\\
= &\; \frac{d}{dt}\left(\int_H e^{i\langle tA^*h+h+\tilde h,x\rangle}\mu(dx)\right)_{|t=0} - \frac12\langle Qh,h\rangle e^{-\frac12\langle Q_\infty(h+\tilde h),h+\tilde h\rangle}\\
= &\; -\left(\langle Q_\infty A^*h,h+\tilde h\rangle + \frac12\langle Qh,h\rangle\right)e^{-\frac12\langle Q_\infty(h+\tilde h),h+\tilde h\rangle}.
\end{aligned}$$

Hence, it follows from Proposition 3.3.6 that

$$\begin{aligned}
\int_H \langle Q_\infty D\tilde\varphi(x), A^*D\varphi(x)\rangle\mu(dx) &= -\langle A^*h, Q_\infty\tilde h\rangle e^{-\frac12\langle Q_\infty(h+\tilde h),h+\tilde h\rangle}\\
&= \int_H L_0\varphi(x)\tilde\varphi(x)\mu(dx).
\end{aligned}$$

In particular, and again by Proposition 3.3.6, we obtain

$$\begin{aligned}
\int_H L_0\varphi(x)\varphi(x)\mu(dx) &= \int_H \langle Q_\infty D\varphi(x), A^*D\varphi(x)\rangle\mu(dx)\\
&= -\frac12\int_H \langle QD\varphi(x), D\varphi(x)\rangle\mu(dx).
\end{aligned}$$

This end the proof of the proposition. □

Remark 3.3.12 *If the Ornstein Uhlenbeck semigroup is symmetric, then it follows from Proposition 3.3.7 that*

$$\int_H L_0\varphi(x)\tilde\varphi(x)\mu(dx) = -\frac12\int_H \langle QD\varphi(x), D\tilde\varphi(x)\rangle\mu(dx) \tag{3.7}$$

for φ, $\tilde\varphi \in \mathcal{E}_A(H)$.

For the proof of the next proposition we need the following lemma.

Lemma 3.3.13 *Assume that* $\ker Q = \{0\}$ *and* $Q_\infty^{\frac12}(H) \subset Q^{\frac12}(H)$. *Then the operator*

$$D_Q : \mathcal{E}_A(H) \subset L^2(H,\mu) \to L^2(H,\mu;H);\ \varphi \mapsto Q^{\frac12}D\varphi$$

is closable.

Proof: From the closed graph theorem we have $K := \overline{Q^{-\frac{1}{2}}Q_\infty^{\frac{1}{2}}}$ is a bounded linear operator on H. Its adjoint is given by $K^* = Q_\infty^{\frac{1}{2}}Q^{-\frac{1}{2}}$.
Let $(\varphi_n) \subset \mathcal{E}_A(H)$ and $F \in L^2(H,\mu;H)$ with $\lim_{n\to\infty} \|\varphi_n\|_{L^2(H,\mu)} = 0$ and $\lim_{n\to\infty} \|D_Q\varphi_n - F\|_{L^2(H,\mu;H)} = 0$. Hence,

$$Q_\infty^{\frac{1}{2}} D\varphi_n = K^* Q^{\frac{1}{2}} D\varphi_n \to K^* F$$

in $L^2(H,\mu;H)$ as $n \to \infty$. Now, it follows from Proposition 3.2.10 that $K^*F \equiv 0$ and therefore $F \equiv 0$. This can be obtain by considering the orthonormal basis of eigenfunctions e_n, $n \in \mathbb{N}$, of Q_∞ and the fact that $\ker Q_\infty = \{0\}$. □

As in Section 2 we define The spaces

$W_Q^{1,2}(H,\mu) := D(\overline{D_Q})$ and
$W_Q^{2,2}(H,\mu) :=$

$$:= \{\varphi \in W_Q^{1,2}(H,\mu) : \varphi \in \bigcap_{h,k\in\mathbb{N}} D(D_h D_k), \int_H \mathrm{Tr}(QD^2\varphi(x))^2 \mu(dx) < \infty\}.$$

In the following result we obtain that $D((-L_2)^{\frac{1}{2}}) = W_Q^{1,2}(H,\mu)$ for symmetric Ornstein-Uhlenbeck semigroups on $L^2(H,\mu)$.

Proposition 3.3.14 *Suppose (H3), (H4),* $\ker Q = \{0\}$*, and* $Q_\infty^{\frac{1}{2}}(H) \subseteq Q^{\frac{1}{2}}(H)$*. Then,*

$$D(L_2) \subset W_Q^{1,2}(H,\mu).$$

Moreover, for any $\varphi \in D(L_2)$*,*

$$\int_H L_2\varphi(x)\varphi(x)\mu(dx) - \frac{1}{2}\int_H \langle QD\varphi(x), D\varphi(x)\rangle \mu(dx).$$

In the case where $(R_t)_{t\geq 0}$ *is symmetric, one has*

$$D((-L_2)^{\frac{1}{2}}) = W_Q^{1,2}(H,\mu).$$

Proof: Let $\varphi \in D(L_2)$. It follows from Proposition 3.3.10 that there is $(\varphi_n) \subset \mathcal{E}_A(H)$ with

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^2(H,\mu)} = 0 \text{ and } \lim_{n\to\infty} \|L_0\varphi_n - L_2\varphi\|_{L^2(H,\mu)} = 0.$$

By Proposition 3.3.11, we have

$$\begin{aligned} &\int_H \langle Q^{\frac{1}{2}} D(\varphi_n - \varphi_m)(x), Q^{\frac{1}{2}} D(\varphi_n - \varphi_m)(x)\rangle \mu(dx) \\ = \quad & -2\int_H L_0(\varphi_n - \varphi_m)(x)(\varphi_n - \varphi_m)(x)\mu(dx). \end{aligned}$$

Now, one can apply Lemma 3.3.13 and hence $\varphi \in W_Q^{1,2}(H,\mu)$ and

$$\int_H L_2\varphi(x)\varphi(x)\mu(dx) - \frac{1}{2}\int_H \langle QD\varphi(x), D\varphi(x)\rangle\mu(dx).$$

On the other hand the last assertion follows from

$$\int_H |(-L_2)^{\frac{1}{2}}\varphi(x)|^2\mu(dx) = \int_H |Q^{\frac{1}{2}}D\varphi(x)|^2\mu(dx).$$

□

Remark 3.3.15 *The bilinear form*

$$a(\varphi,\tilde{\varphi}) := \int_H \langle Q_\infty D\tilde{\varphi}(x), A^*D\varphi(x)\rangle\mu(dx), \quad \varphi,\, \tilde{\varphi} \in \mathcal{E}_A(H)$$

is not always continuous on $W_Q^{1,2}(H,\mu) \times W_Q^{1,2}(H,\mu)$ and therefore not in general a Dirichlet form. The continuity of the bilinear form a can be proved under some additional conditions (see [3] or [17]). In [9] it is proved that a is a Dirichlet form provided that $Q = I$, which implies that $AQ_\infty \in \mathcal{L}(H)$.

Suppose now that the assumptions of Corollary 3.3.9 are satisfied. Then $Q_\infty = -\frac{1}{2}A^{-1}$. Let consider an orthonormal system $(e_n) \subset H$ and $(\alpha_n) \subset (0,\infty)$ such that

$$Ae_n = -\alpha_n e_n, \quad n \in \mathbb{N}.$$

The following proposition is the main tool used for the characterization of the domain of L_2.

Proposition 3.3.16 *Suppose that the assumptions of Corollary 3.3.9 are satisfied. Then,*

$$\frac{1}{2}\int_H \mathrm{Tr}\left((D^2\varphi(x))^2\right)\mu(dx) + \int_H |(-A)^{\frac{1}{2}}D\varphi(x)|^2\mu(dx) = 2\int_H (L_2\varphi(x))^2\mu(dx)$$

for $\varphi \in \mathcal{E}_A(H)$.

Proof: For $\varphi \in \mathcal{E}_A(H)$ we have $D_j(L_2\varphi) = L_2D_j\varphi - \alpha_jD_j\varphi$. Hence, by Proposition 3.3.14,

$$\begin{aligned}
&\int_H D_j\varphi(x)D_j(L_2\varphi)(x)\mu(dx)\\
= &\int_H D_j\varphi(x)L_2(D_j\varphi)(x)\mu(dx) - \alpha_j\int_H |D_j\varphi(x)|^2\mu(dx)\\
= &-\frac{1}{2}\int_H \langle DD_j\varphi(x), DD_j\varphi(x)\rangle\mu(dx) - \alpha_j\int_H |D_j\varphi(x)|^2\mu(dx).
\end{aligned}$$

Now, if we take the sum over $j \in \mathbb{N}$, we obtain

$$\frac{1}{2}\int_H \mathrm{Tr}\left((D^2\varphi(x))^2\right)\mu(dx) + \int_H |(-A)^{\frac{1}{2}}D\varphi(x)|^2\mu(dx)$$
$$= -\int_H \langle D\varphi(x), D(L_2\varphi)(x)\rangle\mu(dx).$$

Since $L_2\varphi \in W^{1,2}(H,\mu)$, it follows from Remark 3.3.12 that

$$\int_H \langle D\varphi(x), D(L_2\varphi)(x)\rangle\mu(dx) = -2\int_H |L_2\varphi(x)|^2\mu(dx).$$

Thus,

$$\frac{1}{2}\int_H \mathrm{Tr}\left((D^2\varphi(x))^2\right)\mu(dx)+\int_H |(-A)^{\frac{1}{2}}D\varphi(x)|^2\mu(dx) = 2\int_H |L_2\varphi(x)|^2\mu(dx).$$

□

For the characterization of the domain of L_2 we need the following space

$$W^{1,2}_{(-A)}(H,\mu) := \{\varphi \in W^{1,2}(H,\mu) : \int_H |(-A)^{\frac{1}{2}}D\varphi(x)|^2\mu(dx) = \sum_{k\in\mathbb{N}}\int_H \alpha_k|D_k\varphi(x)|^2\mu(dx) < \infty\}.$$

Endowed with the inner product

$$\langle\varphi,\psi\rangle_{W^{1,2}_{(-A)}(H,\mu)} := \varphi,\psi\rangle_{L^2(H,\mu)} + \int_H \langle(-A)^{\frac{1}{2}}D\varphi(x), (-A)^{\frac{1}{2}}D\psi(x)\rangle\mu(dx),$$

$W^{1,2}_{(-A)}(H,\mu)$ is Hilbert space.

Theorem 3.3.17 *Assume that the assumptions of Corollary 3.3.9 hold. Then,*

$$D(L_2) = W^{2,2}(H,\mu)\cap W^{1,2}_{(-A)}(H,\mu).$$

Proof: Let $\varphi \in D(L_2)$. By Proposition 3.3.10 there is $(\varphi_n) \subset \mathcal{E}_A(H)$ with $\varphi_n \to \varphi$ and $L_2\varphi_n \to L_2\varphi$ in $L^2(H,\mu)$. For $n,m \in \mathbb{N}$, it follows from Proposition 3.3.16 that

$$2\int_H |L_2(\varphi_n-\varphi_m)(x)|^2\mu(dx) = \frac{1}{2}\int_H \mathrm{Tr}\left((D^2(\varphi_n-\varphi_m)(x))^2\right)\mu(dx) + \int_H |(-A)^{\frac{1}{2}}D(\varphi_n-\varphi_m)(x)|^2\mu(dx).$$

Therefore (φ_n) is a Cauchy sequence in both spaces $W^{2,2}(H,\mu)$ and $W^{1,2}_{(-A)}(H,\mu)$. This implies that

$$D(L_2) \subseteq W^{2,2}(H,\mu)\cap W^{1,2}_{(-A)}(H,\mu).$$

Now, if $\varphi \in W^{2,2}(H,\mu) \cap W^{1,2}_{(-A)}(H,\mu)$ then one can find a sequence $(\varphi_n) \subset \mathcal{E}_A(H)$ such that φ_n converges to φ in both spaces $W^{2,2}(H,\mu)$ and $W^{1,2}_{(-A)}(H,\mu)$. The other inclusion follows now from Proposition 3.3.16. □

In the more general assumptions given in Corollary 3.3.8 one has to prove the formula

$$\frac{1}{2}\int_H \mathrm{Tr}\left((QD^2\varphi(x))^2\right)\mu(dx) + \int_H \langle(-AQ)D\varphi(x), D\varphi(x)\rangle\mu(dx) = \\ = 2\int_H (L_2\varphi(x))^2\mu(dx). \quad (3.8)$$

The proof of (3.8) is similar to that of Proposition 3.3.16. As in the proof of Theorem 3.3.17, (3.8) implies the following general result.

Theorem 3.3.18 *Suppose that the assumptions of Corollary 3.3.8 hold. Then,*

$$D(L_2) = \{\varphi \in W^{2,2}_Q(H,\mu) : \int_H \langle(-AQ)D\varphi(x), D\varphi(x)\rangle\mu(dx) < \infty\}.$$

Remark 3.3.19 *Theorem 3.3.17 and 3.3.18 are due to Da Prato [10]. In the finite dimensional case Lunardi [24] proved first that $D(L_2) = W^{2,2}(\mathbb{R}^N,\mu)$, by making heavy use of interpolation theory. A simpler proof of the same result can be found in [11]. Recently, this result was extended to $p \in (1,\infty)$ (see [25] or [26]).*

Exercise 3.3.20 *Assume (H1) and (H2). Prove that $\mathcal{N}(0,Q_t)$ is $\mathcal{N}(0,Q_\infty)$–absolutely continuous.*

Exercise 3.3.21 *Let $1 < p < \infty$, and $B \in \mathcal{L}_1^+(H)$ with $\ker B = \{0\}$. Show that*

$$\int_H |\langle h, B^{-\frac{1}{2}}y\rangle|^p \mathcal{N}(0,B)(dy) = |h|^p \int_{\mathbb{R}} |y|^p \mathcal{N}(0,1)(dy).$$

This generalizes the case $p = 2$ proved in Proposition 1.3.1.

Exercise 3.3.22 *Assume (H1) and (H2). Show that*

(i) $Q_t^{\frac{1}{2}}(H) = Q_\infty^{\frac{1}{2}}(H)$.

(ii) *For any $t > 0$, $S_0(t) := Q_\infty^{-\frac{1}{2}} e^{tA} Q_\infty^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on H.*

(iii) *Deduce that e^{tA} is a trace class operator on H for every $t > 0$.*

Exercise 3.3.23 *Assume (H2), (H3) and (H4).*

(a) *Show that*

$$Q_t = Q_\infty^{\frac{1}{2}}(I - S_0(t)S_0^*(t))Q_\infty^{\frac{1}{2}}, \quad t \geq 0.$$

(b) *By using the Cameron-Martin formula and the Feldman-Hajek theorem (see Exercise 1.3.6) show that*

$$R_t\varphi(x) = \int_H k(t,x,y)\varphi(y)\mu(dy), \quad \mu - \text{a.a.}\, x \in H,$$

with

$$\begin{aligned} k(t,x,y) &:= \\ &\exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle (I - S_0(t)S_0^*(t))^{-1}S_0(t)Q_\infty^{-\frac{1}{2}}x, Q_\infty^{-\frac{1}{2}}y\rangle\right)\cdot \\ &\cdot\det(I - S_0(t)S_0^*(t))^{-\frac{1}{2}}\cdot \\ &\cdot\exp\left(-\frac{1}{2}\langle S_0(t)S_0^*(t)(I - S_0(t)S_0^*(t))^{-1}Q_\infty^{-\frac{1}{2}}y, Q_\infty^{-\frac{1}{2}}y\rangle\right) \end{aligned}$$

for $t > 0$, and $x, y \in H$.

(c) *Show that, for any $1 < q < \infty$,*

$$\begin{aligned} &\int_H k(t,x,y)^q\mu(dy) = \\ &\quad \det(I - S_0(t)S_0^*(t))^{\frac{1-q}{2}}\det(I + (q-1)S_0(t)S_0^*(t))^{-\frac{1}{2}} \\ &\quad \exp\left(\frac{q(q-1)}{2}\langle (I + (q-1)S_0(t)S_0^*(t))^{-1}Q_\infty^{-\frac{1}{2}}e^{tA}x, Q_\infty^{-\frac{1}{2}}e^{tA}x\rangle\right) \end{aligned}$$

for $t > 0$ and $x \in H$, (see [6, Lemma 3]).

Exercise 3.3.24 *Suppose (H2), (H3) and (H4). Use the formula*

$$\langle DR_t\varphi(x), y\rangle = \int_H \langle \Lambda_t y, Q_t^{-\frac{1}{2}}h\rangle\varphi(e^{tA}x + h)\mathcal{N}(0, Q_t)(dh),$$

which, by Lemma 3.3.4, remains valid for $t > 0$ and $\varphi \in L^p(H,\mu)$ to prove that

$$R_t L^p(H,\mu) \subset W^{1,p}(H,\mu)$$

for $t > 0$ and $1 \leq p < \infty$. Deduce from [7] that the Ornstein-Uhlenbeck semigroup (R_t) is immediately compact in $L^p(H,\mu)$.