Appendix: Tannery's Limiting Theorem

In this monograph, we have frequently refered to the Tannery theorem. This theorem deals with the limiting process on infinite series, which can be reproduced as follows.

For a given infinite series $\{v_k(n)\}_{k\geq 0}$, suppose that the series satisfies the following conditions:

- For any fixed k, there holds $\lim\limits_{n\to\infty} v_k(n) = w_k$;
- For any $k \in \mathbb{N}_0$, we have $|v_k(n)| \leq M_k$ with M_k being independent of n and the series $\sum\limits_{k=0}^{\infty} M_k$ is convergent.

Then we have the following limit relation:

$$\lim_{n\to\infty} \sum_{k=0}^{m(n)} v_k(n) \;=\; \sum_{k=0}^{\infty} w_k \;=\; W$$

where $m(n)$ is an increasing integer valued function which tends steadily to infinity as n does.

PROOF. For any given $\varepsilon > 0$, first choose a number $\ell = \ell(\varepsilon)$ such that $\sum\limits_{k=\ell}^{\infty} M_k < \varepsilon$ and then let n be taken large enough to make $m(n) > \ell$. This leads us consequently to the following inequality: $\Big|\sum\limits_{k=\ell}^{m(n)} v_k\Big| \leq \sum\limits_{k=\ell}^{m(n)} M_k < \varepsilon.$

Noting also that $\Big|\sum_{k=\ell}^{\infty} w_k\Big| \leq \sum_{k=\ell}^{\infty} M_k < \varepsilon$, we can estimate the difference

$$\begin{aligned}\Big|\sum_{k=0}^{m(n)} v_k(n) - W\Big| &\leq \Big|\sum_{k=\ell}^{m(n)} v_k(n)\Big| + \Big|\sum_{k=\ell}^{\infty} w_k\Big| + \Big|\sum_{k=0}^{\ell-1}\big\{v_k(n) - w_k\big\}\Big| \\ &< 2\varepsilon + \Big|\sum_{k=0}^{\ell-1}\big\{v_k(n) - w_k\big\}\Big|.\end{aligned}$$

Remember that so far n has only been restricted by the condition $m(n) > \ell$. Since ℓ is independent of n, we can allow n to tend to infinity and obtain

$$\lim_{n\to\infty}\sum_{k=0}^{\ell-1}\big\{v_k(n) - w_k\big\} = 0 \quad \text{for} \quad \lim_{n\to\infty} v_k(n) = w_k \quad \text{with } k \text{ being fixed.}$$

Hence we have found that for any $\varepsilon > 0$, there holds

$$\lim_{n\to\infty}\Big|\sum_{k=0}^{m(n)} v_k(n) - W\Big| < 2\varepsilon$$

which implies the limit relation:

$$\lim_{n\to\infty}\sum_{k=0}^{m(n)} v_k(n) = W = \sum_{k=0}^{\infty} w_k$$

as anticipated in the Tannery theorem. □