CHAPTER G

The Lagrange Four Square Theorem

Representing natural numbers as sums of squares is an important topic of number theory. Given a general natural number n, denote by $r_\ell(n)$ the number of integer solutions of Diophantine equation

$$n = x_1^2 + x_2^2 + \cdots + x_\ell^2$$

which counts the number of ways in which n can be written as sums of ℓ squares. In ℓ-dimensional space, $r_\ell(n)$ gives also the number of points with integer coordinates on the sphere.

When ℓ is odd, the problem is very difficult. However for the even case, the problem may be treated in a fairly reasonable manner. Combining Ramanujan's ${}_1\psi_1$-bilateral formula with the Jacobi-triple product identity, we present solutions for the two square and four square problems. The six and eight square problems are dealt with similarly by means of Bailey's bilateral ${}_6\psi_6$-series identity.

G1. Representations by two square sums

When $\ell = 2$, the result may be stated as the following q-series identity

$$\left\{ \sum_{n=-\infty}^{+\infty} q^{n^2} \right\}^2 = 1 + 4 \sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}}$$

and the corresponding formula for the numbers of representations by two squares

$$r_2(n) \quad = \quad 4 \sum_{(1+2c)\,|\,n} (-1)^c \quad = \quad 4 \sum_{2\,\nmid\,d\,|\,n} (-1)^{\binom{d}{2}}.$$

Proof. According to the Jacobi triple product identity

$$\sum_{n=-\infty}^{+\infty} q^{n^2} \;=\; \sum_{n=-\infty}^{+\infty} (-1)^n q^{2\binom{n}{2}} (-q)^n \;=\; \left[q^2, -q, -q;\, q^2\right]_\infty$$

we have shifted factorial product expression

$$\Big\{\sum_{n=-\infty}^{+\infty} q^{n^2}\Big\}^2 = \big[q^2, -q, -q;\ q^2\big]_\infty^2$$

which can be reformulated by means of the Euler formula as

$$\big[q^2, -q, -q;\ q^2\big]_\infty^2 = \frac{\big[q^2,\ q^2, -q, -q;\ q^2\big]_\infty}{\big[q,\ q, -q^2, -q^2;\ q^2\big]_\infty}.$$

Recalling Ramanujan's ${}_1\psi_1$-bilateral series identity

$${}_1\psi_1\left[\begin{matrix} a \\ c \end{matrix}\ \Big|\ q;\ z\right] = \left[\begin{matrix} q, & c/a, & az, & q/az \\ c, & q/a, & z, & c/az \end{matrix}\ \Big|\ q\right]_\infty$$

we have

$$\begin{aligned}\Big\{\sum_{n=-\infty}^{+\infty} q^{n^2}\Big\}^2 &= {}_1\psi_1\left[\begin{matrix} -1 \\ -q^2 \end{matrix}\ \Big|\ q^2;\ q\right] \\ &= 1+\sum_{k=1}^{\infty}\Big\{\frac{(-1;q^2)_k}{(-q^2;q^2)_k}q^k+\frac{(-1;q^2)_{-k}}{(-q^2;q^2)_{-k}}q^{-k}\Big\}.\end{aligned}$$

Noting further two relations on shifted factorial fractions:

$$\begin{aligned}\frac{(-1;q^2)_{-k}}{(-q^2;q^2)_{-k}} &= \frac{(-1;q^2)_k}{(-q^2;q^2)_k}q^{2k} \\ \frac{(-1;q^2)_k}{(-q^2;q^2)_k} &= \frac{2}{1+q^{2k}}\end{aligned}$$

we find the following simplified expression

$$\Big\{\sum_{n=-\infty}^{+\infty} q^{n^2}\Big\}^2 = 1+4\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}.$$

Extracting the coefficient of q^n, we establish

$$\begin{aligned} r_2(n) &= [q^n]\Big\{1+4\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}\Big\} \\ &= [q^n]\Big\{1+4\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m q^{k(1+2m)}\Big\} \\ &= 4\sum_{(1+2m)|n}(-1)^m.\end{aligned}$$

This completes the solution of representations by two square sums. □

G2. Representations by four square sums

The Lagrange four square theorem states that every natural number can be expressed as sum of four square numbers. More precisely, we have the following q-series identity

$$\left\{\sum_{n=-\infty}^{+\infty} q^{n^2}\right\}^4 = \left\{\frac{(-q;\,-q)_\infty}{(q;\,-q)_\infty}\right\}^4 = 1+\sum_{n=1}^{\infty}\frac{8nq^n}{1+(-q)^n}$$

and the corresponding formula for the numbers of representations by four squares

$$r_4(n) = 8\sum_{4\,\nmid\, d\,\mid\, n} d \quad\Rightarrow\quad r_4(n)\geq 1 \quad\text{for}\quad n=1,2,\cdots.$$

Its demonstration is similar to that for the case of two squares. Based on Ramanujan's bilateral sum, we have the following limiting relation

$$\begin{aligned}
\left\{\frac{(q;\,q)_\infty}{(-q;\,q)_\infty}\right\}^4 &= \lim_{z\to -q}\frac{2}{1+q/z}\left[\begin{matrix} q, & q, & -z, & -q/z \\ -q, & -q, & z, & q/z\end{matrix}\,\middle|\, q\right]_\infty \\
&= \lim_{z\to -q}\frac{2}{1+q/z}\,{}_1\psi_1\left[\begin{matrix}-1\\ -q\end{matrix}\,\middle|\, q;\ z\right] \\
&= \lim_{z\to -q}\frac{2}{1+q/z}\left\{1+\sum_{n=1}^{\infty}\frac{(-1;q)_n}{(-q;q)_n}z^n+\sum_{n=1}^{\infty}\frac{(-1;q)_{-n}}{(-q;q)_{-n}}z^{-n}\right\} \\
&= \lim_{z\to -q}\frac{2}{1+q/z}\left\{1+\sum_{n=1}^{\infty}\frac{2z^n}{1+q^n}+\sum_{n=1}^{\infty}\frac{2(q/z)^n}{1+q^n}\right\}.
\end{aligned}$$

Reformulating the last sum as

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{2(q/z)^n}{1+q^n} &= \sum_{n=1}^{\infty}\left\{\frac{2(q/z)^n(1+q^n)}{1+q^n}-\frac{2(q^2/z)^n}{1+q^n}\right\} \\
&= \frac{2q/z}{1-q/z}-\sum_{n=1}^{\infty}\frac{2(q^2/z)^n}{1+q^n}
\end{aligned}$$

we may compute the limit explicitly as

$$\begin{aligned}
\left\{\frac{(q;\,q)_\infty}{(-q;\,q)_\infty}\right\}^4 &= \lim_{z\to -q}\frac{2}{1+q/z}\left\{\frac{1+q/z}{1-q/z}+\sum_{n=1}^{\infty}\frac{2z^n}{1+q^n}\left[1-(q/z)^{2n}\right]\right\} \\
&= 1+\sum_{n=1}^{\infty}\frac{8n(-q)^n}{1+q^n}.
\end{aligned}$$

On the other hand, it is not hard to derive

$$\sum_{n=-\infty}^{+\infty}(-1)^n q^{n^2} \;=\; \big[q^2, q, q;\, q^2\big]_\infty \;=\; \frac{(q;\,q)_\infty}{(-q;\,q)_\infty}.$$

Performing the parameter replacement $q \to -q$, we find the following expression:

$$\Big\{\sum_{n=-\infty}^{+\infty} q^{n^2}\Big\}^4 \;=\; \frac{\big(-q;-q\big)_\infty^4}{\big(q;-q\big)_\infty^4} \;=\; 1+\sum_{n=1}^{\infty}\frac{8nq^n}{1+(-q)^n}$$

which is equivalent to the sum for $r_4(n)$. In fact, observe that

$$1+8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n} \;=\; 1+8\sum_{k=1}^{\infty}\frac{2kq^{2k}}{1+q^{2k}}+8\sum_{k=1}^{\infty}\frac{(2k-1)q^{2k-1}}{1-q^{2k-1}}.$$

Noting that

$$\frac{2kq^{2k}}{1+q^{2k}} \;=\; \frac{2k(q^{2k}-q^{4k})}{1-q^{4k}} \;=\; \frac{2kq^{2k}}{1-q^{2k}}-\frac{4kq^{4k}}{1-q^{4k}}$$

we have

$$1+8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n} \;=\; 1+8\sum_{\substack{k=1\\ 4\nmid k}}^{\infty}\frac{kq^k}{1-q^k}.$$

Extracting the coefficient of q^n, we therefore have

$$r_4(n) \;=\; [q^n]\Big\{1+8\sum_{\substack{k=1\\ 4\nmid k}}^{\infty}\frac{kq^k}{1-q^k}\Big\} = [q^n]\Big\{1+8\sum_{\substack{k=1\\ 4\nmid k}}^{\infty}\sum_{m=1}^{\infty}kq^{km}\Big\} \;=\; 8\sum_{\substack{k|n\\ 4\nmid k}}k.$$

This completes the solution of representations by four square sums. $\square$

By means of Bailey's bilateral ${}_6\psi_6$-series identity, we now investigate the representations by six and eight squares.

G3. Representations by six square sums

There hold the following q-series identity

$$\Big\{\sum_{n=-\infty}^{\infty} q^{n^2}\Big\}^6 = \Big\{\frac{(-q;-q)_\infty}{(q;-q)_\infty}\Big\}^6 \;=\; 1+16\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}}$$
$$-\;4\sum_{n=0}^{\infty}(-1)^n\frac{(1+2n)^2q^{1+2n}}{1-q^{1+2n}}$$

and the corresponding formula for the numbers of representations by six squares

$$r_6(n) \;=\; 16\sum_{d\,|\,n} d^2\chi(n/d) - 4\sum_{d\,|\,n} d^2\chi(d)$$

where the quadratic Dirichlet character $\chi(d)$ is defined by

$$\chi(d) \;=\; \begin{cases} +1, & d \equiv_4 +1 \\ -1, & d \equiv_4 -1 \\ 0, & d \equiv_2 \ 0. \end{cases}$$

The proof will be fulfilled in three steps.

G3.1. Recall Bailey's very well-poised non-terminating bilateral series identity

$$\begin{aligned} &{}_6\psi_6\left[\begin{matrix} qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e \\ a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\;\middle|\; q;\ \frac{qa^2}{bcde}\right] \\ &= \left[\begin{matrix} q,\ qa,\ q/a,\ qa/bc,\ qa/bd,\ qa/be,\ qa/cd,\ qa/ce,\ qa/de \\ qa/b,\ qa/c,\ qa/d,\ qa/e,\ q/b,\ q/c,\ q/d,\ q/e,\ qa^2/bcde \end{matrix}\;\middle|\; q\right]_\infty \end{aligned}$$

provided that $\left|qa^2/bcde\right| < 1$.

Specifying with $b = c = d = -1$ and $e \to \infty$, we may restate it as

$$\begin{aligned} \frac{(q;q)_\infty(q/a;q)_\infty(qa;q)_\infty^4}{(-q;q)_\infty^3(-qa;q)_\infty^3} &= 1+\sum_{k=1}^{\infty} a^{2k}\frac{1-q^{2k}a}{1-a}\frac{(-1;q)_k^3}{(-qa;q)_k^3}q^{\binom{1+k}{2}} \\ &\quad+\sum_{k=1}^{\infty} a^{-2k}\frac{1-q^{-2k}a}{1-a}\frac{(-1;q)_{-k}^3}{(-qa;q)_{-k}^3}q^{\binom{1-k}{2}} \\ &= 1+\sum_{k=1}^{\infty}\frac{q^{\binom{1+k}{2}}}{1-a}\Big\{\left(a^{2k}-q^{2k}a^{1+2k}\right)\frac{(-1;q)_k^3}{(-qa;q)_k^3} \\ &\qquad+\left(q^{2k}a^k-a^{1+k}\right)\frac{(-1/a;q)_k^3}{(-q;q)_k^3}\Big\}. \end{aligned}$$

Letting $a \to 1$, we can compute, through L'Hôspital's rule, the limit of the summand as follows:

$$\begin{aligned}
8\frac{1-q^{2k}}{(1+q^k)^3}q^{\binom{1+k}{2}}\Big\{\frac{(1+2k)q^{2k}-2k}{1-q^{2k}}+3\sum_{i=1}^{k}\frac{q^i}{1+q^i}&\\
+\frac{(1+k)-kq^{2k}}{1-q^{2k}}-3\sum_{j=0}^{k-1}\frac{q^j}{1+q^j}\Big\}&\\
=\frac{4q^{\binom{1+k}{2}+k}}{(1+q^k)^3}\Big\{6-(1-2k)q^k-(1+2k)q^{-k}\Big\}&.
\end{aligned}$$

Therefore we have found the expression:

$$\Big\{\frac{(q;q)_\infty}{(-q;q)_\infty}\Big\}^6 = 1+4\sum_{k=1}^{\infty}\frac{q^{\binom{1+k}{2}+k}}{(1+q^k)^3}\Big\{6-(1-2k)q^k-(1+2k)q^{-k}\Big\}.$$

G3.2. Denoting the last sum with respect to k by $\clubsuit(q)$ and then recalling the binomial expansion

$$\frac{q^k}{(1+q^k)^3} = \sum_{\ell=1}^{\infty}(-1)^{1+\ell}\binom{1+\ell}{2}q^{k\ell}$$

we can manipulate $\clubsuit(q)$ in the following manner:

$$\begin{aligned}
\clubsuit(q) &= \sum_{k=1}^{\infty}\frac{q^{\binom{1+k}{2}+k}}{(1+q^k)^3}\Big\{6-(1-2k)q^k-(1+2k)q^{-k}\Big\}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(-1)^{1+\ell}\binom{1+\ell}{2}\Big\{6-(1-2k)q^k-(1+2k)q^{-k}\Big\}q^{\binom{1+k}{2}+k\ell}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{1+\ell}\Big\{6\binom{1+\ell}{2}+(1-2k)\binom{\ell}{2}+(1+2k)\binom{2+\ell}{2}\Big\}q^{\binom{1+k}{2}+k\ell}.
\end{aligned}$$

Rewriting the q-exponent by

$$\binom{1+k}{2}+k\ell = \frac{1}{2}\Big\{k(1+2\ell+k)\Big\}$$

and then simplifying the binomial sum

$$\begin{aligned}
&6\binom{1+\ell}{2}+(1-2k)\binom{\ell}{2}+(1+2k)\binom{2+\ell}{2}\\
=\;&(1+2\ell)\times(1+2k+2\ell) = (1+k+2\ell)^2-k^2
\end{aligned}$$

we can split $\clubsuit$, according to the parity of k, into two double sums:

$$\begin{aligned}
\clubsuit(q) &= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{1+\ell}\Big\{(1+k+2\ell)^2-k^2\Big\}q^{\binom{1+k}{2}+k\ell} && \text{(G3.1a)}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{1+\ell}\Big\{(1+2k+2\ell)^2-(2k)^2\Big\}q^{k(1+2k+2\ell)} && \text{(G3.1b)}\\
&+ \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty}(-1)^{1+\ell}\Big\{(2k+2\ell)^2-(2k-1)^2\Big\}q^{(k+\ell)(2k-1)}. && \text{(G3.1c)}
\end{aligned}$$

G3.3. Putting $n := k+\ell$ and then applying the geometric series, we can reduce (G3.1b) as follows:

$$\begin{aligned}
\mathrm{Eq(G3.1b)} &= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}(-1)^{1+n-k}\Big\{(1+2n)^2-(2k)^2\Big\}q^{k(1+2n)}\\
&= \sum_{n=1}^{\infty}(-1)^{1+n}(1+2n)^2\sum_{k=1}^{n}(-1)^k q^{k(1+2n)}\\
&+ \sum_{k=1}^{\infty}(-1)^k(2k)^2q^k\sum_{n=k}^{\infty}(-1)^n q^{2nk}\\
&= \sum_{n=0}^{\infty}\frac{(1+2n)^2}{1+q^{1+2n}}\Big\{(-1)^nq^{1+2n}-q^{(1+n)(1+2n)}\Big\}\\
&+ \sum_{k=1}^{\infty}\frac{4k^2}{1+q^{2k}}q^{k(1+2k)}.
\end{aligned}$$

We can also treat (G3.1c) analogously:

$$\begin{aligned}
\mathrm{Eq(G3.1c)} &= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}(-1)^{1+n-k}\Big\{(2n)^2-(2k-1)^2\Big\}q^{n(2k-1)}\\
&= \sum_{n=1}^{\infty}(-1)^{1+n}(2n)^2q^{-n}\sum_{k=1}^{n}(-1)^k q^{2nk}\\
&+ \sum_{k=1}^{\infty}(-1)^k(2k-1)^2\sum_{n=k}^{\infty}(-1)^n q^{n(2k-1)}\\
&= \sum_{n=1}^{\infty}\frac{(2n)^2}{1+q^{2n}}\Big\{(-1)^nq^{n}-q^{n(1+2n)}\Big\}\\
&+ \sum_{k=1}^{\infty}\frac{(2k-1)^2}{1+q^{2k-1}}q^{k(2k-1)}.
\end{aligned}$$

Their combination leads us to the following:

$$\begin{aligned} \clubsuit(q) &= \text{Eq(G3.1a)} = \text{Eq(G3.1b)} + \text{Eq(G3.1c)} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{q^{1+2n}}{1+q^{1+2n}}(1+2n)^2 + 4\sum_{n=1}^{\infty}(-1)^n \frac{n^2q^n}{1+q^{2n}}. \end{aligned}$$

Replacing q by $-q$, we have finally established the q-series identity:

$$\begin{aligned} \Big\{\sum_{n=-\infty}^{\infty} q^{n^2}\Big\}^6 &= \Big\{\frac{(-q;-q)_\infty}{(q;-q)_\infty}\Big\}^6 = 1 + 4\clubsuit(-q) \\ &= 16\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}} - 4\sum_{n=0}^{\infty}(-1)^n\frac{(1+2n)^2q^{1+2n}}{1-q^{1+2n}}. \end{aligned}$$

Extracting the coefficient of q^n, we get the formula for $r_6(n)$ stated in the Theorem. This completes the solution of representations by six square sums. $\square$

G4. Representations by eight square sums

Following the same procedure to the last section, we can also show the eight square sum theorem. But the proof is much easier this time.

The theorem states that there hold the q-series identity:

$$\Big\{\sum_{n=-\infty}^{\infty} q^{n^2}\Big\}^8 = \Big\{\frac{(-q;-q)_\infty}{(q;-q)_\infty}\Big\}^8 = 1 + 16\sum_{n=1}^{\infty}\frac{n^3q^n}{1-(-q)^n}$$

and the corresponding formula for the numbers of representations by eight squares

$$r_8(n) = 16\sum_{d\,|\,n}(-1)^{n+d}\,d^3.$$

The proof is divided into two parts.

G4.1. Putting with $b = c = d = e = -1$ in Bailey's very well-poised non-terminating bilateral series identity, we can write the result as

$$\begin{aligned}\frac{(q;q)_\infty(q/a;q)_\infty(qa;q)_\infty^7}{(qa^2;q)_\infty(-q;q)_\infty^4(-qa;q)_\infty^4} &= 1+\sum_{k=1}^{\infty}\frac{1-q^{2k}a}{1-a}\frac{(-1;q)_k^4}{(-qa;q)_k^4}(qa^2)^k\\ &\quad+\sum_{k=1}^{\infty}\frac{1-q^{-2k}a}{1-a}\frac{(-1;q)_{-k}^4}{(-qa;q)_{-k}^4}(qa^2)^{-k}\\ &= 1+\sum_{k=1}^{\infty}\frac{q^k}{1-a}\left\{\left(a^{2k}-q^{2k}a^{1+2k}\right)\frac{(-1;q)_k^4}{(-qa;q)_k^4}\right.\\ &\quad\left.+\left(q^{2k}a^{2k}-a^{1+2k}\right)\frac{(-1/a;q)_k^4}{(-q;q)_k^4}\right\}.\end{aligned}$$

Letting $a \to 1$, we can compute, through L'Hôspital's rule, the limit of the summand as follows:

$$\begin{aligned}&16q^k\frac{1-q^{2k}}{(1+q^k)^4}\left\{\frac{(1+2k)q^{2k}-2k}{1-q^{2k}}+4\sum_{i=1}^{k}\frac{q^i}{1+q^i}\right.\\ &\qquad\left.+\frac{(1+2k)-2kq^{2k}}{1-q^{2k}}-4\sum_{j=0}^{k-1}\frac{q^j}{1+q^j}\right\}\\ &=\frac{16q^{2k}}{(1+q^k)^4}\left\{4-q^k-q^{-k}\right\}.\end{aligned}$$

Therefore we have found the expression:

$$\left\{\frac{(q;q)_\infty}{(-q;q)_\infty}\right\}^8 = 1+16\sum_{k=1}^{\infty}\frac{q^{2k}}{(1+q^k)^4}\left\{4-q^k-q^{-k}\right\}.$$

G4.2. Denoting the last sum with respect to k by $\diamondsuit(q)$ and then recalling the binomial expansion

$$\frac{q^{2k}}{(1+q^k)^4} = \sum_{\ell=2}^{\infty}(-1)^\ell\binom{1+\ell}{3}q^{k\ell}$$

we can manipulate $\diamondsuit(q)$ in the following manner:

$$\begin{aligned}
\diamondsuit(q) &= \sum_{k=1}^{\infty}\frac{q^{2k}}{(1+q^k)^4}\Big\{4-q^k-q^{-k}\Big\}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=2}^{\infty}(-1)^{\ell}\binom{1+\ell}{3}\Big\{4-q^k-q^{-k}\Big\}q^{k\ell}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(-1)^{\ell}\Big\{4\binom{1+\ell}{3}+\binom{\ell}{3}+\binom{2+\ell}{3}\Big\}q^{k\ell}\\
&= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(-1)^{\ell}\ell^3q^{k\ell} = \sum_{\ell=1}^{\infty}(-1)^{\ell}\frac{q^{\ell}}{1-q^{\ell}}\ell^3
\end{aligned}$$

where the following binomial sum has been used

$$4\binom{1+\ell}{3}+\binom{\ell}{3}+\binom{2+\ell}{3} = \ell^3.$$

Now replacing q by $-q$, we derive the q-series identity:

$$\begin{aligned}
\Big\{\sum_{n=-\infty}^{\infty}q^{n^2}\Big\}^8 &= \Big\{\frac{(-q;-q)_\infty}{(q;-q)_\infty}\Big\}^8 = 1+16\diamondsuit(-q)\\
&= 1+16\sum_{n=1}^{\infty}\frac{n^3q^n}{1-(-q)^n}.
\end{aligned}$$

Extracting the coefficient of q^n, we get the formula for $r_8(n)$ stated in the Theorem. This completes the solution of representations by eight square sums. □

G5. Jacobi's identity and q-difference equations

Among q-difference equations, there is a beautiful result due to Jacobi (1829), which will be proved and generalized in this section.

G5.1. Jacobi's q-difference equation. The identity on eight infinite products states that

$$\big(-q;\,q^2\big)_\infty^8-\big(q;\,q^2\big)_\infty^8 = 16q\big(-q^2;\,q^2\big)_\infty^8$$

which has been commented by Jacobi (1829) as "*aequatio identica satis abstrusa*".

Its proof can be fulfilled by means of the Jacobi-triple product identity and Lagrange's four square theorem.

In fact, multiplying both sides by $\left(q^2;\, q^2\right)_\infty^4$ and then noticing that

$$\left[q^2,\, q,\, q;\, q^2\right]_\infty \quad = \quad \sum_{m=-\infty}^{+\infty} (-1)^m\, q^{m^2}$$

we can reformulate the eight product difference equation as follows

$$2q\sum_{m=0}^{\infty} r_4(1+2m)\, q^{2m} \;=\; q\Big\{\sum_{n=-\infty}^{+\infty} q^{n(n+1)}\Big\}^4 \;=\; q\sum_{n=0}^{\infty} s_4(n)\, q^{2n}$$

where $s_4(n)$ is the number of integer solutions of Diophantine equation

$$n \quad = \quad \binom{1+x_1}{2} + \binom{1+x_2}{2} + \binom{1+x_3}{2} + \binom{1+x_4}{2}$$

which counts the number of ways expressing n as sums of four triangles. It is equal to the number of integer solutions of Diophantine equation

$$4+8n \quad = \quad (1+2x_1)^2 + (1+2x_2)^2 + (1+2x_3)^2 + (1+2x_4)^2.$$

The last one is in turn the number of odd integer solutions of Diophantine equation

$$4+8n \quad = \quad y_1^2 + y_2^2 + y_3^2 + y_4^2$$

whose integer solutions enumerated by $r_4(4+8n)$ may be divided into two categories: odd integer solutions counted by $s_4(n)$ and even integer solutions by $r_4(1+2n)$. Therefore we have

$$s_4(n) \quad = \quad r_4(4+8n) - r_4(1+2n) \quad = \quad 2r_4(1+2n)$$

which leads us to Jacobi's q-difference equation.

G5.2. Theorem. Generalizing the Jacobi q-difference equation, we prove, by combining the telescoping method with bailey's bilateral ${}_6\psi_6$-series identity, the following theorem due to Chu (1992).

For five parameters related by multiplicative relation $A^2 = bcde$, there holds

$$\begin{aligned} & \langle A/b;q\rangle_\infty\, \langle A/c;q\rangle_\infty\, \langle A/d;q\rangle_\infty\, \langle A/e;q\rangle_\infty && \text{(G5.1a)} \\ - \quad & \langle b;q\rangle_\infty\, \langle c;q\rangle_\infty\, \langle d;q\rangle_\infty\, \langle e;q\rangle_\infty && \text{(G5.1b)} \\ = \quad & b\, \langle A;q\rangle_\infty\, \langle A/bc;q\rangle_\infty \langle A/bd;q\rangle_\infty\, \langle A/be;q\rangle_\infty && \text{(G5.1c)} \end{aligned}$$

where the q-shifted factorial for $|q|<1$ is defined by

$$(x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n) \quad \text{and} \quad \langle x;q\rangle_\infty \;=\; (x;q)_\infty \times (q/x;q)_\infty.$$

This identity reduces to Jacobi's equation under parameter replacements

$$q \to q^2:\; A = -q^2 \quad \text{and} \quad b=c=d=e=-q.$$

Proof. Define the factorial fractions by

$$T_k := \left[\begin{matrix} b, & c, & d, & e \\ A/b, & A/c, & A/d, & A/e \end{matrix}\,\Big|\, q\right]_k.$$

It is trivial to check factorization

$$\begin{aligned} & (1-q^kA/b)(1-q^kA/c)(1-q^kA/d)(1-q^kA/e) \\ - & (1-q^kb)(1-q^kc)(1-q^kd)(1-q^ke) \\ = & bq^k(1-q^{2k}A)(1-A/bc)(1-A/bd)(1-A/be) \end{aligned}$$

which leads us to the following difference relation:

$$\begin{aligned} T_k - T_{k+1} &= \left[\begin{matrix} b, & c, & d, & e \\ qA/b, & qA/c, & qA/d, & qA/e \end{matrix}\,\Big|\, q\right]_k \\ &\times \frac{\left\{\begin{matrix}(1-q^kA/b)(1-q^kA/c)(1-q^kA/d)(1-q^kA/e) \\ -(1-q^kb)\,(1-q^kc)\,(1-q^kd)\,(1-q^ke)\end{matrix}\right\}}{(1-A/b)(1-A/c)(1-A/d)(1-A/e)} \\ &= \left[\begin{matrix} b, & c, & d, & e \\ qA/b, & qA/c, & qA/d, & qA/e \end{matrix}\,\Big|\, q\right]_k \\ &\times \frac{bq^k(1-q^{2k}A)(1-A/bc)(1-A/bd)(1-A/be)}{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}. \end{aligned}$$

Reformulating the last relation as

$$\begin{aligned} & \frac{1-q^{2k}A}{1-A}\left[\begin{matrix} b, & c, & d, & e \\ qA/b, & qA/c, & qA/d, & qA/e \end{matrix}\,\Big|\, q\right]_k q^k \\ = & \left\{T_k - T_{k+1}\right\}\frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)} \end{aligned}$$

and then applying the telescoping method, we derive the bilateral finite summation formula:

$$\sum_{k=m}^{n-1}\frac{1-q^{2k}A}{1-A}\left[\begin{matrix} b, & c, & d, & e \\ qA/b, & qA/c, & qA/d, & qA/e \end{matrix}\,\Big|\, q\right]_k q^k \tag{G5.2a}$$

$$= \frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)}\sum_{k=m}^{n-1}\left\{T_k - T_{k+1}\right\} \tag{G5.2b}$$

$$= \frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)}\left\{T_m - T_n\right\}. \tag{G5.2c}$$

Recalling the definition of T_k and keeping in mind of $A^2 = bcde$, we have

$$\begin{aligned}
\lim_{n\to+\infty} T_n &= \left[\begin{matrix} b, & c, & d, & e \\ A/b, & A/c, & A/d, & A/e \end{matrix} \,\middle|\, q\right]_\infty \\
\lim_{m\to-\infty} T_m &= \lim_{m\to+\infty} \left[\begin{matrix} b, & c, & d, & e \\ A/b, & A/c, & A/d, & A/e \end{matrix} \,\middle|\, q\right]_{-m} \\
&= \lim_{m\to+\infty} \left[\begin{matrix} qb/A, & qc/A, & qd/A, & qe/A \\ q/b, & q/c, & q/d, & q/e \end{matrix} \,\middle|\, q\right]_m \\
&= \left[\begin{matrix} qb/A, & qc/A, & qd/A, & qe/A \\ q/b, & q/c, & q/d, & q/e \end{matrix} \,\middle|\, q\right]_\infty .
\end{aligned}$$

Now letting $m \to -\infty$ and $n \to +\infty$ in (G5.2), we get a closed formula for the non-terminating bilateral convergent series:

$$\begin{aligned}
&{}_6\psi_6 \left[\begin{matrix} q\sqrt{A}, & -q\sqrt{A}, & b, & c, & d, & e \\ \sqrt{A}, & -\sqrt{A}, & qA/b, & qA/c, & qA/d, & qA/e \end{matrix} \,\middle|\, q; q\right] \\
&= \frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)} \\
&\times \left\{ \left[\begin{matrix} qb/A, & qc/A, & qd/A, & qe/A \\ q/b, & q/c, & q/d, & q/e \end{matrix} \,\middle|\, q\right]_\infty \right. \\
&\qquad \left. - \left[\begin{matrix} b, & c, & d, & e \\ A/b, & A/c, & A/d, & A/e \end{matrix} \,\middle|\, q\right]_\infty \right\}.
\end{aligned}$$

Alternatively, the last bilateral sum can be evaluated by Bailey's ${}_6\psi_6$-series identity with $A^2 = bcde$ as follows:

$$\begin{aligned}
&{}_6\psi_6 \left[\begin{matrix} qA^{1/2}, & -qA^{1/2}, & b, & c, & d, & e \\ A^{1/2}, & -A^{1/2}, & qA/b, & qA/c, & qA/d, & qA/e \end{matrix} \,\middle|\, q; \frac{qA^2}{bcde}\right] \\
&= \left[\begin{matrix} qA, q/A, qA/bc, qA/bd, qA/be, qA/cd, qA/ce, qA/de \\ qA/b,\ qA/c,\ qA/d,\ qA/e,\ q/b,\ q/c,\ q/d,\ q/e \end{matrix} \,\middle|\, q\right]_\infty .
\end{aligned}$$

Equating the right members of both results, we get the following relation:

$$\begin{aligned}
&b \left[\begin{matrix} A,\ q/A,\ A/bc,\ A/bd,\ A/be,\ qA/cd,\ qA/ce,\ qA/de \\ A/b,\ A/c,\ A/d,\ A/e,\ q/b,\ q/c,\ q/d,\ q/e \end{matrix} \,\middle|\, q\right]_\infty \\
&= \left[\begin{matrix} qb/A, qc/A, qd/A, qe/A \\ q/b,\ q/c,\ q/d,\ q/e \end{matrix} \,\middle|\, q\right]_\infty - \left[\begin{matrix} b,\ c,\ d,\ e \\ A/b, A/c, A/d, A/e \end{matrix} \,\middle|\, q\right]_\infty
\end{aligned}$$

which is equivalent to the q-difference equation

$$\begin{aligned}
&b\ \langle A; q\rangle_\infty\ \langle A/bc; q\rangle_\infty \langle A/bd; q\rangle_\infty\ \langle A/be; q\rangle_\infty \\
&= \langle A/b; q\rangle_\infty\ \langle A/c; q\rangle_\infty\ \langle A/d; q\rangle_\infty\ \langle A/e; q\rangle_\infty \\
&- \langle b; q\rangle_\infty\ \langle c; q\rangle_\infty\ \langle d; q\rangle_\infty\ \langle e; q\rangle_\infty .
\end{aligned}$$

This proves the q-difference equation stated in Theorem G5.2. $\square$

G5.3. A trigonometric identity. The discovery of Theorem G5.2 has been inspired by the following interesting fact. Given five parameters related instead by additive relation $2A = b + c + d + e$, it can be verified that

$$(A-b)(A-c)(A-d)(A-e) \;-\; bcde \;=\; A(A-b-c)(A-b-d)(A-b-e).$$

Surprisingly, it is also true even if we replace each linear factor with its sine function:

$$\begin{aligned} & \sin(A-b)\ \sin(A-c)\ \sin(A-d)\ \sin(A-e) \;-\; \sin b\ \sin c\ \sin d\ \sin e \\ = \;& \sin A\ \sin(A-b-c)\ \sin(A-b-d)\ \sin(A-b-e). \end{aligned}$$

The q-difference equation displayed in Theorem G5.2 may be considered as the q-analogue of this trigonometric identity.

According to the factorial fraction

$$\frac{\langle x; q\rangle_\infty}{(1-q)(q;\,q)_\infty^2} \;=\; \frac{1-x}{1-q}\prod_{n=1}^{\infty}\frac{(1-q^n x)(1-q^n/x)}{(1-q^n)^2}$$

we have the following limit relation

$$\begin{aligned} \lim_{q\to 1}\frac{\langle q^x; q\rangle_\infty}{(1-q)(q;\,q)_\infty^2} \;&=\; \lim_{q\to 1}\frac{1-q^x}{1-q}\prod_{n=1}^{\infty}\frac{(1-q^{n+x})(1-q^{n-x})}{(1-q^n)^2} \\ &=\; x\prod_{n=1}^{\infty}\Big\{1-\frac{x^2}{n^2}\Big\} \;=\; \frac{\sin(\pi x)}{\pi}. \end{aligned}$$

Replacing first b, c, d, e respectively by q^b, q^c, q^d, q^e in the q-difference equation stated in Theorem G5.2, then dividing both sides by $(1-q)^4(q;q)_\infty^8$ and finally letting $q \to 1$, we get the following trigonometric formula:

$$\begin{aligned} & \sin\pi A\ \sin\pi(A-b-c)\ \sin\pi(A-b-d)\ \sin\pi(A-b-e) \\ = \;& \sin\pi(A-b)\ \sin\pi(A-c)\ \sin\pi(A-d)\ \sin\pi(A-e) \\ - \;& \sin\pi b\ \sin\pi c\ \sin\pi d\ \sin\pi e, \quad (b+c+d+e=2A) \end{aligned}$$

which is the equivalent form of the trigonometric identity to be proved. $\square$