CHAPTER F

# Bilateral Basic Hypergeometric Series

This chapter deals with the bilateral basic hypergeometric series. The Ramanujan ${}_1\psi_1$-summation formula and Bailey's very-well-poised ${}_6\psi_6$-series identity will be established. We shall also investigate the non-terminating bilateral $q$-analogue of Dixon's theorem on cubic-sum of binomial coefficients, partial fraction decomposition method on basic hypergeometric series with integral differences between numerator parameters and denominator parameters.

## F1. Definition and notation

**F1.1. Definition and convergence.** Let $\{a_i\}_{i=1}^{r}$ and $\{b_j\}_{j=1}^{s}$ be complex numbers subject to the condition that $a_i \neq q^m$ and $b_j \neq q^{-n}$ with $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ for all $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$. Then the bilateral $q$-hypergeometric series with variable $z$ is defined by

$${}_r\psi_s\left[\begin{matrix}a_1,\, a_2,\, \cdots,\, a_r\\ b_1,\, b_2,\, \cdots,\, b_s\end{matrix}\,\middle|\, q;\, z\right] = \sum_{n=-\infty}^{+\infty}\Big\{(-1)^n q^{\binom{n}{2}}\Big\}^{s-r}\left[\begin{matrix}a_1, a_2, \cdots, a_r\\ b_1, b_2, \cdots, b_s\end{matrix}\,\middle|\, q\right]_n z^n.$$

For $m \in \mathbb{Z}$, we find by shifting the summation index $n \to m+n$, that the bilateral ${}_r\psi_s$-series satisfies relation

$${}_r\psi_s\left[\begin{matrix}a_1, a_2, \cdots, a_r\\ b_1, b_2, \cdots, b_s\end{matrix}\,\middle|\, q;\, z\right] = \left[\begin{matrix}a_1, & a_2, & \cdots, & a_r\\ b_1, & b_2, & \cdots, & b_s\end{matrix}\,\middle|\, q\right]_m z^m \quad \text{(F1.1a)}$$

$$\times\, {}_r\psi_s\left[\begin{matrix}q^m a_1,\, q^m a_2,\, \cdots,\, q^m a_r\\ q^m b_1,\, q^m b_2,\, \cdots,\, q^m b_s\end{matrix}\,\middle|\, q; z\right]. \quad \text{(F1.1b)}$$

When ${}_r\psi_s$ has no zero parameters, we can reverse the summation order and get another equivalent expression for the bilateral ${}_r\psi_s$-series:

$${}_r\psi_s\left[\begin{matrix}a_1,\, a_2,\, \cdots,\, a_r\\ b_1,\, b_2,\, \cdots,\, b_s\end{matrix}\,\middle|\, q;\, z\right] = \sum_{m=-\infty}^{+\infty}\left[\begin{matrix}q/b_1, q/b_2, \cdots, q/b_s\\ q/a_1, q/a_2, \cdots, q/a_r\end{matrix}\,\middle|\, q\right]_m \left\{\frac{\mathcal{B}}{\mathcal{A}z}\right\}^m \quad \text{(F1.2)}$$

where $\mathcal{A} := a_1 a_2 \cdots a_r$ and $\mathcal{B} := b_1 b_2 \cdots b_s$ have been defined for brevity, and the shifted factorial with negative integer order has been inverted as follows:

$$(x;q)_{-n} = \frac{(x;q)_\infty}{(q^{-n}x;q)_\infty} = \frac{1}{(q^{-n}x;q)_n} = \frac{q^{\binom{n+1}{2}}(-1/x)^n}{(q/x;q)_n}, \quad (n \in \mathbb{N}_0).$$

Splitting the bilateral series ${}_r\phi_s$ into two infinite series:

$$\begin{aligned} {}_r\psi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\middle|\, q;\, z\right] &= \sum_{n=0}^{+\infty} \Big\{(-1)^n q^{\binom{n}{2}}\Big\}^{s-r} \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\middle|\, q\right]_n z^n \\ &+ \sum_{m=1}^{+\infty} \left[\begin{matrix} q/b_1, q/b_2, \cdots, q/b_s \\ q/a_1, q/a_2, \cdots, q/a_r \end{matrix} \,\middle|\, q\right]_m \Big\{\frac{\mathcal{B}}{\mathcal{A}z}\Big\}^m \end{aligned}$$

we can check without difficulty that for $|q| < 1$ and $\mathcal{R} = \big|\mathcal{B}/\mathcal{A}\big|$, the convergence condition of the bilateral series is determined as follows:

- if $r < s$, the series converges for $|z| > \mathcal{R}$;
- if $r > s$, the series diverges for all $z \in \mathbb{C}$ except for $z = 0$;
- if $r = s$, which is the most important case, the series converges for $\mathcal{R} < |z| < 1$.

**F1.2. Ordinary bilateral hypergeometric series.** Similarly, we can define the (ordinary) bilateral hypergeometric series.

Let $\{a_i\}_{i=1}^r$ and $\{b_j\}_{j=1}^s$ be complex numbers subject to the condition that $a_i \neq m$ and $b_j \neq -n$ with $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ for all $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$. Then the (ordinary) bilateral hypergeometric series with variable $z$ is defined by

$${}_rH_s\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \,\middle|\, z\right] = \sum_{n=-\infty}^{+\infty} \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}\right]_n z^n.$$

Only when $r = s$ and $|z| = 1$, the bilateral ${}_rH_r$-series is of some interest. Writing it in the sum of two unilateral series

$$\begin{aligned} {}_rH_r\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_r \end{matrix} \,\middle|\, z\right] &= \sum_{n=0}^{+\infty} \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_r \end{matrix}\right]_n z^n \\ &+ \sum_{m=1}^{+\infty} \left[\begin{matrix} 1-b_1, 1-b_2, \cdots, 1-b_r \\ 1-a_1, 1-a_2, \cdots, 1-a_r \end{matrix}\right]_m z^{-m} \end{aligned}$$

we can determine the convergence condition of ${}_rH_r$-series as follows:

- if $z = +1$, the series converges for $|\Re(B - A)| > 1$;

- if $z=-1$, the series converges for $|\Re(B-A)|>0$.

**F1.3. Examples.** Here we shall review the Jacobi triple and the quintuple product identities derived in C2.4 and C2.6 respectively.

- **The Jacobi triple product identity** can be stated in terms of bilateral series as follows:

$$ {}_0\psi_1\left[\begin{matrix} - \\ 0 \end{matrix}\,\middle|\, q;x\right] = \sum_{n=-\infty}^{+\infty}(-1)^n\,q^{\binom{n}{2}}\,x^n = [q,x,q/x;q]_\infty . \tag{F1.3}$$

- **An alternative form** can be obtained by separating the sum into two according to the parity of summation index $n$:

$$\begin{aligned}
\left[q^2,\,qy,\,q/y;q^2\right]_\infty &= \sum_{n=-\infty}^{+\infty}(-1)^n\,q^{n^2}\,y^n && \text{(F1.4a)}\\
&= \sum_{n=-\infty}^{+\infty} q^{4n^2}\Big\{1-yq^{1+4n}\Big\}\,y^{2n} && \text{(F1.4b)}\\
&= \sum_{n=-\infty}^{+\infty} q^{4n^2}\Big\{1-(q/y)^{1+4n}\Big\}\,y^{2n}. && \text{(F1.4c)}
\end{aligned}$$

- **Quintuple product identity** can have different forms such as

$$\begin{aligned}
& [q,\,z,\,q/z;q]_\infty\,\left[qz^2,\,q/z^2;q^2\right]_\infty && \text{(F1.5a)}\\
=& \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}}\,\Big\{1-zq^n\Big\}\,\big(qz^3\big)^n && \text{(F1.5b)}\\
=& \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}}\,\Big\{1-z^{1+6n}\Big\}\,\big(q^2/z^3\big)^n && \text{(F1.5c)}\\
=& \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}}\,\Big\{1-(q/z^2)^{1+3n}\Big\}\,\big(qz^3\big)^n && \text{(F1.5d)}\\
=& \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}}\,\Big\{1-(qz^2)^{2+3n}\Big\}\,\big(q^2/z^3\big)^n && \text{(F1.5e)}
\end{aligned}$$

as well as different limiting expressions:

$$\begin{aligned}
\sum_{k=-\infty}^{+\infty}\big(1+6k\big)\,q^{3\binom{k}{2}+2k} &= [q,\,q,\,q;q]_\infty\,\left[q,\,q;q^2\right]_\infty && \text{(F1.6a)}\\
\sum_{k=-\infty}^{+\infty}(1+3k)q^{3\binom{k}{2}+\frac{5}{2}k} &= \Big[q,\,q^{1/2},q^{1/2};q\Big]_\infty\left[q^2,q^2;q^2\right]_\infty . && \text{(F1.6b)}
\end{aligned}$$

PROOF. We need to show only (F1.5d), (F1.5e) and (F1.6b) because the others have been demonstrated in C2.6.

It is obvious that (F1.6b) is the limiting case $z \to q^{1/2}$ of (F1.5a) and (F1.5d). Therefore only (F1.5d) and (F1.5e) remain to be confirmed.

Splitting (F1.5b) into two sums and reversing the later by $n \to -1-n$, we can manipulate the sum as follows:

$$\begin{aligned}\sum_n q^{3\binom{n}{2}}\Big\{1-zq^n\Big\}\big(qz^3\big)^n &= \sum_n \Big\{q^{3\binom{n}{2}+n}z^{3n} - q^{3\binom{n}{2}+2n}z^{1+3n}\Big\}\\ &= \sum_n \Big\{q^{3\binom{n}{2}+n}z^{3n} - q^{3\binom{n}{2}+1+4n}z^{-2-3n}\Big\}\\ &= \sum_n q^{3\binom{n}{2}}\Big\{1-(q/z^2)^{1+3n}\Big\}\big(qz^3\big)^n\end{aligned}$$

which gives the bilateral sum stated in (F1.5d).

Similarly, splitting (F1.5c) into two sums and shifting the summation index $n \to 1+n$ for the latter, we can reformulate the sum as follows:

$$\begin{aligned}\sum_n q^{3\binom{n}{2}}\Big\{1-z^{1+6n}\Big\}\big(q^2/z^3\big)^n &= \sum_n \Big\{q^{3\binom{n}{2}+2n}z^{-3n} - q^{3\binom{n}{2}+2n}z^{1+3n}\Big\}\\ &= \sum_n \Big\{q^{3\binom{n}{2}+2n}z^{-3n} - q^{3\binom{n}{2}+2+5n}z^{4+3n}\Big\}\\ &= \sum_n q^{3\binom{n}{2}}\Big\{1-(qz^2)^{2+3n}\Big\}\big(q^2/z^3\big)^n\end{aligned}$$

which is exactly the sum displayed in (F1.5e). □

## F2. Ramanujan's bilateral ${}_1\psi_1$-series identity

$${}_1\psi_1\left[\begin{matrix}a\\c\end{matrix}\,\middle|\,q;\,z\right] = \left[\begin{matrix}q, & c/a, & az, & q/az\\ c, & q/a, & z, & c/az\end{matrix}\,\middle|\,q\right]_\infty, \quad \big(|c/a|<|z|<1\big). \tag{F2.1}$$

PROOF. For a large natural number $M$, choose three complex parameters

$$\begin{aligned} a &\to aq^{-M}\\ b &\to c/az\\ c &\to cq^{-M}.\end{aligned}$$

Then the $q$-Gauss theorem E2.2 can be restated as

$$ {}_2\phi_1\left[\begin{matrix} aq^{-M}, & c/az \\ & cq^{-M} \end{matrix} \,\middle|\, q;\ z\right] = \frac{(c/a;\ q)_\infty(q^{-M}az;\ q)_\infty}{(z;\ q)_\infty(q^{-M}c;\ q)_\infty}. $$

We can further reformulate it by shifting the summation index $n \to k+M$ as

$$ \sum_{k=-M}^{\infty} \frac{(a;q)_k(q^M c/az;q)_k}{(c;q)_k(q^{1+M};q)_k} z^k = z^{-M}\, \frac{(q;q)_M(q^{-M}c;q)_M}{(c/az;q)_M(q^{-M}a;q)_M}\, \frac{(c/a;q)_\infty(q^{-M}az;q)_\infty}{(z;q)_\infty(q^{-M}c;q)_\infty} $$

$$ = \frac{(q;\ q)_M(q/az;\ q)_M}{(q/a;\ q)_M(c/az;\ q)_M}\, \frac{(c/a;\ q)_\infty(az;\ q)_\infty}{(c;\ q)_\infty(z;\ q)_\infty}. $$

Letting $M \to \infty$, we get the Ramanujan ${}_1\psi_1$-bilateral series identity

$$ {}_1\psi_1\left[\begin{matrix} a \\ c \end{matrix} \,\middle|\, q; z\right] = \sum_{k=-\infty}^{+\infty} \frac{(a;q)_k}{(c;q)_k} z^k = \left[\begin{matrix} az,\ q/az,\ q,\ c/a \\ z,\ c/az,\ c,\ q/a \end{matrix} \,\middle|\, q\right]_\infty,\ |c/a| < |z| < 1 $$

where the convergence condition is figured out by analytical continuation.

In fact, the series along the positive direction

$$ \sum_{k=0}^{+\infty} \frac{(a;\ q)_k}{(c;\ q)_k} z^k $$

converges when $|z| < 1$. While the series along the negative direction

$$ \sum_{k=1}^{+\infty} \frac{(a;\ q)_{-k}}{(c;\ q)_{-k}} z^{-k} = \sum_{k=1}^{+\infty} \frac{(q/c;\ q)_k}{(q/a;\ q)_k} \Big(\frac{c}{az}\Big)^k $$

converges when $|z| > |c/a|$. □

### F3. Bailey's bilateral ${}_6\psi_6$-series identity

For complex parameters $a, b, c, d, e$ satisfying the condition $\big|qa^2/bcde\big| < 1$, there holds Bailey's very well-poised non-terminating bilateral series identity (cf. [**56**, §7.1]):

$$ {}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \,\middle|\, q; \frac{qa^2}{bcde}\right] \tag{F3.1a} $$

$$ = \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \,\middle|\, q\right]_\infty. \tag{F3.1b} $$

Here we reproduce a recent proof provided by Schlosser (2003).

**F3.1. Lemma.** For three complex parameters $a$, $b$ and $d$ with $\big|qa/bd\big| < 1$, there holds the following summation formula:

$$\left(\frac{a}{b}\right)^n \left[\begin{matrix} q, & b \\ a, & qa/b \end{matrix}\,\middle|\, q\right]_n = \left[\begin{matrix} q, & q/a, & qd/b, & qa/bd \\ q/b, & qa/b, & q/d, & qd/a \end{matrix}\,\middle|\, q\right]_\infty$$
$$\times\, {}_6\phi_5\left[\begin{matrix} d/a,\, q\sqrt{d/a}, -q\sqrt{d/a},\; b/a,\; q^{-n}d/a, \quad q^n d \\ \sqrt{d/a}, \quad -\sqrt{d/a}, qd/b, \quad q^{1+n}, \quad q^{1-n}/a \end{matrix}\,\middle|\, q;\, \frac{qa}{bd}\right].$$

Proof. According to the $q$-Dixon-Dougall formula, we have

$$ {}_6\phi_5\left[\begin{matrix} d/a,\, q\sqrt{d/a}, -q\sqrt{d/a},\; b/a,\; q^{-n}d/a, \quad q^n d \\ \sqrt{d/a}, \quad -\sqrt{d/a}, qd/b, \quad q^{1+n}, \quad q^{1-n}/a \end{matrix}\,\middle|\, q;\, \frac{qa}{bd}\right]$$
$$= \left[\begin{matrix} q^{1+n}a/b,\, q^{1-n}/b,\, q/d,\, qd/a \\ q^{1+n},\, q^{1-n}/a,\, qd/b,\, qa/bd \end{matrix}\,\middle|\, q\right]_\infty .$$

Separating the factorials dependent in $n$:

$$\frac{(q^{1+n}a/b;\, q)_\infty}{(q^{1+n};\, q)_\infty} = \frac{(q;\, q)_n}{(qa/b;\, q)_n} \times \frac{(qa/b;\, q)_\infty}{(q;\, q)_\infty}$$
$$\frac{(q^{1-n}/b;\, q)_\infty}{(q^{1-n}/a;\, q)_\infty} = \left(\frac{a}{b}\right)^n \frac{(b;\, q)_n}{(a;\, q)_n} \times \frac{(q/b;\, q)_\infty}{(q/a;\, q)_\infty}$$

we can restate the last series as

$$ {}_6\phi_5\left[\begin{matrix} d/a,\, q\sqrt{d/a}, -q\sqrt{d/a},\; b/a,\; q^{-n}d/a, \quad q^n d \\ \sqrt{d/a}, \quad -\sqrt{d/a}, qd/b, \quad q^{1+n}, \quad q^{1-n}/a \end{matrix}\,\middle|\, q;\, \frac{qa}{bd}\right]$$
$$= \left[\begin{matrix} q/b,\, q/d,\, qa/b,\, qd/a \\ q,\, qd/b,\, q/a,\, qa/bd \end{matrix}\,\middle|\, q\right]_\infty \times \left[\begin{matrix} b, & q \\ qa/b, & a \end{matrix}\,\middle|\, q\right]_n \left(\frac{a}{b}\right)^n$$

which is equivalent to the formula displayed in Lemma F3.1. □

**F3.2.** Now we are ready to prove Bailey's very well-poised non-terminating bilateral ${}_6\psi_6$-series identity.

Recalling the definition of bilateral series

$$\text{Eq(F3.1a)} = \sum_{n=-\infty}^{+\infty} \frac{1-q^{2n}a}{1-a} \left[\begin{matrix} b, & c, & d, & e \\ qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\,\middle|\, q\right]_n \left(\frac{qa^2}{bcde}\right)^n$$

and then replacing the factorial faction related to $b$ and $d$ by Lemma F3.1

$$\left(\frac{a}{b}\right)^n \left[\begin{matrix} b, & d \\ qa/b, & qa/d \end{matrix}\,\middle|\, q\right]_n = \left[\begin{matrix} a, & d \\ q, & qa/d \end{matrix}\,\middle|\, q\right]_n \left[\begin{matrix} q,\, q/a,\, qd/b,\, qa/bd \\ q/b,\, qa/b,\, q/d,\, qd/a \end{matrix}\,\middle|\, q\right]_\infty$$
$$\times\, {}_6\phi_5\left[\begin{matrix} d/a,\, q\sqrt{d/a}, -q\sqrt{d/a},\; b/a,\; q^{-n}d/a, \quad q^n d \\ \sqrt{d/a}, \quad -\sqrt{d/a}, qd/b, \quad q^{1+n}, \quad q^{1-n}/a \end{matrix}\,\middle|\, q;\, \frac{qa}{bd}\right]$$

we can express the ${}_6\psi_6$-series as the following double sum:

$$\begin{aligned}
\mathrm{Eq(F3.1a)} \;=\; & \left[\begin{matrix} q, & q/a, & qd/b, & qa/bd \\ q/b, & qa/b, & q/d, & qd/a \end{matrix}\,\middle|\, q\right]_\infty \\
\times & \sum_{n=-\infty}^{+\infty} \frac{1-q^{2n}a}{1-a} \left[\begin{matrix} a, & c, & d, & e \\ q, & qa/c, & qa/d, & qa/e \end{matrix}\,\middle|\, q\right]_n \left(\frac{qa}{cde}\right)^n \\
\times & \sum_{k=0}^{\infty} \frac{1-q^{2k}d/a}{1-d/a} \left[\begin{matrix} d/a, & b/a, & q^{-n}d/a, & q^n d \\ q, & qd/b, & q^{1+n}, & q^{1-n}/a \end{matrix}\,\middle|\, q\right]_k \left(\frac{qa}{bd}\right)^k.
\end{aligned}$$

Interchanging the summation order and then combining the following factorial fractions

$$\begin{aligned}
\frac{(d;q)_n}{(q;q)_n} \times \frac{(q^n d;q)_k}{(q^{1+n};q)_k} &= \frac{(d;q)_{n+k}}{(q;q)_{n+k}} \\
\frac{(a;q)_n}{(qa/d;q)_n} \times \frac{(q^{-n}d/a;q)_k}{(q^{1-n}/a;q)_k} &= \left(\frac{d}{q}\right)^k \frac{(a;q)_{n-k}}{(qa/d;q)_{n-k}}
\end{aligned}$$

we can further reformulate the equation (F3.1a) as follows:

$$\begin{aligned}
\mathrm{Eq(F3.1a)} \;=\; & \left[\begin{matrix} q, & q/a, & qd/b, & qa/bd \\ q/b, & qa/b, & q/d, & qd/a \end{matrix}\,\middle|\, q\right]_\infty \\
\times & \sum_{k=0}^{\infty} \frac{1-q^{2k}d/a}{1-d/a} \left[\begin{matrix} d/a, b/a \\ q, qd/b \end{matrix}\,\middle|\, q\right]_k \left(\frac{a}{b}\right)^k \\
\times & \sum_{n=-\infty}^{+\infty} \frac{1-q^{2n}a}{1-a} \left[\begin{matrix} c, & e \\ qa/c, & qa/e \end{matrix}\,\middle|\, q\right]_n \\
\times & \frac{(d;q)_{n+k}}{(q;q)_{n+k}} \frac{(a;q)_{n-k}}{(qa/d;q)_{n-k}} \left(\frac{qa}{cde}\right)^n.
\end{aligned}$$

**F3.3.** The last sum with respect to $n$ begins in effect with $n=-k$ because the shifted factorial $1/(q;q)_{n+k}$ is equal to zero when $n<-k$. Indicate with $\Omega$ the last sum. Therefore it can be reformulated through $j:=n+k$ as follows:

$$\Omega \;=\; \sum_{j=0}^{+\infty} \frac{1-q^{2j-2k}a}{1-a} \left[\begin{matrix} c,\ e \\ qa/c,\ qa/e \end{matrix}\,\middle|\, q\right]_{j-k} \frac{(d;q)_j}{(q;q)_j} \frac{(a;q)_{j-2k}}{(qa/d;q)_{j-2k}} \left(\frac{qa}{cde}\right)^{j-k}.$$

By means of two relations

$$\begin{aligned}
\frac{(a;q)_{j-2k}}{(qa/d;q)_{j-2k}} &= \frac{(a;q)_{-2k}}{(qa/d;q)_{-2k}} \frac{(q^{-2k}a;q)_j}{(q^{1-2k}a/d;q)_j} \\
\left[\begin{matrix} c,\ e \\ qa/c,\ qa/e \end{matrix}\,\middle|\, q\right]_{j-k} &= \left[\begin{matrix} c,\ e \\ qa/c,\ qa/e \end{matrix}\,\middle|\, q\right]_{-k} \left[\begin{matrix} q^{-k}c,\ q^{-k}e \\ q^{1-k}a/c,\ q^{1-k}a/e \end{matrix}\,\middle|\, q\right]_j
\end{aligned}$$

we can express $\Omega$ in terms of the $q$-hypergeometric series:

$$\begin{aligned}\Omega &= \frac{1-q^{-2k}a}{1-a}\frac{(a;q)_{-2k}}{(qa/d;q)_{-2k}}\left[\begin{matrix}c,\ e\\ qa/c,\ qa/e\end{matrix}\,\Big|\, q\right]_{-k}\Big(\frac{cde}{qa}\Big)^k\\ &\times\sum_{j=0}^{+\infty}\frac{1-q^{2j-2k}a}{1-q^{-2k}a}\left[\begin{matrix}q^{-k}c, & q^{-k}e, & d, & q^{-2k}a\\ q^{1-k}a/c, & q^{1-k}a/e, & q, & q^{1-2k}a/d\end{matrix}\,\Big|\, q\right]_j\Big(\frac{qa}{cde}\Big)^j\\ &= \frac{(qa;q)_{-2k}}{(qa/d;q)_{-2k}}\left[\begin{matrix}c,\ e\\ qa/c,\ qa/e\end{matrix}\,\Big|\, q\right]_{-k}\Big(\frac{cde}{qa}\Big)^k\\ &\times\ {}_6\phi_5\left[\begin{matrix}q^{-2k}a,\ q^{1-k}\sqrt{a},\ -q^{1-k}\sqrt{a},\ q^{-k}c,\ q^{-k}e,\ d\\ q^{-k}\sqrt{a},\ -q^{1-k}\sqrt{a},\ q^{1-k}a/c,\ q^{1-k}a/e,\ q^{1-2k}a/d\end{matrix}\,\Big|\, q;\frac{qa}{cde}\right].\end{aligned}$$

When $|qa/cde| < 1$, the last series can be evaluated by $q$-Dixon-Dougall formula (E7.4a-E7.4b) as follows:

$$\begin{aligned}&{}_6\phi_5\left[\begin{matrix}q^{-2k}a,\ q^{1-k}\sqrt{a},\ -q^{1-k}\sqrt{a},\ q^{-k}c,\ q^{-k}e,\ d\\ q^{-k}\sqrt{a},\ -q^{1-k}\sqrt{a},\ q^{1-k}a/c,\ q^{1-k}a/e,\ q^{1-2k}a/d\end{matrix}\,\Big|\, q;\frac{qa}{cde}\right]\\ &= \left[\begin{matrix}q^{1-2k}a, & qa/ce, & q^{1-k}a/cd, & q^{1-k}a/de\\ q^{1-k}a/c, & q^{1-k}a/e, & q^{1-2k}a/d, & qa/cde\end{matrix}\,\Big|\, q\right]_\infty\\ &= \frac{(qa/d;q)_{-2k}}{(qa;q)_{-2k}}\left[\begin{matrix}qa/c, & qa/e\\ qa/cd, & qa/de\end{matrix}\,\Big|\, q\right]_{-k}\left[\begin{matrix}qa,\ qa/cd,\ qa/ce,\ qa/de\\ qa/c,\ qa/d,\ qa/e,\ qa/cde\end{matrix}\,\Big|\, q\right]_\infty\end{aligned}$$

which leads us consequently to the closed form for $\Omega$:

$$\begin{aligned}\Omega &= \Big(\frac{cde}{qa}\Big)^k\left[\begin{matrix}c, & e\\ qa/cd, & qa/de\end{matrix}\,\Big|\, q\right]_{-k}\left[\begin{matrix}qa,\ qa/cd,\ qa/ce,\ qa/de\\ qa/c,\ qa/d,\ qa/e,\ qa/cde\end{matrix}\,\Big|\, q\right]_\infty\\ &= \Big(\frac{cde}{qa}\Big)^k\left[\begin{matrix}q^{1-k}a/cd, & q^{1-k}a/de\\ q^{-k}c, & q^{-k}e\end{matrix}\,\Big|\, q\right]_k\left[\begin{matrix}qa,\ qa/cd,\ qa/ce,\ qa/de\\ qa/c,\ qa/d,\ qa/e,\ qa/cde\end{matrix}\,\Big|\, q\right]_\infty\\ &= \Big(\frac{qa}{cde}\Big)^k\left[\begin{matrix}cd/a, & de/a\\ q/c, & q/e\end{matrix}\,\Big|\, q\right]_k\left[\begin{matrix}qa,\ qa/cd,\ qa/ce,\ qa/de\\ qa/c,\ qa/d,\ qa/e,\ qa/cde\end{matrix}\,\Big|\, q\right]_\infty.\end{aligned}$$

**F3.4.** Summing up, we can state the bilateral ${}_6\psi_6$-series displayed in (F3.1a) in terms of another $q$-series:

$$\begin{aligned}\mathrm{Eq(F3.1a)} &= \left[\begin{matrix}q, qa, q/a, qd/b, qa/bd, qa/cd, qa/ce, qa/de\\ q/b, q/d, qd/a, qa/b, qa/c, qa/d, qa/e, qa/cde\end{matrix}\,\Big|\, q\right]_\infty\\ &\times\sum_{k=0}^{\infty}\frac{1-q^{2k}d/a}{1-d/a}\left[\begin{matrix}d/a, & b/a, & cd/a, & de/a\\ q, & qd/b, & q/c, & q/e\end{matrix}\,\Big|\, q\right]_k\Big(\frac{qa^2}{bcde}\Big)^k.\end{aligned}$$

The last series can be again evaluated by the very well poised non-terminating ${}_6\phi_5$-summation formula as follows:

$$
\begin{aligned}
&{}_6\phi_5\left[\begin{matrix} d/a,\ q\sqrt{d/a},\ -q\sqrt{d/a},\ b/a,\ cd/a,\ de/a \\ \sqrt{d/a},\ -\sqrt{d/a},\ qd/b,\ q/c,\ q/e \end{matrix}\,\Big|\, q;\ \frac{qa^2}{bcde}\right] \\
&= \left[\begin{matrix} qd/a,\ qa/bc,\ qa/be,\ qa/cde \\ qd/b,\ q/c,\ q/e,\ qa^2/bcde \end{matrix}\,\Big|\, q\right]_\infty .
\end{aligned}
$$

We therefore have established the following

$$
\begin{aligned}
\mathrm{Eq(F3.1a)} &= \left[\begin{matrix} q, qa, q/a, qd/b, qa/bd, qa/cd, qa/ce, qa/de \\ qd/a, q/b, q/d, qa/b, qa/c, qa/d, qa/e, qa/cde \end{matrix}\,\Big|\, q\right]_\infty \\
&\times \left[\begin{matrix} qd/a,\ qa/bc,\ qa/be,\ qa/cde \\ qd/b,\ q/c,\ q/e,\ qa^2/bcde \end{matrix}\,\Big|\, q\right]_\infty \\
&= \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix}\,\Big|\, q\right]_\infty
\end{aligned}
$$

which corresponds exactly to (F3.1b).

This completes the proof of Bailey's very-well-poised ${}_6\psi_6$-series identity. □

**F3.5. The quintuple product identity.** The identity displayed in (F1.5a) and (F1.5b) is a limiting case of Bailey's very-well-poised ${}_6\psi_6$-series identity.

In fact, letting $b = -\sqrt{a}$ and $c, d, e \to \infty$, we can state (F3.1a-F3.1b) as

$$
\begin{aligned}
\sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}}\left\{1 - q^n\sqrt{a}\right\}\left(qa^{3/2}\right)^n &= (1-\sqrt{a})\frac{[q,\ qa,\ q/a; q]_\infty}{[-q\sqrt{a},\ -q/\sqrt{a}; q]_\infty} \\
&= (q;q)_\infty \frac{\left[qa,\ a,\ q/a,\ q^2/a; q^2\right]_\infty}{[-\sqrt{a},\ -q/\sqrt{a}; q]_\infty} \\
&= \left[q,\ \sqrt{a},\ q/\sqrt{a}; q\right]_\infty \left[qa,\ q/a; q^2\right]_\infty .
\end{aligned}
$$

Replacing $a$ by $z^2$, this becomes the quintuple product identity displayed in (F1.5a) and (F1.5b). □

## F4. Bilateral $q$-analogue of Dixon's theorem

For the cubic-sums of binomial coefficients, there is Dixon's well-known theorem, which states that

$$
\sum_{k=-n}^{n+\delta} (-1)^k \binom{2n+\delta}{n+k}^3 = \begin{cases} \binom{3n}{n,n,n}, & \delta = 0 \\ 0, & \delta = 1. \end{cases} \tag{F4.1}
$$

Its terminating $q$-analogue was first found by Jackson [**44**] and subsequently generalized by Bailey [**8**]. Following Bailey's derivation, we will establish two general well-poised bilateral series identities:

$$ {}_4\psi_4\left[\begin{matrix} qw, & b, & c, & d \\ w, & q/b, & q/c, & q/d \end{matrix} \,\Big|\, q; \frac{q}{bcd}\right] = \left[\begin{matrix} q,\ q/bc,\ q/bd,\ q/cd \\ q/b,\ q/c,\ q/d,\ q/bcd \end{matrix} \,\Big|\, q\right]_\infty \tag{F4.2} $$

$$ {}_5\psi_5\left[\begin{matrix} qu, & qv, & b, & c, & d \\ u, & v, & 1/b, & 1/c, & 1/d \end{matrix} \,\Big|\, q; \frac{q^{-1}}{bcd}\right] = \left[\begin{matrix} q,\ 1/bc,\ 1/bd,\ 1/cd \\ q/b, q/c, q/d, q^{-1}/bcd \end{matrix} \,\Big|\, q\right]_\infty \tag{F4.3} $$

$$ \times \quad \frac{1-1/quv}{(1-1/u)\,(1-1/v)}. \tag{F4.4} $$

Further bilateral identities of this type and applications can be found in Chu [**26**], where a systematic treatment of basic almost-poised hypergeometric series has been presented.

**F4.1. Proof of** (F4.2)**.** Recall the non-terminating very-well-poised ${}_6\phi_5$ summation identity (E7.4a-E7.4b):

$$ {}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d \end{matrix} \,\Big|\, q; \frac{qa}{bcd}\right] $$
$$ = \left[\begin{matrix} qa,\ qa/bc,\ qa/bd,\ qa/cd \\ qa/b,\ qa/c,\ qa/d,\ qa/bcd \end{matrix} \,\Big|\, q\right]_\infty, \qquad \left|\frac{qa}{bcd}\right| < 1. $$

Letting $a \to 1$, we can restate the result as

$$ \begin{aligned} \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix} \,\Big|\, q\right]_\infty &= 1 + \sum_{k=1}^{+\infty} \{1+q^k\} \left[\begin{matrix} b,\ c,\ d \\ q/b, q/c, q/d \end{matrix} \,\Big|\, q\right]_k \left(\frac{q}{bcd}\right)^k \\ &= 1 + \sum_{k=1}^{+\infty} \left[\begin{matrix} b,\ c,\ d \\ q/b, q/c, q/d \end{matrix} \,\Big|\, q\right]_k \left(\frac{q}{bcd}\right)^k \\ &+ \sum_{k=1}^{+\infty} \left[\begin{matrix} b,\ c,\ d \\ q/b, q/c, q/d \end{matrix} \,\Big|\, q\right]_{-k} \left(\frac{q}{bcd}\right)^{-k} \\ &= \sum_{k=-\infty}^{+\infty} \left[\begin{matrix} b,\ c,\ d \\ q/b, q/c, q/d \end{matrix} \,\Big|\, q\right]_k \left(\frac{q}{bcd}\right)^k. \end{aligned} $$

In terms of bilateral series, this becomes the following well-poised summation identity

$$ {}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} \,\Big|\, q; \frac{q}{bcd}\right] = \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix} \,\Big|\, q\right]_\infty \tag{F4.5} $$

whose reversal reads as

$$ {}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} \,\Big|\, q; \frac{q^2}{bcd}\right] = \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix} \,\Big|\, q\right]_\infty. \tag{F4.6} $$

In view of the fact that

$$\frac{(qw;q)_k}{(w;q)_k} = \frac{1-wq^k}{1-w} = \frac{1}{1-w} - \frac{w}{1-w}q^k$$

the linear combination of (F4.5) and (F4.6) leads us to bilateral identity (F4.2) with an extra $w$-parameter:

$${}_4\psi_4\left[\begin{matrix} qw, & b, & c, & d \\ w, & q/b, & q/c, & q/d \end{matrix}\,\middle|\, q; \frac{q}{bcd}\right] = \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix}\,\middle|\, q\right]_\infty.$$

**F4.2. Proof of (F4.3-F4.4).** Instead, if taking $a = q$ in non-terminating very-well-poised ${}_6\phi_5$ summation identity (E7.4a-E7.4b) and then multiplying both sides by $1-q$, we get

$$\begin{aligned}
\left[\begin{matrix} q, q^2/bc, q^2/bd, q^2/cd \\ q^2/b, q^2/c, q^2/d, q^2/bcd \end{matrix}\,\middle|\, q\right]_\infty &= \sum_{k=0}^{+\infty}\{1-q^{1+2k}\}\left[\begin{matrix} b,\ c,\ d \\ q^2/b, q^2/c, q^2/d \end{matrix}\,\middle|\, q\right]_k\left(\frac{q^2}{bcd}\right)^k \\
&= \sum_{k=0}^{+\infty}\left[\begin{matrix} b,\ c,\ d \\ q^2/b, q^2/c, q^2/d \end{matrix}\,\middle|\, q\right]_k\left(\frac{q^2}{bcd}\right)^k \\
&\quad - \sum_{k=1}^{+\infty}\left[\begin{matrix} b,\ c,\ d \\ q^2/b, q^2/c, q^2/d \end{matrix}\,\middle|\, q\right]_{-k}\left(\frac{q^2}{bcd}\right)^{-k} \\
&= \sum_{k=-\infty}^{+\infty}\left[\begin{matrix} b,\ c,\ d \\ q^2/b, q^2/c, q^2/d \end{matrix}\,\middle|\, q\right]_k\left(\frac{q^2}{bcd}\right)^k
\end{aligned}$$

where we have performed the summation index substitution $k \to k-1$ for the second sum.

In terms of bilateral series, this reads as the following well-poised summation identity

$${}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q^2/b, & q^2/c, & q^2/d \end{matrix}\,\middle|\, q; \frac{q^2}{bcd}\right] = \left[\begin{matrix} q, q^2/bc, q^2/bd, q^2/cd \\ q^2/b, q^2/c, q^2/d, q^2/bcd \end{matrix}\,\middle|\, q\right]_\infty. \quad \text{(F4.7)}$$

Shifting the summation index by $k \to k-1$ and then performing parameter replacement $b \to qb$, $c \to qc$ and $d \to qd$, we can derive the following equivalent result:

$${}_3\psi_3\left[\begin{matrix} b, & c, & d \\ 1/b, & 1/c, & 1/d \end{matrix}\,\middle|\, q; \frac{q^{-1}}{bcd}\right] = \frac{-1}{q}\left[\begin{matrix} q, 1/bc, 1/bd, 1/cd \\ 1/b, 1/c, 1/d, q^{-1}/bcd \end{matrix}\,\middle|\, q\right]_\infty. \quad \text{(F4.8)}$$

Its reversal can be stated, after some little modification, as

$${}_3\psi_3\left[\begin{matrix} b, & c, & d \\ 1/b, & 1/c, & 1/d \end{matrix}\,\middle|\, q; \frac{q}{bcd}\right] = \left[\begin{matrix} q, 1/bc, 1/bd, 1/cd \\ 1/b, 1/c, 1/d, q^{-1}/bcd \end{matrix}\,\middle|\, q\right]_\infty. \quad \text{(F4.9)}$$

Note further that

$$ {}_3\psi_3\left[\begin{matrix} b, & c, & d \\ 1/b, & 1/c, & 1/d \end{matrix} \,\Big|\, q; \frac{1}{bcd}\right] = 0 \tag{F4.10} $$

which is the case $\kappa = 1$ of the following general statement:

$$ {}_{1+2\kappa}\psi_{2\kappa+1}\left[\begin{matrix} c_1, & c_2, \cdots, & c_{1+2\kappa} \\ 1/c_1, & 1/c_2, \cdots, & 1/c_{1+2\kappa} \end{matrix} \,\Big|\, q; \frac{1}{c_1 c_2 \cdots c_{1+2\kappa}}\right] = 0. \tag{F4.11} $$

In fact, denote by $\Theta$ the bilateral $\psi$-series on the left hand side. Its reversal with the summation index shifted by $k \to k-1$ can be stated as

$$ \begin{aligned} \Theta &= {}_{1+2\kappa}\psi_{2\kappa+1}\left[\begin{matrix} qc_1, & qc_2, \cdots, & qc_{1+2n} \\ q/c_1, & q/c_2, \cdots, & q/c_{1+2\kappa} \end{matrix} \,\Big|\, q; \prod_{\iota=1}^{1+2\kappa} \frac{1}{c_\iota}\right] \\ &= {}_{1+2\kappa}\psi_{2\kappa+1}\left[\begin{matrix} c_1, & c_2, \cdots, & c_{1+2\kappa} \\ 1/c_1, & 1/c_2, \cdots, & 1/c_{1+2\kappa} \end{matrix} \,\Big|\, q; \prod_{\iota=1}^{1+2\kappa} \frac{1}{c_\iota}\right] \\ &\times \prod_{\iota=1}^{1+2\kappa} \frac{1-1/c_\iota}{1-c_\iota} \left\{\prod_{\iota=1}^{1+2\kappa} \frac{1}{c_\iota}\right\}^{-1}. \end{aligned} $$

Simplifying the last factor-product, we find that

$$ \Theta = (-1)^{1+2\kappa}\Theta = 0 $$

which is exactly (F4.11).

By means of three terms relation

$$ \begin{aligned} \frac{(qu;q)_k}{(u;q)_k}\frac{(qv;q)_k}{(v;q)_k} &= \frac{1-uq^k}{1-u}\frac{1-vq^k}{1-v} \\ = \frac{1}{(1-u)(1-v)} &- \frac{u+v}{(1-u)(1-v)}q^k + \frac{uv}{(1-u)(1-v)}q^{2k} \end{aligned} $$

we can establish from the combination of (F4.8), (F4.9) and (F4.10) the following general identity with two extra free-parameters:

$$ \begin{aligned} {}_5\psi_5\left[\begin{matrix} qu, & qv, & b, & c, & d \\ u, & v, & 1/b, & 1/c, & 1/d \end{matrix} \,\Big|\, q; \frac{q^{-1}}{bcd}\right] &= \frac{1-1/quv}{(1-1/u)\,(1-1/v)} \\ &\times \left[\begin{matrix} q,\ 1/bc,\ 1/bd,\ 1/cd \\ q/b, q/c, q/d, q^{-1}/bcd \end{matrix} \,\Big|\, q\right]_\infty \end{aligned} $$

which is the bilateral identity stated in (F4.3-F4.4).

**F4.3. $q$-Analogue of Dixon's theorem.** Putting $b = c = d = q^{-n}$ in (F4.5), we can state the result in terms of $q$-binomial sum

$$ \sum_{k=-n}^{n} (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}^3 q^{k(3k-1)/2} = \frac{(q;q)_{3n}}{(q;q)_n^3} \tag{F4.12} $$

which is the $q$-analogue of (F4.1) corresponding to $\delta = 0$.

Similarly, taking $b = c = d = q^{-n}$ in (F4.7), we can express the result as another $q$-binomial sum

$$\sum_{k=-n}^{1+n} (-1)^k \begin{bmatrix} 1+2n \\ n+k \end{bmatrix}^3 q^{k(3k-1)/2} = \frac{(q;q)_{1+3n}}{(q;q)_n^3} \tag{F4.13}$$

which is the $q$-analogue of (F4.1) corresponding to $\delta = 1$.

## F5. Partial fraction decomposition method

For terminating hypergeometric series with integral differences between numerator parameters and denominator parameters, an interesting identity was first discovered by Minton (1970) and slightly extended by Karlsson (1971). Their $q$-analogue was then established by Gasper (1984). The bilateral non-terminating forms of these formulae have been found by Chu (1994), who has further generalized these results into basic bilateral very well-poised hypergeometric summation identities.

The purpose of this section is to present the formulae of Chu-Karlsson-Minton for basic hypergeometric series with integral parameter differences. For the terminating cases and their dual formulae, refer to Chu (1998).

**F5.1. Theorem.** For complex numbers $a$, $c$, $d$, and $\{x_k\}_{k=1}^{\ell}$ with $\lambda$ and $\{n_k\}_{k=1}^{\ell}$ being nonnegative integers and $n = \sum_{k=1}^{\ell} n_k$, there holds the bilateral series identity with integral parameter differences

$$_{\ell+2}\psi_{2+\ell}\left[\begin{matrix} a, & d, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell \\ c, & qd, & x_1, & x_2, & \cdots, & x_\ell \end{matrix} \,\middle|\, q; q^{1-\lambda}/a\right] \tag{F5.1a}$$

$$= \left[\begin{matrix} q, & q, & c/d, & qd/a \\ q/a, & c, & qd, & q/d \end{matrix} \,\middle|\, q\right]_\infty d^\lambda \prod_{k=1}^{\ell} \frac{(x_k/d;q)_{n_k}}{(x_k;q)_{n_k}} \tag{F5.1b}$$

provided that $(|q/a| < |q^\lambda| < |q^n/c|)$. This identity contains, in particular, the results due to Gasper [**33**] on unilateral $q$-series as special cases.

Proof. Consider the function of complex variable $z$ defined in terms of $q$-shifted factorials by

$$f(z) = z^{1+\lambda} \left[\begin{matrix} q, \; q, \; c/z, \; qz/a \\ c, \; q/a, \; z, \; q/z \end{matrix} \,\middle|\, q\right]_\infty \prod_{k=1}^{\ell} \frac{(x_k/z;q)_{n_k}}{(x_k;q)_{n_k}}. \tag{F5.2}$$

It is not hard to verify that $f(z)$ has simple poles $z = q^j$ for $j = 0, \pm 1, \pm 2, \cdots$ with residues

$$-\Big\{q^{-1-\lambda}/a\Big\}^j \frac{(a;q)_j}{(c;q)_j} \prod_{k=1}^{\ell} \frac{(q^{n_k} x_k;q)_j}{(x_k c;q)_j}. \tag{F5.3}$$

Denote by $\delta_m$ and $\partial_m$ the circles with the center at the origin and radii $|q|^{m+1/2}$ and $|q|^{-m-1/2}$, respectively. It is clear that there is no pole of $f(z)$ passing through $\delta_m$ or $\partial_m$. Then the residue theorem (cf. [**58**, §3.1]) states that

$$\frac{f(z)}{z} + \sum_{\tau} \frac{\mathrm{Res}\,[f(z)]_{z=\tau}}{\tau\,(\tau - z)} = \frac{1}{2\pi i}\int_{\partial_m} \frac{f(t)}{t\,(t-z)}\mathrm{dt} \tag{F5.4a}$$

$$- \frac{1}{2\pi i}\int_{\delta_m} \frac{f(t)}{t\,(t-z)}\mathrm{dt} \tag{F5.4b}$$

where the summation runs over all poles $\{\tau\}$ of $f(z)$ between contours $\delta_m$ and $\partial_m$. For sufficiently large $m$, we can estimate that

$$\int_{\delta_m} \frac{f(t)}{t(t-z)}\mathrm{dt} = \mathcal{O}\left\{q^{(1+\lambda-n)m} \frac{(q^{-m-1/2}c;q)_m}{(q^{-m+1/2};q)_m}\right\} \tag{F5.5a}$$

$$= \mathcal{O}\big(|q^{\lambda-n}c|^m\big) \tag{F5.5b}$$

$$\int_{\partial_m} \frac{F(t)}{t(t-z)}\mathrm{dt} = \mathcal{O}\left\{q^{-m\lambda} \frac{(q^{-m+1/2}/a;q)_m}{(q^{-m-1/2};q)_m}\right\} \tag{F5.5c}$$

$$= \mathcal{O}\big(|q^{1-\lambda}/a|^m\big). \tag{F5.5d}$$

When $|q/a| < |q^\lambda| < |q^n/c|$, both integrals displayed in (F5.4a-F5.4b) tend to zero as $m \to \infty$. Therefore we can express (F5.4a-F5.4b) as a bilateral summation identity:

$$\begin{aligned} &{}_{\ell+2}\psi_{2+\ell}\left[\begin{matrix} a, & z, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell \\ c, & qz, & x_1, & x_2, & \cdots, & x_\ell \end{matrix} \,\Big|\, q;\ q^{1-\lambda}/a\right] \\ &= \left[\begin{matrix} q, & q, & c/z, & qz/a \\ q/a, & c, & qz, & q/z \end{matrix} \,\Big|\, q\right]_\infty z^\lambda \prod_{k=1}^{\ell} \frac{(x_k/z;q)_{n_k}}{(x_k;q)_{n_k}} \end{aligned}$$

whose convergent condition coincides with $|q/a| < |q^\lambda| < |q^n/c|$. Rewriting the last identity with $z$ being replaced by $d$, we confirm the formula displayed in (F5.1a-F5.1b). □

Two interesting special cases of (F5.1a-F5.1b) are worth to mention:

$${}_3\psi_3\left[\begin{matrix} a, & qx, & z \\ c, & x, & qz \end{matrix} \,\Big|\, q; q/a\right] = \frac{1-x/z}{1-x}\left[\begin{matrix} q,\ q,\ c/z,\ qz/a \\ c,\ q/a,\ qz,\ q/z \end{matrix} \,\Big|\, q\right] \tag{F5.6a}$$

$${}_3\psi_3\left[\begin{matrix} a, & qx, & z \\ c, & x, & qz \end{matrix} \,\Big|\, q; 1/a\right] = \frac{1-z/x}{1-1/x}\left[\begin{matrix} q,\ q,\ c/z,\ qz/a \\ c,\ q/a,\ qz,\ q/z \end{matrix} \,\Big|\, q\right] \tag{F5.6b}$$

where their convergence conditions are given respectively by $|c| < |q| < |a|$ and $|c| < 1 < |a|$.

**F5.2. Corollary.** Replacing each parameter by its $q$-exponential in equality (F5.1a-F5.1b) and then letting $q \to 1$, we can state the limit as bilateral hypergeometric summation formula ($n + \Re(a-c) < 0$):

$$ {}_{\ell+2}H_{2+\ell}\left[\begin{matrix} a, & d, & x_1+n_1, & x_2+n_2, & \cdots, & x_\ell+n_\ell \\ c, & 1+d, & x_1, & x_2, & \cdots, & x_\ell \end{matrix}\,\middle|\, 1\right] \tag{F5.7a} $$

$$ = \Gamma\left[\begin{matrix} 1-a,\ c \\ 1-a+d,\ c-d \end{matrix}\right] \frac{\pi d}{\sin \pi d} \prod_{k=1}^{\ell} \frac{(x_k-d)_{n_k}}{(x_k)_{n_k}}. \tag{F5.7b} $$

This generalizes the identity due to Karlsson [**46**]

$$ {}_{\ell+2}F_{1+\ell}\left[\begin{matrix} a, & d, & x_1+n_1, & x_2+n_2, & \cdots, & x_\ell+n_\ell \\ & 1+d, & x_1, & x_2, & \cdots, & x_\ell \end{matrix}\,\middle|\, 1\right] \tag{F5.8a} $$

$$ = \frac{\Gamma(1-a)\Gamma(1+d)}{\Gamma(1-a+d)} \prod_{k=1}^{\ell} \frac{(x_k-d)_{n_k}}{(x_k)_{n_k}}, \quad (n+\Re(a)<1) \tag{F5.8b} $$

which is a non-terminating extension of an earlier result due to Minton [**50**].

**F5.3. Transformation of bilateral series into unilateral series.** For $|w| < 1$, recalling the $q$-Gauss summation formula

$$ \frac{(z;q)_k}{(wz;q)_k} = \frac{1}{1-zq^k}\left[\begin{matrix} z, w \\ q, wz \end{matrix}\,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix} q/w, q^k z \\ q^{1+k} z \end{matrix}\,\middle|\, q; w\right] $$

we can consider the series composition ($|q/a| < |q^\lambda| < |q^{1+n}/cw|$):

$$ \begin{aligned} & {}_{\ell+2}\psi_{2+\ell}\left[\begin{matrix} a, & z, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell \\ c, & wz, & x_1, & x_2, & \cdots, & x_\ell \end{matrix}\,\middle|\, q;\ q^{1-\lambda}/a\right] \times \left[\begin{matrix} q, wz \\ z, w \end{matrix}\,\middle|\, q\right]_\infty \\ &= \sum_{k=-\infty}^{+\infty} \frac{\left(q^{1-\lambda}/a\right)^k}{1-zq^k}\left[\begin{matrix} a, & \{q^{n_\kappa}x_\kappa\} \\ c, & \{x_\kappa\} \end{matrix}\,\middle|\, q\right]_k {}_2\phi_1\left[\begin{matrix} q/w, q^k z \\ q^{1+k} z \end{matrix}\,\middle|\, q; w\right] \\ &= \sum_{i=0}^{+\infty} \frac{w^i}{1-zq^i} \frac{(q/w;q)_i}{(q;q)_i} \sum_{k=-\infty}^{+\infty} \left[\begin{matrix} a, & q^i z, & \{q^{n_\kappa}x_\kappa\} \\ c, & q^{1+i}z, & \{x_\kappa\} \end{matrix}\,\middle|\, q\right]_k \left(q^{1-\lambda}/a\right)^k. \end{aligned} $$

Under the condition $|q/a| < |q^\lambda| < |q^n/c|$, the last series can be evaluated by (F5.1a-F5.1b) as

$$\begin{aligned}
&{}_{\ell+2}\psi_{2+\ell}\left[\begin{matrix}a, & q^iz, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell\\ c, & q^{1+i}z, & x_1, & x_2, & \cdots, & x_\ell\end{matrix}\,\Big|\, q;\, q^{1-\lambda}/a\right]\\
&= \left[\begin{matrix}q, & q, & q^{-i}c/z, & q^{1+i}z/a\\ q/a, & c, & q^{1+i}z, & q^{1-i}/z\end{matrix}\,\Big|\, q\right]_\infty q^{i\lambda}z^\lambda\prod_{k=1}^{\ell}\frac{(q^{-i}x_k/z;q)_{n_k}}{(x_k;q)_{n_k}}\\
&= \left[\begin{matrix}q, & q, & c/z, & qz/a\\ q/a, & c, & qz, & q/z\end{matrix}\,\Big|\, q\right]_\infty z^\lambda\prod_{k=1}^{\ell}\frac{(x_k/z;q)_{n_k}}{(x_k;q)_{n_k}}\\
&\times \left(q^{\lambda-1-n}c\right)^i\left[\begin{matrix}qz/c, qz\\ qz/a,\ z\end{matrix}\,\Big|\, q\right]_i\prod_{k=1}^{\ell}\frac{(qz/x_k;q)_i}{(q^{1-n_k}z/x_k;q)_i}.
\end{aligned}$$

Substituting this result into the last expression, we get the following transformation:

$${}_{\ell+2}\psi_{2+\ell}\left[\begin{matrix}a, & z, & q^{n_1}x_1, & q^{n_2}x_2, & \cdots, & q^{n_\ell}x_\ell\\ c, & wz, & x_1, & x_2, & \cdots, & x_\ell\end{matrix}\,\Big|\, q;\, q^{1-\lambda}/a\right] \quad \text{(F5.9a)}$$

$$= \left[\begin{matrix}q, & w, & c/z, & qz/a\\ q/a, & c, & wz, & q/z\end{matrix}\,\Big|\, q\right]_\infty z^\lambda\prod_{k=1}^{\ell}\frac{(x_k/z;q)_{n_k}}{(x_k;q)_{n_k}} \quad \text{(F5.9b)}$$

$$\times \quad {}_{\ell+2}\phi_{1+\ell}\left[\begin{matrix}q/w, & qz/c, & \{qz/x_\kappa\}\\ & qz/a, & \{q^{1-n_\kappa}z/x_\kappa\}\end{matrix}\,\Big|\, q;\, q^{\lambda-1-n}cw\right] \quad \text{(F5.9c)}$$

provided that $|q/a| < |q^\lambda| < |q^{1+n}/cw|$, which guarantees that both nonterminating series are convergent.

When $q \to 1$, we write down the transformation for ordinary hypergeometric series $(1 + n + \Re(a - c) < 1 \leq \Re(w))$:

$$\begin{aligned}
&{}_{\ell+2}H_{2+\ell}\left[\begin{matrix}a, & z, & n_1+x_1, & n_2+x_2, & \cdots, & n_\ell+x_\ell\\ c, & w+z, & x_1, & x_2, & \cdots, & x_\ell\end{matrix}\,\Big|\, 1\right]\\
&= \Gamma\left[\begin{matrix}1-a, c, w+z, 1-z\\ w, c-z, 1-a+z\end{matrix}\right]\prod_{k=1}^{\ell}\frac{(x_k-z)_{n_k}}{(x_k)_{n_k}}\\
&\times \quad {}_{\ell+2}F_{1+\ell}\left[\begin{matrix}1-w, & 1-c+z, & \{1+z-x_\kappa\}\\ & 1-a+z, & \{1+z-x_\kappa-n_\kappa\}\end{matrix}\,\Big|\, 1\right].
\end{aligned}$$

In particular, putting $n = 0$ and then evaluating the ${}_2\phi_1$-series by the Gauss summation theorem, we recover the well-known Dougall formula:

$${}_2H_2\left[\begin{matrix}a, & b\\ c, & d\end{matrix}\,\Big|\, 1\right] = \Gamma\left[\begin{matrix}1-a, 1-b, c, d, c+d-a-b-1\\ c-a, c-b, d-a, d-b\end{matrix}\right]$$

provided that $\Re(c+d-a-b) > 1$ for the convergence of the bilateral series.

**F5.4. Bilateral basic very-well-poised summation formula.** The partial decomposition method can further be applied to derive the following bilateral very-well-poised summation formula (Chu, 1998).

For complex numbers $a$, $b$, $c$, $d$, and $\{x_k,\ y_k\}_{k=1}^{\ell}$ satisfying $x_k y_k = aq^{1+n_k}$ $(k = 1, 2, \cdots, \ell)$ with $\{n_k\}_{k=1}^{\ell}$ being nonnegative integers and $n = \sum_{k=1}^{\ell} n_k$, there holds

$$ {}_{2\ell+6}\psi_{6+2\ell}\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, & b,\ c, & d,\ a/d, & \{x_k, \quad y_k\} \\ \sqrt{a},\ -\sqrt{a}, & qa/b, qa/c, & qa/d, qd, & \{qa/x_k, qa/y_k\} \end{matrix}\middle| q; \frac{q^{1-n}a}{bc}\right] \tag{F5.10a}$$

$$= \left[\begin{matrix} q,\ q,\ qa,\ q/a,\ qa/bd,\ qa/cd,\ qd/b,\ qd/c \\ qa/d,\ qd/a,\ qd,\ q/d,\ q/b,\ q/c,\ qa/b,\ qa/c \end{matrix}\middle| q\right]_\infty \tag{F5.10b}$$

$$\times \quad (a/d)^n \prod_{k=1}^{\ell} \left[\begin{matrix} qd/x_k,\ qd/y_k \\ qa/x_k,\ qa/y_k \end{matrix}\middle| q\right]_{n_k} \tag{F5.10c}$$

provided that $|q^{1-n}a/bc| < 1$ and the bilateral series is well-defined.

Proof. Consider the meromorphic function of complex variable $z$ defined by

$$F(z) \quad = \quad \left[\begin{matrix} q,\ q,\ a,\ q/a,\ qa/bz,\ qa/cz,\ qz/b,\ qz/c \\ a/z,\ qz/a,\ z,\ q/z,\ q/b,\ q/c,\ qa/b,\ qa/c \end{matrix}\middle| q\right]_\infty \tag{F5.11a}$$

$$\times \quad (a/z)^n \prod_{k=1}^{\ell} \left[\begin{matrix} qz/x_k,\ qz/y_k \\ qa/x_k,\ qa/y_k \end{matrix}\middle| q\right]_{n_k}. \tag{F5.11b}$$

We can check without difficulty that $F(z)$ satisfies the multiplicative reflection property $F(z) = F(a/z)$. It has simple poles $z = q^j$ and $z = aq^j$ $(j = 0, \pm 1, \pm 2, \cdots)$ with residues

$$\left\{\frac{qa}{bc}\right\}^j \left[\begin{matrix} b,\ c \\ qa/b, qa/c \end{matrix}\middle| q\right]_j \prod_{k=1}^{\ell} \left[\begin{matrix} x_k,\ y_k \\ qa/x_k, qa/y_k \end{matrix}\middle| q\right]_j \tag{F5.12a}$$

$$\times \quad \begin{cases} -q^{-j(1+n)} & \text{at} \quad z = q^{-j} \\ +aq^{j(1-n)} & \text{at} \quad z = aq^j. \end{cases} \tag{F5.12b}$$

Write $|a| = |q|^A$ and $A = \epsilon \pmod 1$. Denote by $\delta_m$ and $\partial_m$ the circles with the center at the origin and radii $|q|^{m+(1+\epsilon)/2}$ and $|q|^{-m+(1+\epsilon)/2}$, respectively. It is clear that there is no pole of $F(z)$ passing through $\delta_m$ or $\partial_m$. Then the residue theorem (cf. [**58**, §3.1]) states that

$$\frac{F(z)}{z} + \sum_\tau \frac{\operatorname{Res}\,[F(z)]_{z=\tau}}{\tau\,(\tau - z)} \quad = \quad \frac{1}{2\pi i}\int_{\partial_m} \frac{F(t)}{t\,(t-z)}\mathrm{dt} \tag{F5.13a}$$

$$- \quad \frac{1}{2\pi i}\int_{\delta_m} \frac{F(t)}{t\,(t-z)}\mathrm{dt} \tag{F5.13b}$$

where the summation runs over all poles $\{\tau\}$ of $F(z)$ between contours $\delta_m$ and $\partial_m$. For sufficiently large $m$, we can estimate that

$$\begin{aligned}
\int_{\delta_m}\frac{F(t)}{t(t-z)}\mathrm{dt} &= \mathcal{O}\left\{q^{-mn}\left[\begin{matrix}q^{-m+(1-\epsilon)/2}a/b,q^{-m+(1-\epsilon)/2}a/c\\ q^{-m+(1-\epsilon)/2},q^{-m-(1+\epsilon)/2}a\end{matrix}\,\Big|\, q\right]_m\right\} && \text{(F5.14a)}\\
&= \mathcal{O}\left\{(q^{1-n}a/bc)^m\right\} && \text{(F5.14b)}\\
\int_{\partial_m}\frac{F(t)}{t(t-z)}\mathrm{dt} &= \mathcal{O}\left\{q^{m-mn}\left[\begin{matrix}q^{-m+(3+\epsilon)/2}/b,q^{-m+(3+\epsilon)/2}/c\\ q^{-m+(1+\epsilon)/2},q^{-m+(3+\epsilon)/2}/a\end{matrix}\,\Big|\, q\right]_m\right\} && \text{(F5.14c)}\\
&= \mathcal{O}\left\{(q^{2-n}a/bc)^m\right\}. && \text{(F5.14d)}
\end{aligned}$$

Note first that for $m\to\infty$, both integrals displayed in (F5.13a-F5.13b) tend to zero under condition $|q^{1-n}a/bc|<1$. Write then the residue-sum displayed on the left hand side of (F5.13a) explicitly

$$\begin{aligned}
\sum_{\tau}\frac{\mathrm{Res}\,[F(z)]_{z=\tau}}{\tau\,(\tau-z)} = \sum_{j=-\infty}^{+\infty}\left\{\frac{qa}{bc}\right\}^j\left[\begin{matrix}b,\,c\\ qa/b,qa/c\end{matrix}\,\Big|\, q\right]_j\prod_{k=1}^{\ell}\left[\begin{matrix}x_k,\,y_k\\ qa/x_k,qa/y_k\end{matrix}\,\Big|\, q\right]_j\\
\times\left\{\frac{aq^{j(1-n)}}{aq^j(aq^j-z)}-\frac{q^{-j(1+n)}}{q^{-j}(q^{-j}-z)}\right\}
\end{aligned}$$

where the difference of two fractions in the last line can be simplified as

$$\frac{aq^{j(1-n)}}{aq^j(aq^j-z)}-\frac{q^{-j(1+n)}}{q^{-j}(q^{-j}-z)} = \frac{-(1-q^{2j}a)q^{-jn}}{z(1-q^jz)(1-q^ja/z)}.$$

We can therefore reformulate the limit of integral-sum (F5.13a-F5.13b) as the following bilateral series identity:

$$\begin{aligned}
F(z)\frac{(1-z)(1-a/z)}{1-a} &= \sum_{j=-\infty}^{+\infty}\frac{1-z}{1-q^jz}\,\frac{1-a/z}{1-q^ja/z}\left[\begin{matrix}b,\,c\\ qa/b,qa/c\end{matrix}\,\Big|\, q\right]_j\\
&\times\frac{1-q^{2j}a}{1-a}\prod_{k=1}^{\ell}\left[\begin{matrix}x_k,\,y_k\\ qa/x_k,qa/y_k\end{matrix}\,\Big|\, q\right]_j\left\{\frac{q^{1-n}a}{bc}\right\}^j
\end{aligned}$$

where the bilateral series on the right converges under the same condition $|q^{1-n}a/bc|<1$.

Replacing $z$ by $d$, we see that this identity becomes exactly the basic very well-poised bilateral hypergeometric formula (F5.10a-F5.10b). Unfortunately, this very well-poised evaluation is not a proper extension of (F5.1), even though we have expected that. □

**Remark** When $n=0$, we recover from the formula (F5.10a-F5.10b-F5.10c) a special case of Bailey's bilateral ${}_6\psi_6$-series identity ($\left|qa^2/bcde\right|<1$):

$$
{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \,\Big|\, q; \frac{qa^2}{bcde}\right]
$$

$$
= \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \,\Big|\, q\right]_\infty .
$$

**F5.5. Ordinary hypergeometric counterparts.** For $q\to 1$, we derive from the limit of the last bilateral basic hypergeometric series identity the ordinary hypergeometric summation formula.

For complex numbers $a$, $b$, $c$, $d$, and $\{x_k,\ y_k\}_{k=1}^{\ell}$ satisfying the condition $x_k+y_k = 1+a+n_k$ $(k=1,2,\cdots,\ell)$ with $\{n_k\}_{k=1}^{\ell}$ being nonnegative integers and $n=\sum_{k=1}^{\ell} n_k$, there holds

$$
{}_{2\ell+5}H_{5+2\ell}\left[\begin{matrix} 1+\frac{a}{2}, & b, & c, & d, & a-d, & \{x_k, \quad y_k\} \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+d, & \{1+a-x_k, 1+a-y_k\} \end{matrix} \,\Big|\, 1\right] \tag{F5.15a}
$$

$$
= \frac{\sin\pi a}{\pi a}\,\frac{\pi d}{\sin\pi d}\,\frac{\pi(a-d)}{\sin\pi(a-d)}\prod_{k=1}^{\ell}\left[\begin{matrix} 1+d-x_k,\ 1+d-y_k \\ 1+a-x_k,\ 1+a-y_k \end{matrix}\right]_{n_k} \tag{F5.15b}
$$

$$
\times\quad \Gamma\left[\begin{matrix} 1+a-b, & 1+a-c, & 1-b, & 1-c \\ 1+a-b-d, & 1+a-c-d, & 1-b+d, & 1-c+d \end{matrix}\right] \tag{F5.15c}
$$

provided that $n+\Re(b+c-a)<1$ and the bilateral series is well-defined.

The Chu-Karlsson-Minton formulae (F5.7-F5.8) may be regarded as its limiting case of $a\to\infty$ after replacing $c$ by $1+a-c$. When $n=0$, it reduces to a special case of the Dougall formula (cf. [**56**, §6.1]):

$$
{}_5H_5\left[\begin{matrix} 1+\frac{a}{2}, & b, & c, & d, & e \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix} \,\Big|\, 1\right] \tag{F5.16a}
$$

$$
= \Gamma\left[\begin{matrix} 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e \\ 1+a-b-c,\ 1+a-b-d,\ 1+a-c-d,\ 1-a \end{matrix}\right] \tag{F5.16b}
$$

$$
\times\quad \Gamma\left[\begin{matrix} 1-b,\ 1-c,\ 1-d,\ 1-e,\ 1+2a-b-c-d-e \\ 1+a-b-e,\ 1+a-c-e,\ 1+a-d-e,\ 1+a \end{matrix}\right] \tag{F5.16c}
$$

where $\Re(1+2a-b-c-d-e)>0$.