Chapter V

Groups of Lie type

1 Lie Algebras

Our main references here will be [10] and the book of R. Carter[5].

(1.1) Definition *A* Lie algebra L *is a vector space* L*, over a field* $\mathbb{F}$*, endowed with a bilinear map* $L \times L \to L$:

$$(x, y) \mapsto [xy] \quad \text{(Lie product)}$$

for which the following conditions hold. For all $x, y, z \in L$*:*

(1) $[xx] = 0$*;*

(2) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ *(*Jacobi identity*).*

By (1) any Lie product is anticommutative, namely $[xy] = -[yx]$. Indeed:

$$0 = [(x + y)(x + y)] = [xx] + [xy] + [yx] + [yy] = [xy] + [yx].$$

(1.2) Definition *Let* $\mathcal{B} = \{x_1, \ldots, x_n\}$ *be a basis of* L *over* $\mathbb{F}$*. The* structure constants *of* L *(with respect to* $\mathcal{B}$*) are the elements* $a_{ij}^k \in \mathbb{F}$ *defined by:*

$$[x_i x_j] = \sum_{k=1}^{n} a_{ij}^k x_k.$$

Every Lie product over L is determined by its structure constants by the bilinearity.

(1.3) Definition

(1) *A subspace* I *of* L *is called an* ideal *if* $[ix] \in I$ *for all* $i \in I$*,* $x \in L$*;*

(2) *L is* simple *if $L \neq \{0\}$ and it has no proper ideal.*

(1.4) Definition *A linear map $\delta : L \to L$ is called a* derivation *if it satisfies*

$$\delta([yz]) = [\delta(y)z] + [y\delta(z)], \ \forall \ y, z \in L.$$

(1.5) Example *For each $x \in L$ the derivation* $\operatorname{ad} x : L \to L$ *defined by:*

$$\operatorname{ad} x(y) := [xy], \forall \ y \in L.$$

The linearity of $\operatorname{ad} x$ is an immediate consequence of the bilinearity of the Lie product. The map $\operatorname{ad} x$ is a derivation by axioms (1) and (2) of Definition 1.1 of Lie product.

(1.6) Definition *Let L, L' be Lie algebras over $\mathbb{F}$. A map $\varphi : L \to L'$ is called a* homomorphism *if, for all $x, y \in L$:*

$$\varphi([xy]) = [\varphi(x)\varphi(y)].$$

An isomorphism *is a bijective homomorphism. An isomorphism $\varphi : L \to L$ is called an* automorphism *of L. The group of automorphisms of L is indicated by* $\operatorname{Aut}(L)$.

2 Linear Lie Algebras

An *associative algebra* A, over a field $\mathbb{F}$, is a ring A, which is a vector space over $\mathbb{F}$, satisfying the following axiom. For all $\lambda \in \mathbb{F}$ and for all $x, y \in A$:

$$\lambda(xy) = (\lambda x)y = x(\lambda y).$$

(2.1) Lemma *Let A be an associative algebra over $\mathbb{F}$. Then A is a Lie algebra with respect to the product defined by:*

$$[x, y] := xy - yx, \quad \forall \ x, y \in A. \tag{2.2}$$

Proof Routine calculation. ■

(2.3) Definition *Let V be a vector space over $\mathbb{F}$.*

(1) *The associative algebra $End_{\mathbb{F}}(V)$, considered as a Lie algebra with respect to the product* (2.2), *is called the* general linear *Lie algebra and indicated by $\mathcal{GL}(V)$;*

(2) *the matrix algebra* $\mathrm{Mat}_n(\mathbb{F})$, *considered as a Lie algebra with respect to* (2.2), *is indicated by* $\mathcal{GL}_n(\mathbb{F})$*;*

(3) $\mathcal{GL}_n(\mathbb{F})$ *and its subalgebras are called the* linear *Lie algebras.*

Let $\mathcal{B}$ be a fixed basis of $V = \mathbb{F}^n$. The map $\Phi_{\mathcal{B}} : \mathcal{GL}(V) \simeq \mathcal{GL}_n(\mathbb{F})$ such that $\Phi_{\mathcal{B}}(\alpha)$ is the matrix of α with respect to $\mathcal{B}$ is an isomorphism of Lie algebras. Thus:

$$\mathcal{GL}(\mathbb{F}^n) \simeq \mathcal{GL}_n(\mathbb{F}).$$

A basis of $\mathcal{GL}_n(\mathbb{F})$ consists of the matrices having 1 in one position and 0 elsewhere, namely the matrices:

$$\{e_{ij} \mid 1 \leq i, j \leq n\}.$$

The structure constants, with respect to this basis, are all ± 1 or 0. More precisely:

$$[e_{ij}, e_{k\ell}] := e_{ij}e_{k\ell} - e_{k\ell}e_{ij} = \delta_{jk}e_{i\ell} - \delta_{\ell i}e_{kj}. \tag{2.4}$$

Conjugation by a fixed element of $\mathrm{GL}_n(\mathbb{F})$ is an automorphism of the associative algebra $\mathrm{Mat}_n(\mathbb{F})$ and also of the Lie algebra $\mathcal{GL}_n(\mathbb{F})$, as shown in the following:

(2.5) Lemma *For a fixed* $g \in \mathrm{GL}_n(\mathbb{F})$*, let* $\gamma_g : \mathcal{GL}_n(\mathbb{F}) \to \mathcal{GL}_n(\mathbb{F})$ *be defined by:*

$$\gamma_g(m) := g^{-1}mg, \quad \forall\, m \in \mathcal{GL}_n(\mathbb{F}).$$

Then γ_g *is an automorphism of the Lie algebra* $\mathcal{GL}_{\ell+1}(\mathbb{F})$*.*

Proof γ_g is linear since, for all $m_1, m_2, m \in \mathrm{GL}_n(\mathbb{F})$, $\lambda \in \mathbb{F}$:

$$\begin{array}{l} g^{-1}(m_1 + m_2)g = g^{-1}m_1g + g^{-1}m_2g \\ g^{-1}(\lambda m)g = \lambda g^{-1}mg \end{array}.$$

γ_g preserves the Lie product, i.e., $[g^{-1}m_1g, g^{-1}m_2g] = g^{-1}[m_1, m_2]g$. In fact:

$$g^{-1}m_1gg^{-1}m_2g - g^{-1}m_2gg^{-1}m_1g = g^{-1}\left(m_1m_2 - m_2m_1\right)g.$$

γ_g is bijective having $\gamma_{g^{-1}}$ as its inverse. ■

(2.6) Lemma *The trace map* $\mathrm{tr} : \mathcal{GL}_n(\mathbb{F}) \to \mathcal{GL}_1(\mathbb{F})$ *is a Lie algebras homomorphism. In particular its kernel is a subalgebra, indicated by* $\mathbf{A}_\ell$*.*

Proof For all $a, b \in \mathcal{GL}_n(\mathbb{F})$, $\lambda \in \mathbb{F}$:

$\mathrm{tr}(a + b) = \mathrm{tr}(a) + \mathrm{tr}(b)$,

$\mathrm{tr}(\lambda a) = \lambda\,\mathrm{tr}(a)$,

$\mathrm{tr}([a, b]) = \mathrm{tr}(ab - ba) = \mathrm{tr}(ab) - \mathrm{tr}(ba) = 0 = [\mathrm{tr}(a), \mathrm{tr}(b)]$. ■

3 The classical Lie algebras

We give an explicit description of the *classical* Lie algebras over $\mathbb{C}$.

3.1 The special linear algebra $\mathbf{A}_\ell$

$\mathbf{A}_\ell$ is the subalgebra of $\mathcal{GL}_{\ell+1}(\mathbb{C})$ consisting of the matrices of trace 0, namely the kernel of the trace homomorphism $\mathrm{tr} : \mathcal{GL}_{\ell+1}(\mathbb{C}) \to \mathcal{GL}_1(\mathbb{C})$.
A basis of $\mathbf{A}_\ell$ is given by the matrices:

$$\{e_{i,i} - e_{i+1,i+1} \mid 1 \leq i \leq \ell\} \ \cup \ \{e_{ij} | 1 \leq i \neq j \leq \ell + 1\}\,. \tag{3.1}$$

Thus, for the dimension of the special linear algebra, we get:

$$\dim_{\mathbb{C}} (\mathbf{A}_\ell) = (\ell+1)\ell + \ell = \ell^2 + 2\ell. \tag{3.2}$$

(3.3) Theorem $\mathrm{PGL}_{\ell+1}(\mathbb{C}) \leq \mathrm{Aut}(\mathbf{A}_\ell)$.

Proof By Lemma 2.5, for all $g \in \mathrm{GL}_{\ell+1}(\mathbb{C})$, the inner automorphism

$$\gamma_g : \mathcal{GL}_{\ell+1}(\mathbb{C}) \to \mathcal{GL}_{\ell+1}(\mathbb{C})$$

is an automorphism of the Lie algebra $\mathcal{GL}_{\ell+1}(\mathbb{C})$. For all $m \in \mathbf{A}_\ell$ we have $\mathrm{tr}(\gamma_g(m)) = \mathrm{tr}(m) = 0$, i.e., $\gamma_g(\mathbf{A}_\ell) \leq \mathbf{A}_\ell$. Since $\mathbf{A}_\ell$ has finite dimension and γ_g is injective, we get $\gamma_g(\mathbf{A}_\ell) = \mathbf{A}_\ell$. So the restriction of γ_g to $\mathbf{A}_\ell$ is an automorphism of $\mathbf{A}_\ell$. Hence we may consider the homomorphism $\gamma : \mathrm{GL}_{\ell+1}(\mathbb{C}) \ \to \ \mathrm{Aut}\,(\mathbf{A}_\ell)$ defined by: $g \mapsto \gamma_g$. The kernel of γ is the subgroup Z of scalar matrices. We conclude that:

$$\mathrm{PGL}_{\ell+1}(\mathbb{C}) := \frac{\mathrm{GL}_{\ell+1}(\mathbb{C})}{Z} \quad \simeq \quad \mathrm{Im}\ \gamma \leq \mathrm{Aut}(\mathbf{A}_\ell).$$

■

3.2 The symplectic algebra $\mathbf{C}_\ell$

Let us consider the antisymmetric, non-singular matrix:

$$s = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}. \tag{3.4}$$

The symplectic algebra $\mathbf{C}_\ell$ is the subalgebra of $\mathcal{GL}_{2\ell}(\mathbb{C})$ defined by:

$$\mathbf{C}_\ell := \{x \in \mathcal{GL}_{2\ell}(\mathbb{C}) \mid sx = -x^T\, s\}.$$

Partitioning x into $\ell \times \ell$ blocks, we have that $x \in \mathbf{C}_\ell$ if and only if it has shape:

$$x = \begin{pmatrix} m & n \\ p & -m^T \end{pmatrix} \quad \text{with} \quad n = n^T\,,\ p = p^T \ \text{symmetric.}$$

Thus, a basis of $\mathbf{C}_\ell$ is given by the matrices:

$$\left\{ \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix} \mid 1 \le i, j \le \ell \right\} \cup \tag{3.5}$$

$$\left\{ \begin{pmatrix} 0 & e_{ii} \\ 0 & 0 \end{pmatrix} \mid 1 \le i \le \ell \right\} \cup \left\{ \begin{pmatrix} 0 & e_{ij} + e_{ji} \\ 0 & 0 \end{pmatrix} \mid 1 \le i < j \le \ell \right\} \cup \tag{3.6}$$

$$\{\text{the transposes of (3.6)}\}. \tag{3.7}$$

So, for the dimension of the symplectic algebra, we obtain:

$$\dim_{\mathbb{C}} \mathbf{C}_\ell = \ell^{\mathbf{2}} + \mathbf{2}\left(\mathbf{1} + \frac{\ell(\ell - \mathbf{1})}{\mathbf{2}}\right) = \mathbf{2}\ell^{\mathbf{2}} + \ell. \tag{3.8}$$

(3.9) Theorem $\mathrm{PSp}_{\ell+1}(\mathbb{C}) \le \mathrm{Aut}(\mathbf{C}_\ell)$.

Proof Let $\mathrm{Sp}_{2\ell}(\mathbb{C})$ be the group of isometries of s in (3.4). Thus

$$sg = \left(g^{-1}\right)^T s, \quad \forall\, g \in \mathrm{Sp}_{2\ell}(\mathbb{C}).$$

Take γ_g as in Lemma 2.5. Then $\gamma_g(x) = g^{-1}xg \in \mathbf{C}_\ell$, for all $x \in \mathbf{C}_\ell$. Indeed:

$$s\left(g^{-1}xg\right) = g^T sxg = g^T\left(-x^T s\right)g = -g^T x^T \left(g^{-1}\right)^T s = -\left(g^{-1}xg\right)^T s.$$

So the restriction of γ_g to $\mathbf{C}_\ell$ is an automorphism of $\mathbf{C}_\ell$. Hence we may consider the homomorphism $\gamma : \mathrm{Sp}_{2\ell}(\mathbb{C}) \to \mathrm{Aut}\left(\mathbf{C}_\ell\right)$ defined by: $g \mapsto \gamma_g$. The kernel of γ is the subgroup $\langle -I \rangle$ of symplectic scalar matrices. We conclude that:

$$\mathrm{PSp}_{\ell+1}(\mathbb{C}) := \frac{\mathrm{Sp}_{2\ell}(\mathbb{C})}{\langle -I \rangle} \quad \simeq \quad \mathrm{Im}\,\gamma \le \mathrm{Aut}(\mathbf{C}_\ell).$$

■

3.3 The orthogonal algebra $\mathbf{B}_\ell$

Let us consider the symmetric, non-singular matrix:

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}. \tag{3.10}$$

The orthogonal algebra $\mathbf{B}_\ell$ is the subalgebra of $\mathcal{GL}_{2\ell+1}(\mathbb{C})$ defined by:

$$\mathbf{B}_\ell := \{x \in \mathcal{GL}_{2\ell+1}(\mathbb{C}) \mid sx = -x^T s\}.$$

Partitioning x into blocks, one has that $x \in \mathbf{B}_\ell$ if and only if it has shape

$$x = \begin{pmatrix} 0 & -v_1^T & -v_2^T \\ v_2 & m & n \\ v_1 & p & -m^T \end{pmatrix} \quad \text{with} \quad n = -n^T\,,\ p = -p^T \ \text{antisymmetric.}$$

Thus the orthogonal algebra $\mathbf{B}_\ell$ has basis:

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ij} & 0 \\ 0 & 0 & -e_{ji} \end{pmatrix} | 1 \le i, j \le \ell \right\} \cup \tag{3.11}$$

$$\left\{ \begin{pmatrix} 0 & -e_i^T & 0 \\ 0 & 0 & 0 \\ e_i & 0 & 0 \end{pmatrix} | 1 \le i \le \ell \right\} \cup \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{ij} - e_{ji} \\ 0 & 0 & 0 \end{pmatrix} | 1 \le i < j \le \ell \right\} \tag{3.12}$$

$$\cup\ \{\text{the transposes of 3.12}\}\,.$$

We conclude that the dimension of this orthogonal algebra is given by:

$$\dim_{\mathbb{C}} \mathbf{B}_\ell = \ell^2 + 2\left(l + \frac{\ell(\ell-1)}{2}\right) = 2\ell^2 + \ell. \tag{3.13}$$

(3.14) Theorem *Let $G \le \mathrm{GL}_{2\ell+1}(\mathbb{C})$ be the group of isometries of s in* (3.10). *Then*

$$\frac{ZG}{Z} \le \mathrm{Aut}(\mathbf{B}_\ell)$$

where Z denotes the group of scalar matrices.

The proof is the same as that of Theorem 3.9.

3.4 The orthogonal algebra $\mathbf{D}_\ell$

Let us consider the symmetric, non-singular matrix:

$$s = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}. \tag{3.15}$$

The orthogonal algebra $\mathbf{D}_\ell$ is the subalgebra of $\mathcal{GL}_{2\ell}(\mathbb{C})$ defined by:

$$\mathbf{D}_\ell := \{x \in \mathcal{GL}_{2\ell}(\mathbb{C}) \mid sx = -x^T s\}.$$

Partitioning x into blocks, one has that $x \in \mathbf{D}_\ell$ if and only if it has shape:

$$x = \begin{pmatrix} m & n \\ p & -m^T \end{pmatrix} \quad \text{with} \quad n = -n^T, \ p = -p^T \ \text{antisymmetric.}$$

Thus the orthogonal algebra $\mathbf{D}_\ell$ has basis:

$$\left\{ \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix} | 1 \leq i, j \leq \ell \right\} \cup \tag{3.16}$$

$$\left\{ \begin{pmatrix} 0 & e_{ij} - e_{ji} \\ 0 & 0 \end{pmatrix} | 1 \leq i < j \leq \ell \right\} \cup \{\text{their transposes}\}. \tag{3.17}$$

We conclude that the dimension of this orthogonal algebra is given by:

$$\dim_{\mathbb{C}} \mathbf{D}_\ell = \ell^{\mathbf{2}} + \mathbf{2}\frac{\ell(\ell - \mathbf{1})}{\mathbf{2}} = \mathbf{2}\ell^{\mathbf{2}} - \ell. \tag{3.18}$$

(3.19) Theorem *Let $G \leq \mathrm{GL}_{2\ell}(\mathbb{C})$ be the group of isometries of s in* (3.15)*. Then*

$$\frac{ZG}{Z} \leq \mathrm{Aut}(\mathbf{D}_\ell)$$

where Z denotes the group of scalar matrices.

The proof is the same as that of Theorem 3.9.

4 Root systems

Let L be a finite dimensional simple Lie algebras over $\mathbb{C}$. By the classification due to Killing and Cartan, L is one of the 9 algebras denoted respectively by:

$$\mathbf{A}_\ell, \mathbf{B}_\ell, \ \mathbf{C}_\ell, \ \mathbf{D}_\ell, \ \mathbf{E}_6, \ \mathbf{E}_7, \ \mathbf{E}_8, \ \mathbf{F}_4, \ \mathbf{G}_2. \tag{4.1}$$

There exists a set $\Phi = \Phi(L)$ such that L admits a decomposition

$$L \ = \ \mathcal{H} \ \oplus \bigoplus_{r \in \Phi} L_r \quad \text{(Cartan decomposition)} \tag{4.2}$$

where $\mathcal{H}$ is an ℓ-dimensional abelian subalgebra (namely $[h_1 h_2] = 0$ for all $h_1, h_2 \in \mathcal{H}$) and, for each $r \in \Phi$, the following conditions hold:

(1) $L_r = \mathbb{C}v_r$ for some $v_r \in L$, i.e., L_r is a 1-dimensional space;

(2) $[hv_r] = r(h)v_r$ with $r(h) \in \mathbb{C}$, for all $h \in \mathcal{H}$;

(3) the map $\operatorname{ad} v_r : L \to L$ is nilpotent;

(4) there exists a unique $s \in \Phi$ (denoted by $-r$) such that $0 \neq [v_r v_s] \in \mathcal{H}$.

(4.3) Remark *Fix $y \in L$. Recalling that $\operatorname{ad} y(x) := [yx]$, for all $x \in L$, we have:*

- $\operatorname{ad} h(\mathcal{H}) = \{0\}$ *for all $h \in \mathcal{H}$ since $\mathcal{H}$ is abelian.*
- *v_r is an eigenvector of $\operatorname{ad} h$, with eigenvalue $r(h)$, by point (2) above.*

Every $r \in \Phi$ may be identified with the linear map $r : \mathcal{H} \to \mathbb{C}$ defined by $h \mapsto r(h)$. Clearly r is an element of the dual space $\mathcal{H}^*$ of $\mathcal{H}$, by the bilinearity of the Lie product. Moreover different elements of Φ give rise to different maps. So:

$$\Phi \subseteq \mathcal{H}^*.$$

Now, consider the bilinear, symmetric form: $L \times L \to \mathbb{C}$ defined by

$$(x, y) := \operatorname{tr}(\operatorname{ad} x \ \operatorname{ad} y) \quad \text{(Killing form)}.$$

Since this form is non-degenerate, its restriction to $\mathcal{H} \times \mathcal{H}$ induces the isomorphism of vector spaces $\varphi : \mathcal{H} \to \mathcal{H}^*$ where, for each $\overline{h} \in \mathcal{H}$:

$$\varphi(\overline{h})\,(h) := \operatorname{tr}(\operatorname{ad} \overline{h} \ \operatorname{ad} h), \quad \forall\, h \in \mathcal{H}.$$

Identifying each $r \in \Phi$ with its preimage in $\mathcal{H}$, we may assume:

$$\Phi \subseteq \mathcal{H}.$$

It can be shown that Φ contains a $\mathbb{C}$-basis

$$\Pi = \{r_1, \ \ldots, \ r_\ell\} \quad \text{(fundamental system)}$$

of $\mathcal{H}$ such that every $r \in \Phi$:

(1) is a linear combination of elements in Π with *rational* coefficients;

(2) these coefficients are either all positive, or all negative.

Property (2) defines an obvious partition of Φ into positive and negative roots:

$$\Phi \; = \; \Phi^+ \,\dot\cup\, \Phi^-.$$

By property (1), Φ is a subset of the real vector space:

$$\mathcal{H}_{\mathbb{R}} \; := \; \mathbb{R}r_1 \oplus \; \cdots \oplus \; \mathbb{R}r_\ell \; \simeq \mathbb{R}^\ell.$$

$\mathcal{H}_{\mathbb{R}}$ is an *euclidean space* with respect to the Killing form as scalar product:

$$(x,y) := \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y), \quad \forall\; x, y \in \mathcal{H}_{\mathbb{R}}.$$

The *length* of a vector $x \in \mathcal{H}_{\mathbb{R}}$ and the *angle* $\widehat{xy}$ for $x, y \in \mathcal{H}_{\mathbb{R}} \setminus \{0\}$ are defined by:

$$|x| := \sqrt{(x,x)}, \quad \cos\widehat{xy} := \frac{(x,y)}{|x|\,|y|}.$$

(4.4) Definition *The numbers A_{rs} are defined by:*

$$A_{rs} := \frac{2(r,s)}{(r,r)}, \quad \forall\; r, s \in \Phi.$$

It turns out that all A_{rs} are in $\mathbb{Z}$. In particular, if $r, s \in \Phi$ are linearly independent and $r+s \in \Phi$, then $A_{rs} = p - q$ where $0 \le p, q \in \mathbb{N}$ and

$$-pr + s, \ldots, s, \ldots, qr + s \tag{4.5}$$

is the longest chain of roots through s involving r.

(4.6) Example *Take the root system Φ with $\Phi^+ = \{r_1, r_2, r_1 + r_2, 2r_1 + r_2\}$.*
Set $s = r_1 + r_2$, $t = 2r_1 + r_2$.

r, s	Longest chain	p, q	A_{rs}
$r_1,\ r_2$	$r_2,\ r_2 + r_1,\ r_2 + 2r_1$	$0, 2$	-2
$r_1,\ r_1 + r_2$	$-r_1 + (r_1 + r_2),\ (r_1 + r_2),\ (r_1 + r_2)\, r_1$	$1, 1$	0
$r_1,\ 2r_1 + r_2$	$-2r_1 + (2r_1 + r_2),\ -r_1 + (2r_1 + r_2),\ t$	$2, 0$	2
$r_2,\ r_1$	$r_1,\ r_1 + r_2$	$0, 1$	-1
$r_2,\ r_1 + r_2$	$-r_2 + (r_1 + r_2),\ (r_1 + r_2)$	$1, 0$	1
$r_2,\ 2r_1 + r_2$	$2r_1 + r_2$	$0, 0$	0 .

The *Cartan matrix* of L, with respect to a basis $\{r_1, \ldots, r_\ell\}$ of $\mathcal{H}_{\mathbb{R}}$, is defined as:

$$A \; := \; \left(\frac{2(r_i, r_j)}{(r_i, r_i)}\right), \quad 1 \le i, j \le \ell. \tag{4.7}$$

A basis $\{r_1, \; \ldots \;, \; r_\ell\}$ of $\mathcal{H}_{\mathbb{R}}$ can be normalized into the basis $\{h_{r_1}, \; \ldots \;, \; h_{r_\ell}\}$, where:

$$h_i := \frac{2r_i}{(r_i, r_i)}, \;\; 1 \le i \le \ell.$$

4.1 Root system of type $\mathbf{A}_\ell$

Let$\{e_1, \ \dots, \ e_{\ell+1}\}$ be an orthonormal basis of the euclidean space $\mathbb{R}^{\ell+1}$.

The following vectors of $\mathbb{R}^{\ell+1}$ form a fundamental system of type $\mathbf{A}_\ell$:

$$\Pi = \left\{ \underbrace{-e_1 + e_2}_{r_1}, \ \underbrace{-e_2 + e_3}_{r_2}, \ \dots, \underbrace{-e_\ell + e_{\ell+1}}_{r_\ell} \right\}.$$

The full root system has order $\ell(\ell+1)$ and is as follows:

$$\Phi = \underbrace{\{-e_i + e_j, \mid 1 \leq i < j \leq \ell+1\}}_{\Phi^+} \ \dot\cup \ \underbrace{\{e_i - e_j, \mid 1 \leq i < j \leq \ell+1\}}_{\Phi^-}.$$

All roots $r \in \Phi$ have the same length $|r| = \sqrt{2}$ (for this root system).

Cartan matrix:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

4.2 Root system of type $\mathbf{B}_\ell$

Let $\{e_1, \ \dots, \ e_\ell\}$ be an orthonormal basis of the euclidean space $\mathbb{R}^\ell$.

The following vectors form a fundamental system of type $\mathbf{B}_\ell$

$$\Pi = \left\{ \underbrace{e_1 - e_2}_{r_1}, \quad \underbrace{e_2 - e_3}_{r_2}, \quad \dots, \quad \underbrace{e_{\ell-1} - e_\ell}_{r_{\ell-1}}, \quad \underbrace{e_\ell}_{r_\ell} \right\}.$$

The full root system has order $2\ell^2$ and is as follows:

$$\Phi = \underbrace{\{e_i \pm e_j, \ e_i \mid 1 \leq i < j \leq \ell\}}_{\Phi^+} \ \dot\cup \ \underbrace{\{-e_i \mp e_j, \ -e_i \mid 1 \leq i < j \leq \ell\}}_{\Phi^-}.$$

For all $r \in \Phi$ we have $|r| \in \{\sqrt{2}, \ 1\}$. So there are *long* and *short* roots. E.g. the r_i-s, $i \leq \ell - 1$, are long, r_ℓ is short.

Cartan matrix:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}.$$

4.3 Root system of type $\mathbf{C}_\ell$

Let $\{e_1, \ldots, e_\ell\}$ be an orthonormal basis of the euclidean space $\mathbb{R}^\ell$.

The following vectors form a fundamental system of type $\mathbf{C}_\ell$

$$\Pi = \left\{ \underbrace{e_1 - e_2}_{r_1}, \quad \underbrace{e_2 - e_3}_{r_2}, \quad \ldots, \quad \underbrace{e_{\ell-1} - e_\ell}_{r_{\ell-1}}, \quad \underbrace{2e_\ell}_{r_\ell} \right\}.$$

The full root system has order $2\ell^2$ and is as follows:

$$\Phi = \underbrace{\{e_i \pm e_j,\ 2e_i \mid 1 \leq i < j \leq \ell\}}_{\Phi^+} \ \dot\cup\ \underbrace{\{-e_i \mp e_j,\ -2e_i,\ \mid 1 \leq i < j \leq \ell\}}_{\Phi^-}.$$

For all $r \in \Phi$ we have $|r| \in \{\sqrt{2},\ 2\}$. Here the r_i-s, $i \leq \ell - 1$, are short, r_ℓ is long.

Cartan matrix:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -1 & 2 & -2 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}.$$

4.4 Root system of type $\mathbf{D}_\ell$

Let $\{e_1, \ldots, e_\ell\}$ be an orthonormal basis of the euclidean space $\mathbb{R}^\ell$.

The following vectors form a fundamental system of type $\mathbf{D}_\ell$

$$\Pi = \left\{ \underbrace{e_1 - e_2}_{r_1}, \quad \underbrace{e_2 - e_3}_{r_2}, \quad \ldots, \quad \underbrace{e_{\ell-1} - e_\ell}_{r_{\ell-1}}, \quad \underbrace{e_{\ell-1} + e_\ell}_{r_\ell} \right\}.$$

The full root system has order $2\ell(\ell - 1)$ and is as follows:

$$\Phi = \underbrace{\{e_i \pm e_j \mid 1 \leq i < j \leq \ell\}}_{\Phi^+} \ \dot\cup\ \underbrace{\{-e_i \mp e_j \mid 1 \leq i < j \leq \ell\}}_{\Phi^-}.$$

As in the case of $\mathbf{A}_\ell$ all roots have the same length. For this system $|r| = \sqrt{2}$.

Cartan matrix:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 0 & 2 \end{pmatrix}.$$

5 Chevalley basis of a simple Lie algebra

Let $L = \mathcal{H} \oplus \bigoplus_{r\in\Phi} L_r$ be a simple Lie algebra over $\mathbb{C}$, with fundamental system Π. Chevalley has proved the existence of a basis of L

(5.1) $$\{h_r \mid r \in \Pi\} \cup \{e_r \mid r \in \Phi\} \quad \text{(Chevalley basis)}$$

where $\mathcal{H} = \bigoplus_{r\in\Pi} \mathbb{C}h_r$ and $L_r = \mathbb{C}e_r$ for each r, satisfying the following conditions:

- $[h_r h_s] = 0$, for all $r, s \in \Pi$;
- $[h_r e_s] = A_{rs} e_s$, for all $r \in \Pi$, $s \in \Phi$, with A_{rs} as in Definition 4.4;
- $[e_r e_{-r}] = h_r$, for all $r \in \Phi$;
- $[e_r e_s] = 0$, for all $r, s \in \Phi$, $r + s \neq 0$ and $r + s \notin \Phi$;
- $[e_r e_s] = \pm(p + 1)e_{r+s}$, if $r + s \in \Phi$, with p as in (4.5).

In particular, with respect to a Chevalley basis, the multiplication constants of L are all in $\mathbb{Z}$, a crucial property for the definition of the groups of Lie type over any field $\mathbb{F}$.

(5.2) Lemma *Suppose that L is linear and that $\mathcal{H}$ consists of diagonal matrices. Then, for each $r \in \Phi$, we have $e_{-r} = e_r^T$.*

Proof For all $h \in \mathcal{H}$, $\operatorname{ad} h\,(e_r) = he_r - e_r h = r(h)e_r$. The condition $h = h^T$ gives:

$$\operatorname{ad} h\left({e_r}^T\right) = h{e_r}^T - {e_r}^T h = (e_r h - h e_r)^T = -r(h){e_r}^T.$$

■

(5.3) Example *Chevalley basis of $\mathbf{A}_1$.*

$$\mathbf{A}_1 \;=\; \mathbb{C}\underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{h_{r_1}} \oplus \mathbb{C}\underbrace{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}_{e_{r_1}} \oplus \mathbb{C}\underbrace{\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}}_{e_{-r_1}}.$$

Let $h = \begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix} \in \mathcal{H}$. With respect to the above basis:

$$(\operatorname{ad} h)_{|\langle e_{r_1}, e_{-r_1}\rangle} = \begin{pmatrix}2a & 0\\ 0 & -2a\end{pmatrix} \quad \Longrightarrow \begin{cases} r_1(h) &= \;\; 2a\\ -r_1(h) &= \; -2a\end{cases}$$

Since $2a = \operatorname{tr}\left(\operatorname{ad}\begin{pmatrix}1/4 & 0\\ 0 & -1/4\end{pmatrix}\operatorname{ad}h\right)$, the Killing form allows the identification:

$$r_1 = \begin{pmatrix}1/4 & 0\\ 0 & -1/4\end{pmatrix}.$$

Normalized basis of $\mathcal{H}$:

$$h_1 := \frac{2r_1}{(r_1, r_1)} = \frac{2r_1}{\operatorname{tr}(\operatorname{ad} r_1)^2} = \frac{2}{1/2}r_1 = 4r_1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Root system: $\Phi = \{r_1,\ -r_1\}$.

(5.4) Example *Chevalley basis of* $\mathbf{A}_2$.

$$\mathbf{A}_2 \;=\; \underbrace{\mathbb{C}h_{r_1} \oplus \mathbb{C}h_{r_2}}_{\mathcal{H}} \oplus \mathbb{C}e_{r_1} \oplus\; \mathbb{C}e_{r_2} \oplus\; \mathbb{C}e_s \oplus\; \mathbb{C}e_{-r_1} \oplus\; \mathbb{C}e_{-r_2} \oplus\; \mathbb{C}e_{-s}$$

where:

$$h_{r_1} = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad h_{r_2} = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad e_{r_1} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad e_{r_2} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

$$e_s = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad e_{-r_1} = e_{r_1}^T, \quad e_{-r_2} = e_{r_2}^T, \quad e_{-s} = e_s^T.$$

We justify and complete the notation. Let $h = \begin{pmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & -a-b\end{pmatrix} \in \mathcal{H}$.

With respect to the above ordered basis:

$$\operatorname{ad} h_{|\langle e_{r_1},\, e_{r_2},\, e_s\rangle} = \begin{pmatrix}a-b & 0 & 0\\ 0 & a+2b & 0\\ 0 & 0 & 2a+b++\end{pmatrix}$$

$$\Longrightarrow \quad \begin{cases} r_1(h) = a-b \\ r_2(h = a+2b \\ s(h) = 2a+b \end{cases} \quad \text{giving} \quad s = r_1 + r_2.$$

Since

$$a-b = \operatorname{tr}\left(\operatorname{ad}\begin{pmatrix}1/6 & 0 & 0\\ 0 & -1/6 & 0\\ 0 & 0 & 0\end{pmatrix}\operatorname{ad}h\right), \quad a+2b = \operatorname{tr}\left(\operatorname{ad}\begin{pmatrix}0 & 0 & 0\\ 0 & 1/6 & 0\\ 0 & 0 & -1/6\end{pmatrix}\operatorname{ad}h\right)$$

the Killing form allows the identifications:

$$r_1 = \begin{pmatrix}1/6 & 0 & 0\\ 0 & -1/6 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad r_2 = \begin{pmatrix}0 & 0 & 0\\ 0 & 1/6 & 0\\ 0 & 0 & -1/6\end{pmatrix}.$$

Normalized basis of $\mathcal{H}$: $\left\{\frac{2r_1}{(r_1,r_1)} = h_{r_1},\ \frac{2r_2}{(r_2,r_2)} = h_{r_1}\right\}$ with h_{r_1}, h_{r_2} as above.
Root system $\Phi = \Phi^+ \cup\ \Phi^-$, with

$$\Phi^+ = \{r_1,\ r_2,\ r_1 + r_2\}\,, \quad \Phi^- = \{\ -r_1,\ -r_2,\ -r_1 - r_2\}\,.$$

(5.5) Example *As fundamental system of* $\mathbf{A}_\ell$ *one may take the* $\ell + 1 \times \ell + 1$ *matrices*

$$e_{r_1} = e_{1,2}, \quad e_{r_2} = e_{2,3}, \quad \ldots, \quad e_{r_\ell} = e_{\ell,\ell+1}.$$

(5.6) Example *Chevalley basis of* $\mathbf{C_2}$.

$$\mathbf{C}_2 \ = \ \underbrace{\mathbb{C}h_{r_1} \oplus \mathbb{C}h_{r_2}}_{\mathcal{H}} \oplus \ \mathbb{C}e_{r_1} \oplus \ \mathbb{C}e_{r_2} \oplus \ \mathbb{C}e_s \oplus \ \mathbb{C}e_t \oplus \ \mathbb{C}e_{-r_1} \oplus \ \mathbb{C}e_{-r_2} \oplus \ \mathbb{C}e_{-s} \oplus \ \mathbb{C}e_{-t}$$

where:

$$h_{r_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_{r_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_{r_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$e_{r_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_s = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_t = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e_{-r_1} = e_{r_1}^T, \quad e_{-r_2} = e_{r_2}^T, \quad e_{-s} = e_s^T, \quad e_{-t} = e_t^T.$$

We justify and complete the notation. Let $h = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}$.

With respect to the above ordered basis:

$$(\operatorname{ad} h)_{\langle e_{r_1}, e_{r_2}, e_s, e_t \rangle} = \begin{pmatrix} a-b & 0 & 0 & 0 \\ 0 & 2b & 0 & 0 \\ 0 & 0 & a+b & 0 \\ 0 & 0 & 0 & 2a \end{pmatrix}$$

$$\Longrightarrow \quad \begin{cases} r_1(h) = a - b \\ r_2(h) = 2b \\ s(h) = a + b \\ t(h) = 2a \end{cases} \quad \text{giving} \quad \begin{cases} s = r_1 + r_2 \\ t = 2r_1 + r_2. \end{cases}$$

Since

$$-a + b = \operatorname{tr}\left(\operatorname{ad} \begin{pmatrix} -1/12 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & 0 \\ 0 & 0 & 1/12 & 0 \\ 0 & 0 & 0 & -1/12 \end{pmatrix} \operatorname{ad} h \right),$$

$$2a = \operatorname{tr}\left(\operatorname{ad} \begin{pmatrix} 1/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \operatorname{ad} h \right)$$

the Killing form allows the identifications:

$$r_1 = \begin{pmatrix} -1/12 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & 0 \\ 0 & 0 & 1/12 & 0 \\ 0 & 0 & 0 & -1/12 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$(r_1, r_1) = \frac{1}{6}$, $(r_2, r_2) = \frac{1}{3}$, $(r_1, r_2) = -\frac{1}{6}$. Cartan matrix $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$.

Normalized basis of $\mathcal{H}$: $\left\{ h_{r_1} = \frac{2r_1}{(r_1,r_1)},\ h_{r_2} = \frac{2r_2}{(r_2,r_2)} \right\}$ with h_{r_1}, h_{r_2} as above.

Root system: $\Phi = \{r_1,\ r_2,\ r_1 + r_2,\ 2r_1 + r_2,\ -r_1,\ -r_2,\ -r_1 - r_2,\ -2r_1 - r_2\}$

The non-trivial products of basis elements are written below. They agree with the conditions for a Chevalley basis given at the beginning of this Section, and also with the values of A_{rs} given in Example 4.6.

[ ]	e_{r_1}	e_{r_2}	$e_{r_1+r_2}$	$e_{2r_1+r_2}$	e_{-r_1}	e_{-r_2}	$e_{-r_1-r_2}$	$e_{-2r_1-r_2}$
h_{r_1}	$2e_{r_1}$	$-2e_{r_2}$	0	$2e_{2r_1+r_2}$	$-2e_{-r_1}$	$2e_{-r_2}$	0	$-2e_{-2r_1-r_2}$
h_{r_2}	$-e_{r_1}$	$2e_{r_2}$	$e_{r_1+r_2}$	0	e_{-r_1}	$-2e_{-r_2}$	$-e_{-r_1-r_2}$	0

[ ]	e_{r_1}	e_{r_2}	$e_{r_1+r_2}$	$e_{2r_1+r_2}$	e_{-r_1}	e_{-r_2}	$e_{-r_1-r_2}$	$e_{-2r_1-r_2}$
e_{r_1}	0	$e_{r_1+r_2}$	$2e_{2r_1+r_2}$	0	h_{r_1}	0	$-2e_{-r_2}$	$-e_{-r_1-r_2}$
e_{r_2}	$-e_{r_1+r_2}$	0	0	0	0	h_{r_2}	e_{-r_1}	0
$e_{r_1+r_2}$	$-2e_{2r_1+r_2}$	0	0	0	$-2e_{r_2}$	e_{r_1}	$h_{r_1+r_2}$	e_{-r_1}
$e_{2r_1+r_2}$	0	0	0	0	$-e_{r_1+r_2}$	0	e_{r_1}	$h_{2r_1+r_2}$
e_{-r_1}	$-h_{r_1}$	0	$2e_{r_2}$	$e_{r_1+r_2}$	0	$-e_{-r_1-r_2}$	$-2e_{-2r_1-r_2}$	0
e_{-r_2}	0	$-h_{r_2}$	$-e_{r_1}$	0	$e_{-r_1-r_2}$	0	0	0
$e_{-r_1-r_2}$	$2e_{-r_2}$	$-e_{-r_1}$	$-h_{r_1+r_2}$	$-e_{r_1}$	$2e_{-2r_1-r_2}$	0	0	0
$e_{-2r_1-r_2}$	$e_{-r_1-r_2}$	0	$-e_{-r_1}$	$-h_{2r_1+r_2}$	0	0	0	0

6 The action of $\exp \operatorname{ad} e$, with e nilpotent

Let L be a linear Lie algebra over $\mathbb{C}$ and $e \in L$. Consider the map $\operatorname{ad} e : L \to L$, defined as $x \mapsto [ex]$. The following identity, which can be verified by induction, holds:

$$\text{(6.1)} \qquad \frac{(\operatorname{ad} e)^k}{k!}(x) = \sum_{i=0}^{k} \frac{e^i}{i!}\, x\, \frac{(-e)^{k-i}}{(k-i)!}, \quad \forall\, k \in \mathbb{N}.$$

In particular, if e is a nilpotent matrix, then $\operatorname{ad} e$ is nilpotent and we may consider the linear map:

$$\exp \operatorname{ad} e := \sum_{k=0}^{\infty} \frac{(\operatorname{ad} e)^k}{k!}.$$

(6.2) Lemma *Let L be a subalgebra of the general linear Lie algebra $\mathcal{G}L_n(\mathbb{C})$ and let $e \in L$ be a nilpotent matrix. Then, for all $x \in L$:*

$$\exp \operatorname{ad} e(x) = (\exp e)\, x\, (\exp e)^{-1}. \tag{6.3}$$

In particular the map $\exp \operatorname{ad} e : L \to L$ is an automorphism of L.

For the proof, based on (6.1), see [5, Lemma 4.5.1, page 66]. The conclusion follows from Lemma 2.5 of this chapter.

In the next two examples we give a proof of (6.3) in the most frequent cases.

(6.4) Example *Let $e^2 = 0$. Then $\exp e = I + e$. Moreover:*

$$\begin{aligned} &\operatorname{ad} e : x \mapsto [e, x] = ex - xe \\ &(\operatorname{ad} e)^2 : x \mapsto [e, ex - xe] = -2(exe) \\ &(\operatorname{ad} e)^3 : x \mapsto [e, -2exe] = 0. \end{aligned}$$

Thus $\exp \operatorname{ad} e = I + \operatorname{ad} e + \frac{1}{2} (\operatorname{ad} e)^2$ *and:*

$$\exp \operatorname{ad} e(x) \;=\; x + (ex - xe) - exe = (I + e)\, x\, (I - e) = (\exp e)\, x\, (\exp e)^{-1}.$$

(6.5) Example *Let $e^3 = 0$. Then $\exp e = I + e + \frac{1}{2}e^2$. Moreover:*

$$\begin{aligned} &\operatorname{ad} e : x \mapsto ex - xe \\ &(\operatorname{ad} e)^2 : x \mapsto [e, ex - xe] = e^2x - 2exe + xe^2 \\ &(\operatorname{ad} e)^3 : x \mapsto [e, e^2x - 2exe + xe^2] = -3e^2xe + 3exe^2 \\ &(\operatorname{ad} e)^4 : x \mapsto [e, -3e^2xe + 3exe^2] = 6e^2xe^2 \\ &(\operatorname{ad} e)^5 : x \mapsto [e, 6e^2xe^2] = 0. \end{aligned}$$

Thus $\exp \operatorname{ad} e = I + \operatorname{ad} e + \frac{1}{2} (\operatorname{ad} e)^2 + \frac{1}{6} (\operatorname{ad} e)^3 + \frac{1}{24} (\operatorname{ad} e)^4$ *and*

$$\exp \operatorname{ad} e(x) \;=\; x + (ex - xe) + \left(\frac{1}{2}e^2x - exe + \frac{1}{2}xe^2\right) - \frac{1}{2}\left(e^2xe - exe^2\right) + \frac{1}{4}e^2xe^2 =$$

$$\left(I + e + \frac{1}{2}e^2\right) x \left(I - e + \frac{1}{2}e^2\right) \;=\; (\exp e)x\, (\exp e)^{-1}.$$

7 Groups of Lie type

Let L be a simple Lie algebra over $\mathbb{C}$, with Chevalley basis as in (5.1):

$$\{h_r \mid r \in \Pi\} \cup \{e_r \mid r \in \Phi\}\,.$$

For all $r \in \Phi$ and for all $t \in \mathbb{C}$, we set

$$x_r(t) := \exp\ (t\,ad)\ e_r \qquad (7.1)$$

(7.2) Definition *The Lie group $L(\mathbb{C})$ is the subgroup of* $\mathrm{Aut}(L)$ *generated by the automorphisms* (7.1)*, namely the group:*

$$L(\mathbb{C})\ :=\ \langle x_r(t) \mid t \in \mathbb{C},\ r \in \Phi\rangle.$$

Since the structure constants are integers, it is possible to define a Lie algebra $\mathbb{F}\otimes_{\mathbb{Z}}L = L_{\mathbb{F}}$ over any field $\mathbb{F}$. The matrix representing $x_r(t)$ with respect to a Chevalley basis has entries of the form at^i where $a \in \mathbb{Z}$ and $i \in \mathbb{N}$. Interpreting a as an element of $\mathbb{F}$, one can identify $x_r(t)$ with an element of $\mathrm{Aut}\,(L_{\mathbb{F}})$ and define the group $L(\mathbb{F})$ as

$$L(\mathbb{F})\ :=\ \langle x_r(t) \mid t \in \mathbb{F},\ r \in \Phi\rangle \quad \text{(the group of type } L \text{ over } \mathbb{F}).$$

The identifications are as follows (see Section 3):

- $\mathbf{A}_\ell(\mathbb{F}) \cong \mathrm{PSL}_{\ell+1}(\mathbb{F})$;
- $\mathbf{B}_\ell(\mathbb{F}) \cong \mathrm{P}\Omega_{2\ell+1}(\mathbb{F}, f)$ where f is the quadratic form: $x_0^2 + \sum_{i=1}^{\ell} x_i x_{-i}$;
- $\mathbf{C}_\ell(\mathbb{F})(\mathbb{F}) \cong \mathrm{PSp}_{2\ell}(\mathbb{F})$;
- $\mathbf{D}_\ell(\mathbb{F}) \cong \mathrm{P}\Omega_{2\ell}(\mathbb{F}, f)$ where f is the quadratic form: $\sum_{i=1}^{\ell} x_i x_{-i}$.
- $^{\mathbf{2}}\mathbf{A}_\ell(\mathbb{F}) \cong \mathrm{PSU}_{\ell+1}(\mathbb{F})$;
- $^{\mathbf{2}}\mathbf{D}_\ell(\mathbb{F}) \cong P\Omega_{2\ell}(\mathbb{F}_0, f)$ where $\mathbb{F}$ has an automorphism σ of order 2, with fixed field $\mathbb{F}_0$, and f is the form $\sum_{i=1}^{\ell-1} x_i x_{-i} + (x_\ell - \alpha x_{-\ell})\,(x_\ell - \alpha^\sigma x_{-\ell})$, $\alpha \in \mathbb{F} \setminus \mathbb{F}_0$.

The consideration of groups of Lie type allows a unified treatment of important classes of groups, like finite simple groups. According to the Classification Theorem, every finite simple group S is isomorphic to one of the following:

- a cyclic group C_p, of prime order p;

- an alternating group $\mathrm{Alt}(n)$, $n \geq 5$;
- a group of Lie type $L(\mathbb{F}_q)$, where L is one of the algebras in (4.1);
- a twisted group of Lie type ${}^iL(\mathbb{F}_q)$, namely the subgroup of $L(\mathbb{F}_{q^i})$ consisting of the elements fixed by an automorphism of order i of $L(\mathbb{F}_{q^i})$;
- one of the 26 sporadic simple groups.

8 Uniform definition of certain subgroups

Let L be a simple Lie algebra over $\mathbb{C}$, with Cartan decomposition

$$L = \mathcal{H} \oplus \bigoplus_{r \in \Phi \subseteq \mathcal{H}} \mathbb{C}e_r.$$

We describe some kinds of important subgroups, which may be defined in a uniform way.

8.1 Unipotent subgroups

For each $r \in \Phi$, the map

(8.1) $$t \mapsto x_r(t) :== \exp\left(t \operatorname{ad} e_r\right)$$

is a monomorphism from the additive group $(\mathbb{F}, +)$ into the multiplicative group $L(\mathbb{F})$.

(8.2) Definition

- *The image of the monomorphism* (8.1) *is denoted by* X_r *and called the* radical subgroup *corresponding to the root r;*
- *the subgroup generated by all radical subgroups corresponding to positive roots is denoted by* U^+ *;*
- *the subgroup generated by all radical subgroups corresponding to negative roots is denoted by* U^-*.*

Thus:

$$\begin{aligned} X_r &= \{x_r(t) \mid t \in \mathbb{F}\} \simeq (\mathbb{F}, +) \\ U^+ &= \langle x_r(t) \mid t \in \mathbb{F}, r \in \Phi^+ \rangle \\ U^- &= \langle x_r(t) \mid t \in \mathbb{F},\ r \in \Phi^- \rangle . \end{aligned}$$

U^+, U^- (and their conjugates in $L(\mathbb{F})$) are called *unipotent* subgroups. By definition

$$L(\mathbb{F}) = \left\langle U^+,\ U^- \right\rangle .$$

(8.3) Example *In* $A_\ell(\mathbb{F})$ *identified with* $\mathrm{PSL}_{\ell+1}(\mathbb{F})$*:*

- *X_r is the projective image of the group $\{I + te_{i,j} \mid t \in \mathbb{F}\}$ for some $i \neq j$,*
- *U^+ is the projective image of the subgroup of upper unitriangular matrices,*
- *U^- is the projective image of the subgroup of lower unitriangular matrices.*

8.2 The subgroup $\langle X_r, X_{-r}\rangle$

For each $r \in \Phi$, the group $\langle X_r, X_{-r}\rangle$ fixes every vector of the Chevalley basis (5.1) except e_r, h_r, e_{-r}. Multiplying e_r by an appropriate scalar, if necessary, we may assume:

- $x_r(t)\,(e_r) = e_r$;
- $x_r(t)\,(h_r) = h_r - 2t\,e_r$;
- $x_r(t)\,(e_{-r}) = -t^2\,e_r + t\,h_r + e_{-r}$;
- $x_{-r}(t)\,(e_r) = e_r - th_r - t^2 e_{-r}$;
- $x_{-r}(t)\,(h_r) = h_r + 2t\,e_r$;
- $x_{-r}(t)\,(e_{-r}) = e_{-r}$.

(8.4) Theorem *There exists an epimorphism $\varphi_r : \mathrm{SL}_2(\mathbb{F}) \to \langle X_r, X_{-r}\rangle$ under which:*

(8.5) $$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_r(t), \qquad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-r}(t).$$

Proof The group $\mathrm{SL}_2(\mathbb{F})$ has a matrix representation of degree 3, deriving from its action on the space of homogeneous polynomials of degree 2 over $\mathbb{F}$ in the indeterminates x, y. With respect to the basis $-x^2, 2xy, y^2$, we have:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -2t & -t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ -t^2 & 2t & 1 \end{pmatrix}.$$

These are the matrices of the action of $x_r(t)$ and $x_{-r}(t)$ restricted to $\langle e_r, r, e_{-r}\rangle$ by the formulas before the statement. ■

8.3 Diagonal and monomial subgroups

In $\mathrm{SL}_2(\mathbb{F})$, for all $\lambda \in \mathbb{F}$ we have:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda^{-1}-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda-1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\lambda^{-1} \\ 0 & 1 \end{pmatrix}.$$

Hence, for all $r \in \Phi$ and all $\lambda \in \mathbb{F}$ we set:

$$h_r(\lambda) := \varphi_r\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right) = x_{-r}(\lambda^{-1}-1)\; x_r(1)\; x_{-r}(\lambda-1)\; x_r(-\lambda^{-1}).$$

(8.6) Definition *The* diagonal *subgroup H of $L(\mathbb{F})$ is defined by*

$$H \;:=\; \langle h_r(\lambda) \mid 0 \neq \lambda \in \mathbb{F},\; r \in \Phi\rangle. \tag{8.7}$$

The group H normalizes both U^+ and U^-.

(8.8) Definition *The product U^+H is called a* Borel subgroup *and is denoted by B^+. Similarly the product U^-H is denoted by B^-.*

(8.9) Example *Identifying $\mathbf{A}_\ell(\mathbb{F})$ with the projective image of $\mathrm{SL}_{\ell+1}(\mathbb{F})$:*

- *B^+ is the image of the group of upper triangular matrices of determinant* 1,
- *B^- is the image of the group of lower triangular matrices of determinant* 1.

In $\mathrm{SL}_2(\mathbb{F})$ we have:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Hence, for all $r \in \Phi$ we set:

$$n_r = \varphi_r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = x_{-r}(-1)\; x_r(1)\; x_{-r}(-1).$$

(8.10) Definition *The (standard)* monomial *subgroup N of $L(\mathbb{F})$ is defined by:*

$$N \;:=\; \langle h_r(\lambda), n_r \mid r \in \Phi, \lambda \in \mathbb{F}\rangle\,. \tag{8.11}$$

H is a normal subgroup of N.

(8.12) Definition *The factor group $W(L) := \frac{N}{H}$ is called the* Weyl group *of L.*

$$\begin{aligned} W(\mathbf{A}_\ell) &\simeq \mathrm{Sym}\,(\ell+1)\,, \\ W(\mathbf{C}_\ell) &\simeq W(\mathbf{B}_\ell) \simeq\ C_2^\ell\ \mathrm{Sym}\,(\ell)\,, \\ W(\mathbf{D}_\ell) &\simeq C_2^{\ell-1}\ \mathrm{Sym}\,(\ell)\,. \end{aligned}$$

(8.13) Example *In the orthogonal algebra B_1 over $\mathbb{C}$, with $\Phi = \{r, -r\}$ and basis*

$$h_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad e_r = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{pmatrix}, \quad e_{-r} = \begin{pmatrix} 0 & 0 & -\sqrt{2} \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we have:

$$x_r(t) = I + te_r + \frac{t^2}{2}e_r^2 = \begin{pmatrix} 1 & \sqrt{2}\,t & 0 \\ 0 & 1 & 0 \\ -\sqrt{2}\,t & -t^2 & 1 \end{pmatrix}; \quad x_{-r}(t) = x_r(t)^T;$$

$$h_r(\lambda) =\ x_{-r}(\lambda^{-1}-1)\ x_r(1)\ x_{-r}(\lambda-1)\ x_r(-\lambda^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{-2} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix};$$

$$n_r = x_r(1)x_{-r}(-1)x_r(1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix};$$

$$h_{-r}(\lambda) = h_r(\lambda)^{-1}, \quad n_r = n_r^{-1};$$

$$H = \langle h_r(\lambda) \mid r \in \Phi,\ \lambda \in \mathbb{C}^* \rangle = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu^{-1} \end{pmatrix} \mid \mu \in \mathbb{C}^* \right\};$$

$$N = \langle h_r(\lambda), n_r \mid r \in \Phi\ \lambda \in \mathbb{C}^* \rangle = \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \mu^{-1} \\ 0 & \mu & 0 \end{pmatrix} \mid \mu \in \mathbb{C}^* \right\};$$

$$W = \frac{N}{H} \cong \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \cong \mathrm{Sym}(2).$$

(8.14) Example *Identifying $\mathbf{A}_\ell(\mathbb{F})$ with the projective image of* $\mathrm{SL}_{\ell+1}(\mathbb{F})$*:*

- *H is the image of the subgroup of diagonal matrices of determinant 1;*
- *N is the image of the subgroup of* monomial *matrices of determinant 1;*
- *the factor group $\frac{N}{H}$ is isomorphic to the symmetric group* $\mathrm{Sym}(\ell+1)$.

9 Exercises

(9.1) Exercise *Let $\varphi : L \to L'$ be a homomorphism of Lie algebras. Show that its kernel is an ideal.*

(9.2) Exercise *Let L be a Lie algebra and $x \in L$. Show that the map $\operatorname{ad} x$ is a derivation.*

(9.3) Exercise *Write a basis of $\mathbf{C}_2$ and a basis of $\mathbf{C}_3$.*

(9.4) Exercise *Show that $\mathbf{C}_\ell(\mathbb{F})$ is a Lie subalgebra of $\mathcal{GL}_{2\ell}(\mathbb{F})$.*

(9.5) Exercise *Write a basis of $\mathbf{B}_1$ and a basis of $\mathbf{B}_2$.*

(9.6) Exercise *Write a basis of $\mathbf{D}_2$.*

(9.7) Exercise *Verify formula* (6.3) *assuming $\mathrm{e}^4 = 0$.*