CHAPTER 4: PROOF OF THE REGULARITY THEOREM

We are now in a position to complete the proof of Theorem 1.9.

We split the demonstration into three steps. First we treat the regularity of the reduced boundary of a set with almost minimal boundary, then we consider sequences of sets with uniformly almost minimal boundaries, and finally we discuss the Hausdorff dimension of the singular points.

A general remark is in order: since the conclusions of Theor. 1.9 are of local character, it is clear that, given a set E with almost minimal boundary in Ω , we can restrict our analysis to a (sufficiently small) neighbourhood of on arbitrary point of Ω (actually, the only interesting case is when that point is in $\partial E \cap \Omega$). Our main assumption will then be

$$\psi(E,B_{x_o,t_o}) \leq \alpha(t)\cdot t^{n-1} \qquad \forall x \in B_{x_o,T_o} \;, \; \forall t\in(0,T_o)$$

with $\alpha(t)$ as in section 1.11. See also the remark in section 3.4.

<u>*Step 1.*</u> Given $n \geq 2$, α as in 1.11, and τ satisfying $0<\tau<\min\{2^{-4},1/2c_2\}$ where c_2 is the constant appearing in (3.47), we indicate by $\sigma^* \in (0,1)$ the constant whose existence is granted by the Main Lemma 3.6.

Let now $E \subset \mathbb{R}^n$, $x_o \in \partial E$, $R_o \in (0,1)$, and $\sigma_o \in (0,\sigma^*]$ be such that:

(4.1) $$\int_o^{R_o} t^{-1}\alpha(t)dt \leqq \omega_{n-1}/2(n-1)$$

(4.2) $$\alpha(R_o) \leqq \sigma_o\, \tau^n/4\, c_1$$

(4.3) $$\psi(E,B_{x,t}) \leqq \alpha(t)\cdot t^{n-1} \qquad \forall x \in B_{x_o,R_o} \text{ and } \forall t\in(0,R_o)$$

$$\omega(E,B_{x_o,R_o}) \leqq \sigma_o R_o^{n-1} \tag{4.4}$$

(Roughly speaking, we are assuming that the excess is small, on a (small) initial ball in which ∂E is almost minimal. Applying the Main Lemma iteratively, we first show that for every integer $h \geq 0$ it holds:

$$\omega(E,B_{x_o,R_h}) \leqq \sigma_h R_h^{n-1} \tag{4.5}$$

where:

$$R_h = \tau^h R_o$$

$$\sigma_h = c_3 \sum_{i=1}^{h} c_4^{i-1} \alpha(R_{h-i}) + c_4^h \sigma_o \tag{4.6}$$

$$c_3 = c_1 \tau^{1-n}, \quad c_4 = c_2 \tau^2$$

and c_1, c_2 are as in (3.47).

In fact, (4.5) reduces simply to (4.4) when $h = 0$. Assuming that (4.5) holds for a certain $h \geqq 0$, and setting

$$F_h = R_h^{-1}(E-x_o), \quad \beta_h(t) = \alpha(R_h t) \quad \text{for} \quad t \in (0,1]$$

we find from (4.3) and (4.5);

$$\psi(F_h,B_{x,t}) \leqq \beta_h(t) \cdot t^{n-1} \qquad \forall x \in B_1, \forall t \in (0,1)$$

$$\omega(F_h,B_1) \leqq \sigma_h$$

Clearly, $\beta_h \prec \alpha$ (section 3.5), while $\sigma_h \leqq \sigma_o \leqq \sigma^*$ $\forall h$: for, if $h \geqq 0$, then

$$\tau^i \alpha(R_{h-i}) \leqq \alpha(R_h) \qquad \forall i = 0,\dots,h$$

since $t^{-1}\alpha(t)$ is non-increasing on $(0,1)$ (recall (α_3) of 1.11); hence, from (4.6) we obtain

$$(4.7)\qquad \sigma_h \leqq c_3\tau^{-1}\alpha(R_h)\sum_{i=1}^{h}(c_2\tau)^{i-1}+c_4^h\sigma_o \leqq 2c_1\tau^{-n}+\tau^h\sigma_o \leqq \sigma_o \quad \forall h \geqq 0,$$

according to (4.2), and our initial assumption $\tau < 1/2c_2$. We are then precisely in the situation covered by the Main Lemma 3.6, from which we derive

$$\omega(F_h,B_\tau) \leqq c_1\beta_h(1)+c_2\sigma_h\tau^{n+1} = (c_3\alpha(R_h)+c_4\sigma_h)\tau^{n-1} = \sigma_{h+1}\tau^{n-1}$$

according to (4.6). In conclusion, we find

$$\omega(E,B_{x_o,R_{n+1}}) \leqq \sigma_{h+1}\ R_{h+1}^{n-1}$$

which is exactly (4.5), with h+1 in place of h.

Next, we show that in the hypotheses (4.1)-(4.4), $x_o \in \partial^*E$. To this aim, we observe that from (3.25) and for every $h,k \geqq 0$:

$$|\nu(E,B_{x_o,R_{h+k}}) - \nu(E,B_{x_o,R_h})| \leqq \sum_{i=o}^{k-1}|\nu(E,B_{x_o,R_{h+i+1}})-\nu(E,B_{x_o,R_{h+i}})|$$

$$(4.8)\qquad \leqq 2\sum_{i=o}^{k-1}\left[\frac{\omega(E,B_{x_o,R_{h+i}})}{|D\phi_E|\ (B_{x_o,R_{h+i+1}})}\right]^{1/2}$$

$$\leqq 2^{3/2}(\omega_{n-1}\tau^{n-1})^{-1/2}\cdot\sum_{i=o}^{k-1}\sigma_{h+i}^{1/2}$$

by virtue of (4.5),(3.31),and (4.1). See section 2.10.

According to (4.7), we have:

$$\sigma_{h+i} \leqq 2c_1 \ \tau^{-n}\alpha(R_{h+i}) + \tau^{h+i} \ \sigma_o$$

whence

$$\sum_{i=o}^{k-1} \sigma_{h+i}^{1/2} \leqq (2c_1\tau^{-n})^{\frac{1}{2}} \cdot \sum_{i=o}^{k-1} \alpha^{1/2}(R_{h+i}) + (\tau^h\sigma_o)^{\frac{1}{2}} \cdot \sum_{i=o}^{k-1} {}^{i/2}$$

(4.9)
$$\leqq (2c_1\tau^{-n})^{\frac{1}{2}} \cdot (1-\tau)^{-1} \cdot \int_{R_{h+k}}^{R_h} t^{-1} \ \alpha^{1/2}(t)dt + 2\tau^{h/2}$$

since $t^{-1}\alpha^{1/2}(t)$ is also non-increasing, by (α_3) of 1.11. By the same reason, we have also:

(4.10) $$\int_o^{R_h} t^{-1}\alpha^{1/2}(t)dt \leqq \tau^{-k}\int_o^{R_{h+k}} t^{-1}\alpha^{1/2}(t)dt \qquad \forall h,k \geqq 0.$$

Thus, substitution of (4.9) into (4.8) yields, for every $h,k \geqq 0$:

(4.11)
$$|\nu(E,B_{x_o,R_{h+k}})-\nu(E,B_{x_o,R_h})| \leqq 4(c_1/\omega_{n-1})^{\frac{1}{2}} \cdot \tau^{1/2-n} \cdot (1-\tau)^{-1} \cdot \int_o^{R_h} t^{-1}\alpha^{1/2}(t)dt + 2^{5/2}(\omega_{n-1}\tau^{n-1})^{-\frac{1}{2}} \cdot \tau^{h/2}$$

which shows that $\{\nu(E,B_{x_o,R_h})\}$ is a Cauchy sequence. Calling ν its limit, we find

$$0 \leqq 1 - |\nu| = \lim_{h\to+\infty} \frac{\omega(E,B_{x_o,R_h})}{|D\phi_E|(B_{x_o,R_h})} \leqq \lim_{h\to+\infty} (2\omega_{n-1}^{-1}\,\sigma_h) = 0$$

by (4.5),(3.31),(4.1) and (4.7).

Now, let $t \in (0,R_o)$, and call $h = h(t)$ the unique, non-negative integer, for wich

$$R_{h+1} \leqq t < R_h$$

Arguing as above (see in particular (4.8), (4.9),(4.10), and (4.11)), we find

(4.12)

$$|\nu-\nu(E,B_{x_o,t})| \leqq |\nu-\nu(E,B_{x_o,R_h})| + 2\left[\frac{\omega(E,B_{x_o,R_h})}{|D\phi_E|(B_{x_o,R_{h+1}})}\right]^{1/2}$$

$$\leqq 4(c_1/\omega_{n-1})^{\frac{1}{2}}.(2-\tau).\tau^{1/2-n}.(1-\tau)^{-1}.\int_o^{R_h} r^{-1}\alpha^{1/2}(r)dr+3.2^{3/2}\tau^{n-1})^{-\frac{1}{2}}\tau^{h/2}$$

$$\leqq c_5 \int_o^{R_{h+1}} r^{-1}\alpha^{1/2}(r)dr + c_6\,\tau^{(h+1)/2}$$

$$\leqq c_5 \int_o^{t} r^{-1}\alpha^{1/2}(r)dr + c_6(t/R_o)^{\frac{1}{2}},$$

where c_5,c_6 depend only on n and τ.

In conclusion, see (3.2), we have $\nu = \nu_E(x_o)$, i.e. $x_o \in \partial^*E$ as claimed. Similarly, in the same hypotheses (4.1)-(4.4) we can prove that $\partial E = \partial^*E$ in a neighborhood of x_o.

For, let $N \geqq 1$ be such that $\sigma_N \leqq \sigma_o \tau^{n-1}$ (see (4.7)), and set

$$\delta = (1-\tau)\tau^N R_o < R_o$$

Then, for every $x \in B_{x_o,\delta}$ we have $B_{x,\tau^{N+1}R_o} \subset B_{x_o,\tau^N R_o}$, whence:

$$(\tau^{N+1}R_o)^{1-n} \cdot \omega(E,B_{x,\tau^{N+1}R_o}) \leqq (\tau^N R_o)^{1-n} \cdot \tau^{1-n} \cdot \omega(E,B_{x_o,\tau^N R_o}) \leqq \tau^{1-n}\sigma_N \leqq \sigma_o$$

by virtue of (4.5). Accordingly, we are again in the situation considered at the very beginning of Step 1, i.e. (4.1)-(4.4) all hol with x_o and R_o replaced by any $x \in B_{x_o,\delta} \cap \partial E$ and, respective

$R = \tau^{N+1} R_o < R_o$. It follows from the preceding discussion that

$x \in \partial^* E$, for any such x. Moreover, see (4.12), for every $x \in \partial E \cap B_{x_o,\delta}$

and every $t \in (0,R)$, we have:

$$(4.13) \qquad |\nu_E(x) - \nu(E,B_{x,t})| \leqq c_5 \int_0^t r^{-1}\alpha^{1/2}(r)dr + c_6(t/R)^{1/2}$$

Using (4.13), we can easily show that ν_E varies smoothly on ∂E near x_o. To this aim, we put $\delta_1 = \tau^2 R/2 < \delta/2$ and, given $x,y \in \partial E \cap B_{x_o,\delta_1}$

with $x \neq y$, we denote by h the unique, positive integer for which

$$(4.14) \qquad \tau^{h+2} R \leqq |x-y| < \tau^{h+1} R.$$

Then we define $s = (1-\tau)\cdot\tau^h R$, $t = \tau^h R$, so that $B_{x,s} \subset B_{y,t}$. It

follows from (3.25) that

$$|\nu(E,B_{x,s}) - \nu(E,B_{y;t})| \leq 2 \left[\frac{\omega(E,B_{y,t})}{|D\phi_E|(B_{x,s})}\right]^{1/2}$$

Hence, repeating the preceding argument, and using (4.13), we get

$$|\nu_E(x) - \nu_E(y)| \leq c_7 \int_o^t r^{-1}\alpha^{1/2}(r)dr + c_8(t/R)^{1/2}$$

where, as usual, c_7 and c_8 depend only on n and τ. Finally, recalling that $t = \tau^h R$, we find from (4.10) and (4.14):

$$|\nu_E(x) - \nu_E(y)| \leq c_7\tau^{-2} \int_o^{|x-y|} r^{-1}\alpha^{1/2}(r)dr + c_8\tau^{-1}(|x-y|/R)^{\frac{1}{2}}$$

wich proves the continuity of the normal vector ν_E on $\partial E \cap B_{x_o,\delta_1}$. In particular, when $\alpha(t) \leq \text{const.} \cdot t^\alpha$ for $\alpha \in (0,1)$, we obtain that ν_E is of class $C^{o,\alpha/2}$ (see also section 1.12).

To conclude with the first part of the Regularity Theorem, we have only to show that in the case when ∂E is almost minimal in Ω and $x_o \in \partial^* E \cap \Omega$, then it is possible to pick R_o and σ_o such that (4.1)-(4.4) all hold. This is certainly true, because of almost minimality (see sections 1.5, 1.11, and 1.13), and since $t^{1-n}\omega(E,B_{x,t})$ tends to zero as $t \to o^+$, whenever $x \in \partial^* E$ (recall (2.26)).

Step 2. Now, given α as in 1.11, $T_o \in (0,1)$, and $x_o \in \mathbb{R}^n$, we suppose that

(4.15) $\quad \psi(E_h, B_{x,t}) \leq \alpha(t) \cdot t^{n-1} \quad \forall x \in B_{x_o,T_o}, \ \forall t \in (0,T_o), \ \forall h \geq 1$

Moreover, we assume $E_h \to E_\infty$ on $B_{x_o,2T_o}$. If $x_h \in \partial E_h$ and $x_h \to x_\infty \in B_{x_o,T_o}$, then clearly $B_{x_h,r} \cap E_h \to B_{x_\infty,r} \cap E_\infty$ $\forall r \in (0,d)$, with $d = T_o - |x_o - x_\infty|$. Furthermore, $B_{x_h,r} \subset B_{x_\infty,d}$ whenever $r < d/2$ and h is large enough. From (3.32) we get immediately $x_\infty \in \partial E_\infty$, as required. Next, we assume $x_\infty \in \partial^* E_\infty$, and fix τ and σ^* as in Step 1. Reasoning possibly on subsequences of $\{E_h\}$, we can choose $r \in (0,d)$ and $h_o \geqq 1$ such that $\forall h \geqq h_o$:

$$\int_o^r t^{-1}\alpha(t)dt \leqq \omega_{n-1}/2(n-1)$$

$$\alpha(r) \leqq \sigma^* \tau^n / 4c_1$$

$$r^{1-n}\omega(E_\infty, B_{x_\infty,r}) \leqq 2^{-n-1}\,\sigma^* \tag{4.16}$$

$$|x_h - x_\infty| < r/2$$

$$r^{1-n} \cdot \int_{\partial B_{x_\infty,r}} |\phi_{E_h} - \phi_{E_\infty}| dH_{n-1} \leqq 2^{-n-2}\,\sigma^*$$

As a consequence of the almost minimality of ∂E_h, we derive from (4.16) and (3.17)

$$r^{1-n}\omega(E_h, B_{x_\infty,r}) \leqq 2^{1-n}\sigma^* \quad \text{and} \quad B_{x_h,r/2} \subset B_{x_\infty,r} \quad \forall h \geqq h_o.$$

Hence:

$$\omega(E_h, B_{x_h,r/2}) \leqq \sigma^* \cdot (r/2)^{n-1}$$

by virtue of the monotonicity of ω . Thus, for every $h \geqq h_o$, we see that $E_h, x_h, r/2$, and σ^* are precisely in the situation already discus sed at the beginning of Step 1: we get, in particular, $x_h \in \partial^* E_h \quad \forall h \geqq h_o$, while (see (4.13)):

$$(4.17) \quad |\nu_{E_h}(x_h) - \nu(E_h, B_{x_h,t})| \leqq c_5 \int_0^t s^{-1}\alpha^{1/2}(s)\,ds + c_6(2t/r)^{\frac{1}{2}} \quad \forall h \geqq h_o,\ \forall t \in (0, r/2)$$

Similarly, observing that E_∞ is also almost minimal (because of (4.15) and (3.12)), we obtain

$$(4.18) \quad |\nu_{E_\infty}(x_\infty) - \nu(E_\infty, B_{x_\infty,t})| \leqq c_5 \int_0^t s^{-1}\alpha^{1/2}(s)\,ds + c_6(2t/r)^{\frac{1}{2}} \quad \forall t \in (0, r/2).$$

Moreover, it si not difficult to show that

$$(4.19) \quad \limsup_{h \to +\infty} |\nu(E_h, B_{x_h,t}) - \nu(E_\infty, B_{x_\infty,t})| \leqq c_9 \alpha(t) \text{ for a.e. } t \in (0, r/2).$$

This follows e.g. by inserting

$$\frac{D\phi_{E_h}(B_{x_\infty,t})}{|D\phi_{E_h}|(B_{x_h,t})}, \quad \nu(E_h, B_{x_\infty,t}), \quad \text{and} \quad \frac{D\phi_{E_\infty}(B_{x_\infty,t})}{|D\phi_{E_h}|(B_{x_\infty,t})}$$

as intermediate points between $\nu(E_h, B_{x_h,t})$ and $\nu(E_\infty, B_{x_\infty,t})$, and then by using (3.19),(3.16) and almost minimality to estimate the four partial distances.

Combining (4.17),(4.18), and (4.19) we get immediately the convergence of $\nu_{E_h}(x_h)$ toward $\nu_{E_\infty}(x_\infty)$.

As a by-product of the preceding discussion, we obtain that whenever the open se A contains the singular points of ∂E_∞, then it also contains the singular points of ∂E_h, for h large enough. More precisely, denoting by Σ_h the singular set $\partial E_h \setminus \partial^* E_h$, from the assumption

$$\Sigma_\infty \cap K \subset A$$

(K compact $\subset B_{x_o, T_o}$), we derive immediately that

$$\Sigma_h \cap K \subset A$$

for every sufficiently large h. This, in turn, implies that

$$H_s^\infty(\Sigma_\infty \cap K) \geq \limsup_{h \to +\infty} H_s^\infty(\Sigma_h \cap K) \tag{4.20}$$

where, for every real $s \geq 0$ and every $X \subset \mathbb{R}^n$ we de

$$H_s^\infty(X) = \omega_s 2^{-s} \inf\{\sum_{i=1}^\infty (\operatorname{diam} A_i)^s : A_i \text{ open, } X \subset \bigcup_{i=1}^\infty A_i\}$$

(see [13], p. 767, and [27], 2.6.4).

We end this part by recalling two general facts concerning H_s^∞ (see [12], 2.10.2 and 2.10.19 (2), and [27], 2.6.4):

$$H_s^\infty(X) = 0 \qquad \text{if and only if} \qquad H_s(X) = 0 \tag{4.21}$$

$$\limsup_{t \to 0^+} \omega_s^{-1}\, t^{-s}\, H_s^\infty(X \cap B_{x,t}) \geq 2^{-s} \quad \text{for } H_s\text{-a.e. } x \in X. \tag{4.22}$$

Step 3. To conclude the proof of the Regularity Theorem, we have only to show that $H_s(\Sigma_E \cap \Omega) = 0$, whenever E has almost minimal boundary in $\Omega \subset \mathbb{R}^n$ and $s > n - 8$, with:

$$\Sigma_E = \partial E \setminus \partial^* E.$$

This follows easily by "blowing-up" at singular points (see the final part of Prop. 3.4), and then by using known results concerning the existence and non-existence of singular minimal cones in $\mathbb{R}^n$, for which we refer the reader to [27], sections 2.6 and 2.7.

By (4.22), assuming that E satisfies:

$$(4.23) \qquad \psi(E,B_{x,t}) \le \alpha(t)\cdot t^{n-1} \qquad \forall x \in B_{x_o,T_o}\ ,\ \forall t \in (0,T_o)$$

and that

$$(4.24) \qquad H_s(\Sigma_E \cap B_{x_o,T_o}) > 0,$$

we can choose $x \in \Sigma_E \cap B_{x_o,T_o}$ and a sequence $\{t_h\}$, satisfying

$$(4.25) \qquad t_n \downarrow 0 \quad \text{and} \quad H_s^\infty(\Sigma_E \cap B_{x,t_h}) \geq \omega_s 2^{-s-1} t_h^s \quad \forall h.$$

Setting $E_h = t_h^{-1}(E-x)$, and passing to a subsequence if necessary, we find (in view of Prop. 3.4) that $\{E_h\}$ converges to a minimal cone $C_o \subset \mathbb{R}^n$, for which

$$(4.26) \qquad H_s^\infty(\Sigma_{C_o} \cap B_1) > 0\ ,$$

by virtue of (4.20) and (4.25). This way, starting from a set $E \subset \mathbb{R}^n$ with almost minimal boundary (see (4.23)) and satisfying (4.24), we obtain a *minimal* cone C_o with the same property, namely:

$$(4.27) \qquad C_o \subset \mathbb{R}^n \quad \text{and} \quad H_s(\Sigma_{C_o} \cap B_1) > 0$$

(see 4.26) and (4.21)). Now, it is well known that minimal cones in $\mathbb{R}^n$ have smooth boundary up to dimension 7 (included). Therefore, if (4.24) holds for a certain $s \geqq 0$, then necessarily $n \geqq 8$.

On the account of Simon's cone $C \subset \mathbb{R}^8$ (see 1.4), we see that (4.27) may really hold, when $n = 8$ and $s = 0$.

On the other hand, if (4.27) holds with $s > 0$, then we can repeat the above procedure, blowing-up ∂C_o near a singular point different from the vertex, thus getting a minimal *cylinder* $Q = C_1 \times \mathbb{R}$, with the property that $H_s(\Sigma_Q) > 0$. In such a case however, the transversal section C_1 of Q would likewise be a minimal cone in $\mathbb{R}^{n-1}$, with in addition:

$$H_{s-1}(\Sigma_{C_1}) > 0 .$$

An easy induction then shows, that if (4.24) holds with $s \geqq m$ (m a non-negative integer), then there exists a minimal cone $C_m \subset \mathbb{R}^{n-m}$, satisfying

$$H_{s-m}(\Sigma_{C_m}) > 0.$$

From the preceding discussion, we see that (4.24) implies $s \leqq n-8$. In view of the preceding considerations, this concludes the proof of the Regularity Theorem.