Chapter 5

Quantitative approximation of cosine functions

In the preceding chapters we have studied the possibility of approximating the solutions of a suitable parabolic problems using semigroup's theory. A similar approach can be used for the representation of the solutions of suitable hyperbolic problems using the generation of a cosine function (see [50] and [68] for more details on this approach).

Here we are interested to a cosine version of Trotter's theorem on the approximation of C_0-semigroups and we give a general quantitative estimate of the convergence of the iterates of a sequence of trigonometric polynomials. Moreover we introduce some suitable sequences of linear operators approximating the resolvent operators associated with the generator of the cosine functions. Some applications to particular sequences of classical trigonometric polynomials are also furnished.

The results in this chapter are collected in [43].

5.1 Approximation processes for cosine functions

First, we establish a cosine version of Trotter's approximation theorem [70, Theorem 5.3] and provide a quantitative estimate of the convergence. A partial result on the generation of cosine functions is also stated in [35, Theorem 1.2] without quantitative estimates.

Theorem 5.1.1 *Let E be a Banach space, let $(L_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ be two sequences of linear operators from E in itself and assume that there exists $M \geq 1$ and $\omega \geq 0$ such that*

$$\|L_n^k\| \leq M\, e^{\omega\, k/n}\,, \quad \|M_n^k\| \leq M\, e^{\omega\, k/n}\,, \qquad n, k \geq 1\,. \tag{5.1.1}$$

Moreover, assume that D is a dense subspace of E such that, for every

$u \in D$ and $n \geq 1$, we have

$$\|n(L_n u - u)\| \leq \varphi_n(u) \ , \qquad \|n(M_n u - u)\| \leq \varphi_n(u) \ , \tag{5.1.2}$$

and the following estimates of the Voronovskaja-type formula hold

$$\|n(L_n u - u) - Au\| \leq \psi_n(u) \ , \qquad \|n(M_n u - u) + Au\| \leq \psi_n(u) \ , \tag{5.1.3}$$

where $A : D \to E$ is a linear operator on E and $\varphi_n, \psi_n : D \to [0, +\infty[$ are seminorms on the subspace D such that $\lim_{n\to\infty} \psi_n(u) = 0$ for every $u \in D$.

If $(\lambda - A)(D)$ is dense in E for some $\lambda > \omega$, then the square A^2 of the closure of (A, D) generates a cosine function $(C(t))_{t\in\mathbb{R}}$ in E and, for every $t \geq 0$,

$$C(t) = \frac{1}{2} \lim_{n\to\infty} \left(L_n^{k(n)} + M_n^{k(n)} \right) \ , \tag{5.1.4}$$

where $(k(n)_n)_{n\in\mathbb{N}}$ is a sequence of positive integers such that $\lim_{n\to+\infty} k(n)/n = t$ (in particular, we can take $k(n) = [n\,t]$). Consequently, for every $t \in \mathbb{R}$, we have $\|C(t)\| \leq M\, e^{\omega\,|t|}$.

Moreover, for every $t \geq 0$ and for every increasing sequence $(k(n))_{n\geq 1}$ of positive integers and $u \in D$, we have

$$\begin{aligned} \left\| C(t)u - \frac{1}{2} \left(L_n^{k(n)} u + M_n^{k(n)} u \right) \right\| &\leq M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u) \\ &+ M \left(\exp(\omega\, e^{\omega/n}\, t_n) \left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\, e^{\omega\, k(n)/n}\, \frac{\sqrt{k(n)}}{n} \right. \\ &\left. + \frac{\omega}{n}\, \frac{k(n)}{n}\, \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n} \right) \right) \varphi_n(u) \end{aligned} \tag{5.1.5}$$

where $t_n := \sup\{t, k(n)/n\}$.

Proof. From the classical Trotter's Theorem II.1.1 it follows that the closure of the operators A and $-A$ generate a C_0-semigroup $(T_+(t))_{t\geq 0}$ and respectively $(T_-(t))_{t\geq 0}$ in E. Consequently, the closure of A generates a C_0-group $(G(t))_{t\in\mathbb{R}}$ in E and, for every $t \geq 0$,

$$G(t) = T_+(t) \ , \qquad G(-t) = T_-(t) \ .$$

Moreover, again from Trotter's Theorem, we obtain the representation of the group $(G(t))_{t\in\mathbb{R}}$ in terms of iterates of the operators L_n and M_n; indeed, for every $t \geq 0$ and for every sequence $(k(n)_n)_{n\in\mathbb{N}}$ of positive integers such that $\lim_{n\to+\infty} k(n)/n = t$, we have

$$G(t) = \lim_{n\to+\infty} L_n^{k(n)} \ , \qquad G(-t) = \lim_{n\to+\infty} M_n^{k(n)} \ .$$

Consequently, it follows that the square of the closure of (A, D) generates a cosine function $(C(t))_{t\in\mathbb{R}}$ in E (see [17, Example 3.14.15, p. 217]) and, for every $t \in \mathbb{R}$, $C(t) = (G(t) + G(-t))/2$. Hence the representation of the cosine function is a consequence of the representation of $(G(t))_{t\in\mathbb{R}}$ and the estimate $\|C(|t|)\| \le M\, e^{\omega\, t}$ follows from (5.1.1) and (5.1.4).

Finally, we show the validity of (5.1.5).

Let $t \ge 0$, $(k(n))_{n\ge 1}$ an increasing sequence of positive integers and $u \in D$. From Theorem 1.1.2 we get

$$\begin{aligned}\left\|T_+(t)u - L_n^{k(n)}u\right\| &\le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u)\\ &+M\left(\exp(\omega\, e^{\omega/n}\, t_n)\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\, e^{\omega\, k(n)/n}\, \frac{\sqrt{k(n)}}{n}\right.\\ &\left.+\frac{\omega}{n}\, \frac{k(n)}{n}\, \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n}\right)\right)\, \varphi_n(u)\end{aligned}$$

and

$$\begin{aligned}\left\|T_-(t)u - M_n^{k(n)}u\right\| &\le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u)\\ &+M\left(\exp(\omega\, e^{\omega/n}\, t_n)\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\, e^{\omega\, k(n)/n}\, \frac{\sqrt{k(n)}}{n}\right.\\ &\left.+\frac{\omega}{n}\, \frac{k(n)}{n}\, \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n}\right)\right)\, \varphi_n(u)\ .\end{aligned}$$

Taking into account that

$$\begin{aligned}&\left\|C(t)u - \frac{1}{2}\left(L_n^{k(n)}u + M_n^{k(n)}u\right)\right\|\\ &\qquad= \frac{1}{2}\left\|T_+(t)u + T_-(t)u - L_n^{k(n)}u - M_n^{k(n)}u\right\|\\ &\qquad\le \frac{1}{2}\left(\left\|T_+(t)u - L_n^{k(n)}u\right\| + \left\|T_-(t)u - M_n^{k(n)}u\right\|\right)\end{aligned}$$

from the preceding inequalities the proof is completed. □

Remark 5.1.2 In many applications it is natural to consider the sequence $k(n) = [nt]$ for which $t_n = t$ and $|[nt]/n - t| = nt/n - [nt]/n \le 1/n$. Hence estimate (5.1.5) yields

$$\begin{aligned}\left\|C(t)u - \frac{1}{2}\left(L_n^{[nt]}u + M_n^{[nt]}u\right)\right\| &\le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u) \qquad (5.1.6)\\ +\frac{M}{\sqrt{n}}\left(\frac{\exp(\omega\, e^{\omega/n}\, t)}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\, e^{\omega\, t} + \frac{\omega\, t}{\sqrt{n}}\, \exp\left(\omega\, e^{\omega/n}\, t\right)\right)&\, \varphi_n(u)\ .\end{aligned}$$

□

From the classical theory of the cosine functions (see [68] and [50, Chapter II] for more details) we have that the unique solution of the following second-order Cauchy problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = A^2 u(t,x) \ , & t \in \mathbb{R} \ ; \\ u(0,x) = u_0(x) \ , & x \in \mathbb{R} \ ; \\ \dfrac{\partial}{\partial t} u(t,x)|_{t=0} = u_1(x) \ , & x \in \mathbb{R} \ , \end{cases} \tag{5.1.7}$$

with $u_0,\ u_1 \in D$, is given by

$$\begin{aligned} u(t,x) &= C(t)u_0(x) + \int_0^t C(v)u_1(x)\,dv \qquad (5.1.8) \\ &= \frac{1}{2}\lim_{n\to\infty}\left(L_n^{[n\,t]}u_0 + M_n^{[n\,t]}u_0 + \int_0^t \left(L_n^{[n\,v]}u_1 + M_n^{[n\,v]}u_1\right)\,dv\right) \ , \end{aligned}$$

for every $t \in \mathbb{R}$ and $x \in \mathbb{R}$ (observe that the sequences $(L_n^{[n\,v]}u_1)_{n\geq 1}$ and $(M_n^{[n\,v]}u_1)_{n\geq 1}$ are equibounded for $v \in [0,t]$ and this makes possible to apply the Lebesgue dominated convergence theorem).

Remark 5.1.3 Observe that if (5.1.2) and (5.1.3) hold in a dense subspaces of D, we obtain the validity of (5.1.5) in the same subspace, provided that a quantitative Voronovskaja's formula is satisfied on the larger subspace D.
$\square$

5.2 Quantitative estimate of the resolvent

The next result is concerned with the approximation of the resolvent operator of the generator A^2, which will be denoted by $R(\lambda^2, A^2)$ for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$.

For the sake of simplicity we shall assume that E is a complex Banach space, otherwise we can replace it with its complexification.

We recall that (see [50, p. 30])

$$R(\lambda^2, A^2)u := \frac{1}{\lambda}\int_0^{+\infty} e^{-\lambda\, t} C(t)u\, dt\ , \qquad u \in E\ . \tag{5.2.1}$$

Hence, for every $n \geq 1$ we can define the linear operator $R_{\lambda,n} : E \to E$ as follows

$$R_{\lambda,n}u := \frac{1}{2\lambda}\int_0^{+\infty} e^{-\lambda\, t}\left(L_n^{[n\,t]}u + M_n^{[n\,t]}u\right) dt\ , \qquad u \in E\ .$$

We have the following quantitative estimate on the approximation of the resolvent operator.

Theorem 5.2.1 *Consider the same assumptions of Theorem 5.1.1. Then for every $n \geq 1$, $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega\, e^{\omega/n}$ and $u \in D$, we have*

$$\begin{aligned}\|R(\lambda^2, A^2)u - R_{\lambda,n}u\| \leq{}& \frac{M^2}{\operatorname{Re}\lambda(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2}\,\psi_n(u) \\ &+ \frac{M}{\operatorname{Re}\lambda\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})}\right. \\ &\left.+\frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda - \omega)^{3/2}} + \frac{\omega}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2}\right)\varphi_n(u)\ .\end{aligned} \tag{5.2.2}$$

In particular, the sequence $(R_{\lambda,n})_{n\geq 1}$ strongly converges to $R(\lambda^2, A^2)$.

Proof. As in the proof of Theorem 5.1.1, consider the C_0-semigroup $(T_+(t))_{t\geq 0}$ generated by the closure of A and the C_0-semigroup $(T_-(t))_{t\geq 0}$ generated by he closure of $-A$ and denote by $R(\lambda, A)$ and $R(\lambda, -A)$ their resolvent operators which satisfy, for every $u \in E$ and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$,

$$R(\lambda, A)u = \int_0^{+\infty} e^{-\lambda\, t} T_+(t)u\, dt\ , \qquad R(\lambda, -A)u = \int_0^{+\infty} e^{-\lambda\, t} T_-(t)u\, dt\ ,$$

moreover

$$R(\lambda^2, A^2) = \frac{1}{\lambda}\int_0^{\infty} e^{-\lambda t}\frac{1}{2}\,(T_+(t) + T_-(t)) = \frac{1}{2\lambda}\,(R(\lambda, A) + R(\lambda, -A))\ . \tag{5.2.3}$$

Then, we can apply Theorem 1.2.1 to the resolvent operators $R(\lambda, A)$ and $R(\lambda, -A)$ and taking into account the definition of the operators $R_{\lambda,n}$ we obtain completely the proof. □

5.3 Approximation processes for resolvent operators

In this section we introduce some sequences of linear operators which approximate the resolvent operators associated with the generator of a cosine function. The main aim is the possibility of describing the resolvent operators in terms of classical convolution approximation processes.

Let $(a_n)_{n\geq 1}$ be a sequence of positive integers tending to $+\infty$ and for every $n \geq 1$ consider the linear operator $P_{\lambda,a_n,n} : E \to E$ defined by

$$P_{\lambda,a_n,n}u := \frac{1}{2\lambda n}\sum_{k=0}^{a_n} e^{-\lambda k/n}(L_n^k u + M_n^k u)\ , \qquad u \in E\ . \tag{5.3.1}$$

Theorem 5.3.1 *If the sequence $(a_n)_{n\geq 1}$ satisfies*

$$\lim_{n\to+\infty}\frac{a_n}{n} = +\infty\ , \tag{5.3.2}$$

then $\lim_{n\to+\infty} P_{\lambda,a_n,n}u = R(\lambda^2, A^2)u$ for every $u \in E$.

Proof. First, we observe that the closure of (A, D) generates a C_0-semigroup $(T_+(t))_{t\geq 0}$ in E satisfying $\|T_+(t)\| \leq M\, e^{\omega t}$ and $\lim_{n\to+\infty} L_n^{[nt]} = T_+(t)$ strongly on E for every $t \geq 0$; analogously the closure of $(-A, D)$ generates a C_0-semigroup $(T_-(t))_{t\geq 0}$ in E satisfying $\|T_-(t)\| \leq M\, e^{\omega t}$ and $\lim_{n\to+\infty} M_n^{[nt]} = T_-(t)$ strongly on E (see Theorem II.1.1).

Since $\mathrm{Re}\,\lambda > \omega$ we have $\left\|e^{-\lambda/n}L_n\right\| \leq Me^{-(\mathrm{Re}\,\lambda-\omega)/n}$ and $\left\|e^{-\lambda/n}M_n\right\| \leq Me^{-(\mathrm{Re}\,\lambda-\omega)/n}$ and consequently $\|P_{\lambda,a_n,n}\| \leq M/\left(\mathrm{Re}\,\lambda(1-e^{-(\mathrm{Re}\,\lambda-\omega)})\right)$. This shows that $(P_{\lambda,a_n,n})_{n\geq 1}$ is equibounded and we can establish the convergence property on the dense subspace D. Let $u \in D$; we have

$$\left\|P_{\lambda,a_n,n}u - R(\lambda^2, A^2)u\right\| \leq \|P_{\lambda,a_n,n}u - R_{\lambda,n}u\| + \left\|R_{\lambda,n}u - R(\lambda^2, A^2)u\right\|\ . \tag{5.3.3}$$

The second term converges to zero from Theorem 5.2.1. As regards to the first term we preliminary observe that

$$R_{\lambda,n}u = \frac{1}{2\lambda n}\sum_{k=0}^{\infty} e^{-\lambda k/n}\frac{1-e^{\lambda/n}}{\lambda/n}\left(L_n^k + M_n^k\right)\ ,$$

since L_n^k and M_n^k are constant on each interval $[k/n,(k+1)/n[$. Hence

$$\|P_{\lambda,a_n,n}u - R_{\lambda,n}u\| \tag{5.3.4}$$
$$= \left\| \frac{1}{2\lambda n} \times \left(\sum_{k=0}^{a_n} e^{-\lambda k/n}(L_n^k u + M_n^k u) - \sum_{k=0}^{\infty} e^{-\lambda k/n}\, \frac{1-e^{-\lambda/n}}{\lambda/n}\,(L_n^k u + M_n^k u)\right)\right\|$$
$$\le \frac{M}{\operatorname{Re}\lambda\, n}\,\|u\| \sum_{k=a_n+1}^{+\infty} e^{-(\operatorname{Re}\lambda-\omega)k/n}$$
$$+ \frac{M}{\operatorname{Re}\lambda\, n}\|u\| \left|1-\frac{1-e^{-\lambda/n}}{\lambda/n}\right| \sum_{k=0}^{+\infty} e^{-(\operatorname{Re}\lambda-\omega)k/n}$$
$$\le \frac{M}{\operatorname{Re}\lambda\, n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right)}\,\|u\| \times \left(e^{-(\operatorname{Re}\lambda-\omega)(a_n+1)/n} + \left|1-\frac{1-e^{-\lambda/n}}{\lambda/n}\right|\right)\ ;$$

since $\lim_{n\to+\infty} n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right) = \operatorname{Re}\lambda-\omega$, the assumption (5.3.2) ensures that the first term in (5.3.3) tends to 0. □

Our next aim is to provide a quantitative estimate of the convergence in Theorem 5.3.1.

Theorem 5.3.2 *Assume that* (5.1.2) *and* (5.1.3) *hold.*

Then, for every $n > \omega/\log(\operatorname{Re}\lambda/\omega)$ *(take* $n \ge 1$ *if* $\omega = 0$*) and* $u \in D$, *we have*

$$\begin{aligned}\|P_{\lambda,a_n,n}u - R(\lambda^2,A^2)u\| \le{}& \frac{M^2}{\operatorname{Re}\lambda(\operatorname{Re}\lambda-\omega\, e^{\omega/n})^2}\,\psi_n(u)\\ &+\frac{M}{\operatorname{Re}\lambda\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda-\omega\, e^{\omega/n})}\right.\\ &\left.+\frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda-\omega)^{3/2}}+\frac{\omega}{\sqrt{n}\,(\operatorname{Re}\lambda-\omega\, e^{\omega/n})^2}\right)\varphi_n(u)\\ &+\frac{M\left(e^{-(\operatorname{Re}\lambda-\omega)a_n/n}+\frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}\right)}{\operatorname{Re}\lambda(\operatorname{Re}\lambda-\omega)\left(1-\frac{\operatorname{Re}\lambda-\omega}{n}\right)}\,\|u\|\ .\end{aligned} \tag{5.3.5}$$

Proof. We estimate the two terms at the righthand side of (5.3.3). The estimate of the second term is provided by Theorem 5.2.1 and we have only to estimate the first term.

To this end, we use (1.2.11) and (1.2.12)

$$\left|1-\frac{1-e^{-\lambda/n}}{\lambda/n}\right| \le \frac{|\lambda|^{3/2}}{n\left|\operatorname{Re}\sqrt{\lambda}\right|}\ ,$$

$$\frac{1}{n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right)} \le \frac{1}{(\operatorname{Re}\lambda-\omega)\,(1-(\operatorname{Re}\lambda-\omega)/n)}\ ,$$

and from (5.3.4) we get the following estimate of the first term in (5.3.3)

$$\|P_{\lambda,a_n,n}u - R_{\lambda,n}u\| \le \frac{M\left(e^{-(\operatorname{Re}\lambda-\omega)a_n/n} + \frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}\right)}{\operatorname{Re}\lambda(\operatorname{Re}\lambda-\omega)\,(1-(\operatorname{Re}\lambda-\omega)/n)}\,\|u\|\ , \qquad (5.3.6)$$

which completes the proof of (5.3.5). □

Taking $a_n \ge [n\log n/\operatorname{Re}\lambda]$, estimate (5.3.5) becomes

$$\|P_{\lambda,a_n,n}u - R(\lambda,A)u\| \le C_1(\lambda)\,\psi_n(u) + \frac{C_2(\lambda)}{\sqrt{n}}\,\varphi_n(u) + \frac{C_3(\lambda)}{n}\,\|u\|\ , \quad (5.3.7)$$

for every $\omega \ge 0$, $u \in D$ and $n > \omega/\log(\operatorname{Re}\lambda/\omega)$, where $C_i(\lambda)$, $i = 1, 2, 3$, are suitable constants depending only on λ.

5.4 Applications to Rogosinski operators

Denote by $C_{2\pi}$ the space of all 2π-periodic continuous real functions on $\mathbb{R}$ and put $\Pi := \{\pi + 2k\pi \mid k \in \mathbb{Z}\}$. Moreover, let $a \in C_{2\pi} \cap C^1(\mathbb{R} \setminus \Pi)$ be such that $a \neq 0$ in $]-\pi, \pi[$ and consider the first-order differential operator $(A, D(A))$ defined by

$$Au := au', \qquad u \in D(A) := \left\{u \in C_{2\pi} \cap C^1(\mathbb{R} \setminus \Pi) \mid Au \in C_{2\pi}\right\} .$$

In order to consider the generation of cosine functions, we also consider the operator A^2 on the following domain

$$D(A^2) := \left\{u \in C_{2\pi} \cap C^2(]-\pi, \pi[) \mid a(au')' \in C_{2\pi}\right\} .$$

It is well-known (see e.g. [35, Theorem 1.1]) that $(A^2, D(A^2))$ generates a cosine functions $(C(t))_{t\geq 0}$ in $C_{2\pi}$ if and only if

$$\frac{1}{a} \in L^1(-\pi, 0) , \qquad \frac{1}{a} \in L^1(0, \pi) \tag{5.4.1}$$

or alternatively

$$\frac{1}{a} \notin L^1(-\pi, 0) , \qquad \frac{1}{a} \notin L^1(0, \pi) . \tag{5.4.2}$$

Now, we consider the Rogosinski kernel defined by setting, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$r_n(x) := 1 + 2\sum_{k=1}^{n} \cos\left(\frac{k\pi}{2n+1}\right) \cos(kx) ,$$

and the corresponding n-th *Rogosinski operator* $R_n : C_{2\pi} \to C_{2\pi}$ given by

$$R_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - v) r_n(v)\, dv , \qquad f \in C_{2\pi} ,\ x \in \mathbb{R} .$$

The n-th generalized Rogosinski operator $R_{a,n} : C_{2\pi} \to C_{2\pi}$ introduced in [35] is defined by putting

$$R_{a,n} f(x) = R_n f\left(x + \frac{2\pi}{2n+1}\, a(x)\right) , \qquad f \in C_{2\pi} , \quad x \in \mathbb{R} .$$

From [35, Theorem 2.1] the sequence $(\|R_{a,n}\|)_{n\in\mathbb{N}}$ is equibounded and moreover $\|R_{a,n}^k\| \leq 2\pi$ for every $n, k \geq 1$. Further, there exists a positive constant $C > 0$ such that

$$\|R_{a,n} f - f\| \leq C\ \omega\left(f; \frac{1}{n}\right) , \qquad f \in C_{2\pi} . \tag{5.4.3}$$

Our next aim is to establish a quantitative estimate in order to apply Theorems 5.1.1 and 5.3.2.

Lemma 5.4.1 *Let $0 < \alpha \le 1$. Then, for every $f \in C^{1,\alpha}_{2\pi}$,*

$$\left\| \frac{2n+1}{2\pi} \left(R_{a,n}f - f \right) - Af \right\| \le 24 \left(\|a\| + 1 \right) \left(\frac{2\pi}{2n+1} \right)^{\alpha} .$$

PROOF. For every $f \in C^{1,\alpha}_{2\pi}$ we have

$$\begin{aligned} \left\| \frac{2n+1}{2\pi} \left(R_{a,n}f - f \right) - Af \right\| &= \left\| \frac{2n+1}{2\pi} \left(R_{a,n}f - R_n f \right) - Af \right\| + \\ &+ \left\| \frac{2n+1}{2\pi} \left(R_n f - f \right) \right\| . \end{aligned} \tag{5.4.4}$$

As regards to the first term at the righthand side of (5.4.4), from Lagrange's theorem we can write

$$f(y+t) - f(y) = f'(y)t + \big(f'(\xi) - f'(y) \big)\, t \,, \qquad y, t \in \mathbb{R}$$

where $\xi \in]y, y+t[$. For every $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} &\frac{2n+1}{2\pi}(R_{a,n}f(x) - R_n f(x)) - a(x)f'(x) \\ &= \frac{2n+1}{2\pi}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f\left(x - v + \frac{2\pi}{2n+1}\, a(x) \right) - f(x-v) \right) r_n(v)\, dv \\ &\quad - a(x)f'(x) \\ &= \frac{2n+1}{2\pi}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x-v) \frac{2\pi}{2n+1}\, a(x)\, r_n(v) dv - a(x)f'(x) \\ &\quad + \frac{2n+1}{2\pi}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \big(f'(\xi) - f'(x-v) \big) \frac{2\pi}{2n+1} a(x)\, r_n(v)\, dv \\ &= a(x)(R_n f'(x) - f'(x)) + a(x) \frac{1}{2\pi} \int_{-\pi}^{\pi} \big(f'(\xi) - f'(x-v) \big)\, r_n(v)\, dv \,, \end{aligned}$$

where $\xi \in]x - v, x - v + 2\pi a(x)/(2n+1)[$.

We recall that (see e.g. [21, Theorem 2.4.8, pag 106])

$$\|R_n g - g\| \le (2\pi + 1) E_n(g) + 4\, \omega \left(g; \frac{1}{n} \right) \,, \qquad g \in C_{2\pi} \,,$$

where $E_n(g)$ is the best approximation of the function g by trigonometric polynomials of degree n and hence, from the classical Jackson Theorem,

$$\|R_n g - g\| \le 6(2\pi + 1)\, \omega \left(g; \frac{1}{n} \right) + 4\, \omega \left(g; \frac{1}{n} \right) \le (12\pi + 10) \omega \left(g; \frac{1}{n} \right) .$$

Applying the above inequality to f' and f we get

$$\begin{aligned} &\left| \frac{2n+1}{2\pi}(R_{a,n}f(x) - R_n f(x)) - a(x)f'(x) \right| \le \\ &\le \|a\| \left((12\pi + 10) \omega \left(f'; \frac{1}{n} \right) + \omega \left(f'; \frac{2\pi}{2n+1} \right) \right) \\ &\le \|a\| (12\pi + 11)\, \omega \left(f'; \frac{2\pi}{2n+1} \right) \,, \end{aligned}$$

and consequently

$$\left\| \frac{2n+1}{2\pi} (R_{a,n}f - f) - Af \right\| \leq \|a\|(12\pi + 11)\omega\left(f'; \frac{2\pi}{2n+1}\right) \\ + \frac{2n+1}{2\pi}(12\pi+10)\omega\left(f; \frac{1}{n}\right) .$$

Since $f \in C_{2\pi}^{1,\alpha}$ we have $\omega(f,\delta) \leq \delta^{(1+\alpha)}$ and $\omega(f',\delta) \leq \delta^{\alpha}/2$ so we conclude

$$\begin{aligned} &\left\| \frac{2n+1}{2\pi} (R_{a,n}f - f) - Af \right\| \\ &\quad \leq \|a\| \left((6\pi+5) \left(\frac{2\pi}{2n+1} \right)^{\alpha} \right) + \frac{2n+1}{n\,\pi}(6\pi+5)\frac{1}{n^{\alpha}} \\ &\quad \leq (6\pi+5)\,(\|a\|+1) \left(\frac{2\pi}{2n+1} \right)^{\alpha} . \end{aligned}$$

□

In [35, Theorem 2.7] we have established that besides the generation of the cosine function $(C(t))_{t\geq 0}$, condition (5.4.1) or (5.4.2) also ensures that $C_{2\pi}^1 \cap D(A^2)$ is a core for $(A^2, D(A^2))$ and further, for every $t > 0$,

$$C(t) = \frac{1}{2} \lim_{n\to\infty} \left(R_{a,n}^{k(n)} u + R_{-a,n}^{k(n)} u \right) , \tag{5.4.5}$$

where $(k(n))_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to+\infty} \frac{2\pi k(n)}{2n+1} = t$.

From Lemma 5.4.1, we can take $M = 2\pi$, $\omega = 0$, $\psi_n(u) = 24\,(\|a\| + 1)\left(\frac{2\pi}{2n+1}\right)^{\alpha}$ and $\varphi_n(u) = \psi_n(u) + \|Au\|$ in Theorem 5.1.1 and we directly obtain the following quantitative version of (5.4.5).

Theorem 5.4.2 *Let $a \in C_{2\pi} \cap C^1(\mathbb{R}\setminus\Pi)$. If* (5.4.1) *or alternatively* (5.4.2) *holds, then for every $t \geq 0$ and $u \in C^{1,\alpha} \cap D_{2\pi}(A^2)$*

$$\begin{aligned} &\left\| C(t)u - \frac{1}{2}\left(R_{a,n}^{k(n)} u + R_{-a,n}^{k(n)} u \right) \right\| \leq (2\pi)^2\, t\,(\|a\|+1) \left(\frac{2\pi}{2n+1} \right)^{\alpha} \\ &\quad + 2\pi \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{k(n)}}{n} \right) \\ &\quad \times \left(24\,(\|a\|+1) \left(\frac{2\pi}{2n+1} \right)^{\alpha} + \|Au\| \right) , \end{aligned} \tag{5.4.6}$$

where $(k(n))_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to+\infty} \frac{2\pi k(n)}{2n+1} = t$.

Now, let $(a_n)_{n\geq 1}$ be a sequence of positive integers tending to $+\infty$. For every $n \geq 1$, we consider the linear operator $P_{\lambda,a_n,n} : C_{2\pi} \to C_{2\pi}$ defined by

$$P_{\lambda,a_n,n}u := \frac{1}{2\lambda}\frac{1}{n}\sum_{k=0}^{a_n} e^{-\lambda k/n}(R_{a,n}^{k(n)}u + R_{-a,n}^{k(n)}u) \ , \qquad u \in E \ , \tag{5.4.7}$$

If the sequence $(a_n)_{n\geq 1}$ satisfies $1/a_n = o(1/n)$ as $n \to +\infty$ then from Theorem 5.3.1 we obtain $\lim_{n\to+\infty} P_{\lambda,a_n,n}u = R(\lambda^2, A^2)u$ for every $u \in C_{2\pi}$ and we have the following estimate of the convergence.

Theorem 5.4.3 *Let $a \in C_{2\pi} \cap C^1(\mathbb{R} \setminus \Pi)$. If* (5.4.1) *or alternatively* (5.4.2) *holds, then for every $t \geq 0$ and $u \in C^{1,\alpha} \cap D_{2\pi}(A^2)$ and for every $n \geq 1$ and $u \in D$, we have*

$$\begin{aligned}\|P_{\lambda,a_n,n}u - R(\lambda^2, A^2)u\| \leq{}& 24\,\frac{(2\pi)^{2+\alpha}(\|a\|+1)}{(2n+1)^{\alpha}(\operatorname{Re}\lambda)^3} \\ &+\frac{2\pi}{\operatorname{Re}\lambda\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda)}\right. \\ &\left.+\frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda)^{3/2}} + \frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda)^2}\right)(\psi_n(u) + \|Au\|) \\ &+\frac{2\pi\left(e^{-(\operatorname{Re}\lambda)a_n/n} + \frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}\right)}{(\operatorname{Re}\lambda)^2\left(1-\frac{\operatorname{Re}\lambda}{n}\right)}\|u\| \ . \end{aligned} \tag{5.4.8}$$

Exactly the same procedure can be also applied to other sequences of trigonometric polynomials such as Fejér operators and more general averages of trigonometric interpolating operator considered in [35, 28]. Since in these cases the cosine function is the same, we limit ourselves to observe that (5.4.6) remains still valid when considering these other sequences of trigonometric interpolating operators too.