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Quantitative estimates in the approximation of semigroups and applications

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Contents

Chapter I

Introduction

The general aim of this work is concerned with quantitative estimates of the convergence of suitable sequences of linear operators to the solution of evolution problems of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = Au(\cdot,t)u(x) \ , \\ u(x,0) = u_0(x) \ , \end{cases} \tag{I.1}$$

where $A : D(A) \longrightarrow X$ is a linear second-order differential operator on some Banach space X, independent of the time variable. Possible boundary conditions of problem (I.1) are included in the domain of A. In most cases we shall have a second-order (possibly degenerate) differential operator

$$Au(x) = \alpha(x)u''(x) + \beta(x)u'(x) \ , \qquad x \in I \ , \tag{I.2}$$

where $I :=]r_1, r_2[$ is a real interval and $\alpha, \beta : I \to \mathbb{R}$ are continuous functions with $\alpha(x) > 0$ for every $x \in I$.

It is well-known that the solution of this problem can be represented using the semigroup theory; namely, if the operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a suitable domain $D(A)$ and $u_0 \in D(A)$ the solution of the preceding problem is given by

$$u(t,x) = (T(t)u_0)(x) \ , \qquad t \geq 0 \ . \tag{I.3}$$

This approach motivated Feller to study the general conditions (that are, the domains) under which the operator A generates a C_0-semigroup and in 1952 he completely characterized this property both on the space of real continuous functions than of integrable functions on a real interval (see [52, 54, 53]). After these papers many questions arose concerning with representation (I.3). In particular, the generation of a C_0-semigroup has been deeply investigated on an assigned domain. Necessary and sufficient conditions in order for A to be the generator of a C_0-semigroup in $C(\overline{I})$

have been given by Clément and Timmermans [44] when Ventcel's boundary conditions are imposed at the endpoints, and by Timmermans in [69] on the maximal domain. More recently, the existence of a C_0-semigroup has been characterized in [29] also in the case of Neumann's type boundary conditions at the endpoints. In the space $L^1(I)$ the characterization of the generation of a C_0-semigroup has been completely achieved in [19] on the adjoint maximal domain, on the adjoint Dirichlet domain and on the adjoint Neumann domain. All the results obtained in [44, 69, 29, 19] are very closely related to the pioneer work by Feller [54]; in the preliminary section we give a brief exposition of them.

Another field of interest which was immediately developed after the work of Feller was the possibility of approximating the semigroup $(T(t))_{t\geq 0}$ generated by $(A, D(A))$. In this respect, starting from usual numeric methods for computing the solution of a partial differential equation [57], in 1958 Trotter [70] provided some important results which have been revealed as the main tools for the investigation and the approximation of the solution of (I.1) until today. More precisely, the idea of Trotter was based on the construction of an implicit Euler scheme for the approximation of the semigroup $(T(t))_{t\geq 0}$. He treated the question of convergence of suitable difference operators in an operator-theoretic fashion, in which semigroup theory give a fundamental contribution. The main idea consist in replacing the differential operator with approximating operators, and taking the solution of the resulting equation as an approximation to the solution of the original equation. Namely, the process is made through a discretization in time and space: to be more precise, for each $n \in \mathbb{N}$, let be h_n the time-size, and S_n a spatial grid; the function $u_n(kh_n, x)$ defined on S_n represents the approximation of the solution of (I.1) at the time $t_k = kh_n$, and can be defined recursively in terms of a linear operator L_n

$$\begin{cases} u_n(t_k, x) = L_n u_n\,(t_{k-1}, x) \\ u_n(0, x) = f_n(x)\ , \end{cases} \tag{I.4}$$

note that $u_n(t_k, x) = L_n^k f(x)$.

In this scheme we approximate the operator A with $A_n = \frac{L_n u - u}{h_n}$, and consequently, if we require that A_n converges to A in some suitable sense, the solution $u(t, x)$ of (I.1) will be approximated by $L_n^{[th_n^{-1}]} f(x)$.

Thus, under suitable assumptions, Trotter obtained a sequence $(L_n)_{n\geq 1}$ of linear operators satisfying

$$\lim_{n\to+\infty} L_n^{k(n)} = T(t)$$

whenever $\lim_{n\to+\infty} k(n)/n = t$ (see [70, Theorem 5.1] and Chapter II).

As an immediate consequence of this result we can represent the solution of problem (I.1) as follows

$$u(t, x) = \lim_{n\to+\infty} L_n^{k(n)}(u_0)(x)\ , \qquad t \geq 0\ .$$

This last representation opened the possibility of approximating the solution of parabolic problems through positive operators.

Among the main applications, the parabolic problems under consideration were concerned with diffusion models in population genetics and models in mathematical finance. Diffusion models in population genetics and in particular Wright-Fisher models were indeed the starting point of the investigation of Feller. The evolution problem (I.1) corresponding to this model in absence of selection, mutation and migration becomes

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{x(1-x)}{2} \dfrac{\partial^2 u}{\partial x^2}(x,t) , & \\ & t > 0 , \quad 0 < x < 1 , \\ u(x,0) = u_0(x) , & \qquad\quad 0 \le x \le 1 . \end{cases} \tag{I.5}$$

Karlin and Ziegler [56] showed that the solution of the preceding problem can be expressed in terms of iterates of the classical Bernstein operators

$$B_n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) , \qquad 0 \le x \le 1 ,$$

evaluated at the initial function u_0.

The connection with the evolution problem is provided by the following Voronovskaja's formula

$$\lim_{n \to +\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x) ,$$

which holds uniformly for every $f \in C^2([0,1])$.

Then, applying the Trotter's theorem on the approximation of semigroups, we obtain that the semigroup $(T(t))_{t \ge 0}$ generated by the operator $Au(x) = \frac{x(1-x)}{2} u''(x)$ on the Ventcel's domain

$$D(A) = \{u \in C[0,1] \cap C^2]0,1[\mid \lim_{x \to 0,1} Au(x) = 0\}$$

can be represented as

$$T(t)u(x) = \lim_{n \to +\infty} B_n^{[nt]} u(x)$$

and hence the solution of (I.5) can be written as

$$u(x,t) = T(t)u_0(x) = \lim_{n \to +\infty} B_n^{[nt]} u_0(x) .$$

After this example, models with more than two alleles and recently even with an infinite numbers of alleles have been considered. Different mathematicians from several schools were involved in this project.

Of particular interest in this context is the generalization introduced of Altomare [2], concerning with positive operators of Bernstein type, namely Bernstein-Schnabl operators, in an infinite dimensional setting by associating them to a positive projection.

This result has motivated many subsequent researches. On one hand there was the possibility of applying the same method to different sequences of positive operators (Stancu, Lototsky and so on; see [9, Chapter VI] and [3, 4, 5, 16, 12, 22, 23]) obtaining in this way the possibility of approximating the solutions of an enlarged class of Cauchy problems.

On the other hand, starting with an assigned degenerate parabolic problem, it has also been considered the inverse problem of approximating its solution by means of suitable sequences of discrete or integral positive operators (see e.g. [6, 7, 19, 11, 13, 14, 15, 18, 27, 24, 25, 26, 28, 30, 35]).

Further, this approach has also been extended to a direct study of the diffusion model (see [8]).

However, the lack of a quantitative estimate of the convergence of the iterates of the positive operators to the limit semigroup has constituted one of the main problems in the applications of Trotter's theorem. Some results in this direction have recently been obtained in [55] and [62]. In these papers the limit semigroup is assumed to have a growth bound equal to 0 and applications to classical approximation processes require restriction to the subspaces of C^4 and C^3-functions respectively.

In Chapter 1, we shall be able to state a general result concerning with the order of convergence to the limit semigroup. Our result holds under the same general assumptions of Trotter's theorem. Moreover we don't make any assumptions on the growth bound of the semigroup and the general result yields better estimates even in the case of growth bound equal to 0.

This main result requires only a quantitative estimate in the Voronovskaja-type formula in terms of suitable seminorms. Then we shall be able to evaluate the norm difference between the $k(n)$-iterates of the linear operator and the semigroup. Consequently a suitable choice of the sequence $(k(n))_{n\in\mathbb{N}}$ will ensure the strong convergence of the iterates to the semigroup.

We also point out that our estimate behaves better even for linear functions and does not require any invariant subspace properties.

A representation by means of iterates of positive operators can be obtained for the resolvent operator of semigroup's generator. In [1] it was given a general description of this kind of representation in terms of approximating processes of the resolvent operator and different formulas of independent interest were obtained. These formulas turn out to be useful for some qualitative properties of the semigroup in different meaningful cases.

Along this line, we shall give a quantitative estimate in the approximation of the resolvent operator starting from a quantitative version of a Voronovskaja-type formula.

Moreover, we have studied the possibility of introducing some general sequences of linear operators obtained from classical approximation processes which turn to be useful in the approximation of the resolvent operators. The aim is the possibility of representing the resolvent operators in terms of classical approximation operators.

The results in this chapter are collected in [36, 37, 38].

In Chapter 2, we shall provide several applications of the general quantitative estimates, by considering some classes of Schnabl-type operators and using a quantitative version of the Voronovskaja-type formula obtained in [9].

As regards the finite dimensional setting we shall provide a general result concerning with a quantitative Voronovskaja-type formula which holds for all functions of class $C^{2,\alpha}(K)$, where K is a compact domain in $\mathbb{R}^d$. So we shall be able to give some estimates of the convergence to the limit semigroup in this space extending considerably the class of functions for which such an estimate have been found.

The order of convergence to the second-order differential operator is strictly related to the convergence of $(L_n(\mathrm{id}-x)^2(x))_{n\in\mathbb{N}}$ to 0. In particular, when we describe the solution of the evolution problem associated with a Fleming-Viot diffusion model in population genetics in terms of Bernstein operators, we can furnish the order of convergence equal to $\alpha/2$ in the space $C^{2,\alpha}(K)$.

We shall show a partial converse result which ensures that this order of convergence is the best possible for the Bernstein operators.

Finally we conclude Chapter 2 with an application to suitable combinations of iterates of Stancu operators. These results can been found in [37] and [39].

In Chapter 3 we consider iterates of generalized Steklov operators and study their convergence to a limit semigroup generated by the second-order differential operator $Af(x) = a(x)f''(x)$, where the coefficient a has a degeneration at least of second order at the endpoints. We also give a quantitative version of Voronovskaja's formula in order to apply the quantitative estimates in Chapter 1.

The choice of Steklov operators is motivated by the possibility of considering the same kind of operators in different setting, such as spaces of continuous functions or weighted spaces of continuous functions both on bounded and unbounded real intervals; hence we can study the convergence to the limit semigroups in all these setting.

The results in this chapter have been obtained in collaboration with I. Rasa (Cluj-Napoca, Romania) and published in [33], [34]. At the end of the chapter we briefly describe a possibile extension to the multivariate case. In this setting we have only some partial results which has been published in [42].

The need of adapting sequences of operators to different settings has lead us to study the possibility of considering a combinations of different sequences of operators in the approximation of the same problem.

In Chapter 4 we introduce a general and simple method which consists in constructing a new approximation process starting with a decomposition of an Hilbert space into the direct sum of orthogonal subspaces and associating to each subspace an assigned approximation process.

In this way are able to obtain new Voronovskaja-type formulas from assigned ones, or conversely we can construct particular approximation processes in order to satisfy a prescribed Voronovskaja-type formula, extending the class of differential problems under consideration. Some similar questions have also been considered in [31] and in [32].

Our method can be applied in different settings. We concentrate our attention on some particular positive approximation processes in spaces of L^2-real functions, namely the Bernstein-Kantorovich and the Bernstein-Durrmeyer operators.

The choice of these operators resides on the fact that both for Bernstein-Kantorovich and Bernstein-Durrmeyer operators the Voronovskaja's formula is related to the evolution process associated with some diffusion models in population genetics. Thus, it may happen that different factors, such as selection, mutation and migration affect only some alleles and therefore involve only a face of the simplex whose vertices represent the totality of alleles; in this case the possibility of using different approximation processes on different orthogonal subspaces allow us to make the appropriate choice of the approximating operators on every subspace.

In connection with this problem we have also studied perturbations of operators obtained by modifying some of its components. In concrete examples, this has been performed for Jackson convolution operators. These results are published in [40] (a further investigation has been performed in [41] for Bernstein-Durrmeyer operators and for the sake of brevity is not included in this work).

Following the same approach of the preceding chapters, in Chapter 5 we study the possibility of approximating the solution of suitable hyperbolic problems using the generation of a cosine function (see [68] and [50] for more details on this approach).

Unlike to the case of semigroup theory there has not been a similar development in the approximation of cosine functions and only in [30, 35] we can find some qualitative results on the convergence of iterates of trigonometric polynomials to suitable cosine functions in the setting of spaces of continuous periodic real functions. We provide a general cosine version of the Trotter's approximation theorem together with a quantitative estimate of the convergence. We also introduce suitable sequences of operators approximating the resolvent operators associated with the generator of the cosine function and also in this case we obtain a quantitative estimate of the convergence. We apply our results to some generalized Rogosinski trigonometric operators.

These results are collected in [43].

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Chapter II

Preliminary and auxiliary results

In this chapter we collect some brief recalls on the main topics involved in this work. These recalls are only intended to fix some notation and references for the subsequent chapters and not to furnish a complete treatment of the subject.

II.1 Recalls on semigroup's theory

For the sake of completeness we collect here some basic definitions and results on semigroup's theory which will be frequently used in the sequel. We refer to the monographs [48] and [64] for a complete introduction to the subject.

Let E be a Banach space over the field $\mathbb{K}$. We shall denote by $\mathcal{L}(E)$ the space of all bounded linear operators in E.

A *semigroup* (or a *one-parameter semigroup*) of bounded linear operators on E is a family $(T(t))_{t\geq 0}$ of elements of $\mathcal{L}(E)$ such that

1. $T(0) = I$,
2. $T(s+t) = T(s)T(t)$ for every $s, t \geq 0$,

where I denotes the identity operator on E.

A semigroup is said to be *strongly continuous* (or a C_0*-semigroup*) if for every $t_0 \geq 0$ and $f \in E$

$$\lim_{t\to t_0} \|T(t)f - T(t_0)f\| = 0 \ .$$

The *growth bound* of the semigroup is defined by

$$\omega_0 := \inf \{\, \omega \in \mathbb{R} \mid \text{there exists } M \geq 0 \text{ such that } \|T(t)\| \leq M \exp(\omega t) \\ \text{for every } \ t \geq 0\} \ .$$

The *generator* of a C_0-semigroup is a linear operator $A : D(A) \to E$ defined by

$$Af := \lim_{t\to 0^+} \frac{T(t)f - f}{t} ,$$

on the linear subspace

$$D(A) := \left\{ f \in E \mid \text{there exists } \lim_{t\to 0^+} \frac{T(t)f - f}{t} \in E \right\} .$$

A semigroup is said to by *uniformly continuous* if for every $t_0 \geq 0$

$$\lim_{t\to t_0} \|T(t) - T(t_0)\| = 0 .$$

In this case we have $D(A) = E$ and A is bounded; conversely, if A is bounded, we can set

$$T(t) := \exp(tA) := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \quad (t \geq 0) ,$$

and we obtain a uniformly continuous semigroup whose generator is A.

A linear operator $A : D(A) \to E$ is said to be *closed* if $D(A)$ endowed with the *graph norm*

$$\|f\|_A := \|f\| + \|A(f)\| , \quad f \in D(A) ,$$

becomes a Banach space. In other words, the graph $\{(f, Af) \mid f \in D(A)\}$ is closed in $E \times E$.

We say that a linear operator $B : D(B) \to E$ is *an extension* of A if $D(A) \subset D(B)$ and $Af = Bf$ for every $f \in D(A)$.

A linear operator $A : D(A) \to E$ is *closable* if there exists a closed extension of A. The smallest closed extension $\overline{A} : D(\overline{A}) \to E$ of A is called the *closure* of A.

A *core* for a linear operator $A : D(A) \to E$ is a linear subspace D_0 of $D(A)$ which is dense in $D(A)$ with respect to the graph norm.

If $A : D(A) \to E$ is a closed operator, we shall denote by $\rho(A)$ its *resolvent set*, i.e.,

$$\rho(A) := \{\lambda \in \mathbb{K} \mid \lambda I - A \text{ is invertible}\} .$$

The *spectrum* $\sigma(A)$ is defined as

$$\sigma(A) := \mathbb{K} \setminus \rho(A) .$$

If $\lambda \in \rho(A)$ we shall denote by $R(\lambda, A)$ the inverse of $\lambda I - A$.

If A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ then $\rho(A) \neq \varnothing$. Moreover, if ω_0 denotes the growth bound of $(T(t))_{t\geq 0}$ we have, for every $\lambda \in \mathbb{K}$ such that $\operatorname{Re}\lambda > \omega_0$ and $f \in E$

$$R(\lambda, A)f = \int_0^{\infty} \exp(-\lambda t) T(t) f \, dt .$$

Now we recall a generation and approximation theorem which is very important for our purposes.

Theorem II.1.1 (Trotter's approximation theorem) *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators on E and let $(\rho_n)_{n\in N}$ be a decreasing sequence of positive real numbers tending to 0. Suppose that there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that*

$$\|L_n^k\| \leq M e^{\omega\,\rho_n\,k}\ , \qquad \textit{for every } k,\, n \in \mathbb{N}\ . \tag{II.1.1}$$

Moreover, assume that $\mathcal{D}$ is a dense subspace of E and for every $f \in \mathcal{D}$ the following Voronovskaja-type formula holds

$$Af := \lim_{n\to\infty} \frac{L_n(f) - f}{\rho_n}\ .$$

If $(\lambda - A)(\mathcal{D})$ is dense in E for some $\lambda > \omega$, then the closure of A generates a C_0-semigroup $(T(t))_{t\geq 0}$ and for every $f \in E$ and every sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/\rho_n = t$, we have

$$T(t)f = \lim_{n\to\infty} L_n^{k(n)}(f)\ . \tag{II.1.2}$$

II.2 Feller's classification and generation results

In different applications we need to consider suitable domains where a second-order differential operator generates a C_0-semigroup. In this context, the classification of Feller may help us to decide the right choice of the domain of our operators (see [52, 54, 53]). After the work of Feller, it was also considered the problem of investigating the generation of a C_0-semigroup on an assigned domain. Necessary and sufficient conditions in order for A to be the generator of a C_0-semigroup in $C(\overline{I})$ have been given by Clément and Timmermans [44] when Ventcel's boundary conditions are imposed at the endpoints, and by Timmermans in [69] on the maximal domain. More recently, the existence of a C_0-semigroup has been characterized in [29] also in the case of Neumann's type boundary conditions at the endpoints. In the space $L^1(I)$ the characterization of the generation of a C_0-semigroup has been completely achieved in [19] on the adjoint maximal domain, on the adjoint Dirichlet domain and on the adjoint Neumann domain. However, all the results obtained in [44], [69], [29] and [19] are very closely related to the pioneer work by Feller [54].

In the sequel we consider an interval $I :=]r_1, r_2[$ with $-\infty \leq r_1 < r_2 \leq +\infty$ and two continuous real functions $\alpha, \beta : I \to \mathbb{R}$ with $\alpha > 0$ in I. We define the second-order differential operator

$$Au(x) := \alpha(x)\, u''(x) + \beta(x)\, u'(x)\ , \qquad u \in C(\overline{I}) \cap C^2(I)\ .$$

In order to state the main characterizations, we fix $x_0 \in I$ and define, for every $x \in I$,

$$W(x) := \exp\left(-\int_{x_0}^{x} \frac{\beta(t)}{\alpha(t)}\, dt\right) \tag{II.2.1}$$

and

$$Q(x) := \frac{1}{\alpha(x)W(x)} \int_{x_0}^{x} W(s)\, ds\ , \quad R(x) := W(x) \int_{x_0}^{x} \frac{1}{\alpha(s)W(s)}\, ds\ . \tag{II.2.2}$$

It is also useful to set $I_1 :=]r_1, x_0]$ and $I_2 := [x_0, r_2[$. The endpoint r_i, $i = 1, 2$ is said to be

$$\begin{array}{lll} \textit{a regular boundary} & \text{if} & Q \in L^1(I_i)\ , R \in L^1(I_i)\ ; \\ \textit{an exit boundary} & \text{if} & Q \notin L^1(I_i)\ , R \in L^1(I_i)\ ; \\ \textit{an entrance boundary} & \text{if} & Q \in L^1(I_i)\ , R \notin L^1(I_i)\ ; \\ \textit{a natural boundary} & \text{if} & Q \notin L^1(I_i)\ , R \notin L^1(I_i)\ . \end{array} \tag{II.2.3}$$

The following generation results will be used in the sequel.

First, we consider the maximal domain of A defined as follows

$$D_M(A) = \{u \in C(\bar{I}) \cap C^2(I) \mid Au \in C(\bar{I})\}\ . \tag{II.2.4}$$

Theorem II.2.1 (Feller [54], Timmermans [69]) *The linear operator* $(A, D_M(A))$ *generates a* C_0*-semigroup in* $C(\overline{I})$ *if and only if* r_1 *ed* r_2 *are both entrance or natural endpoints. The semigroup is of positive contractions whenever it exists.*

If we consider Ventcel's boundary condition, we get the following domain of A

$$D_V(A) = \{u \in C(\overline{I}) \cap C^2(I) \mid \lim_{x \to r_1, r_2} Au(x) = 0\} \tag{II.2.5}$$

and we have the following generation result.

Theorem II.2.2 (Feller [54], Clément-Timmermans [44]) *The linear operator* $(A, D_V(A))$ *generates a* C_0*-semigroup in* $C(\overline{I})$ *if and only if both* r_1 *and* r_2 *are not entrance boundary points. The semigroup is of positive contractions whenever it exists.*

II.3 Cosine functions

In this section we briefly collect some definitions and properties of cosine functions. We limit ourselves to those notions which are strictly necessary for the sequel. An organic treatment of the subject can be found in the monographs [50, 17] and in the paper by Sova [68].

A family $(C(t))_{t\in\mathbb{R}}$ of linear operators on a Banach space E is a *cosine function* if the following conditions are satisfied

1. $C(0) = I$,
2. $C(s+t) + C(s-t) = 2\,C(s)C(t)$ for every $s, t \geq 0$.

A cosine function is said to be *strongly continuous* (or a C_0*-cosine function*) if for every $t_0 \geq 0$ and $f \in E$

$$\lim_{t\to t_0} \|C(t)f - C(t_0)f\| = 0\ .$$

The infinitesimal generator of a strongly continuous cosine function is the linear operator $A : D(A) \to E$ defined by

$$A(f) := \lim_{t\to 0} \frac{C(t)u - 2u + C(-t)u}{h^2}$$

on

$$D(A) := \left\{ u \in E \mid \lim_{t\to 0} \frac{C(t)u - 2u + C(-t)u}{h^2} \in E \right\}\ .$$

If a linear operator $A : D(A) \to E$ generates a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$, then the solution of the second-order hyperbolic problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = Au(t,x)\ , & t \in \mathbb{R}\ ; \\ u(0,x) = u_0(x)\ , & x \in \mathbb{R}\ ; \\ \dfrac{\partial}{\partial t} u(t,x)|_{t=0} = u_1(x)\ , & x \in \mathbb{R}\ . \end{cases}$$

is given by

$$u(t,x) = C(t)u_0(x) + \int_0^t C(s)u_1(x)\,ds\ .$$

If $(C(t))_{t\in\mathbb{R}}$ is a cosine function on E, there exist $\omega \in \mathbb{R}$ and $C_0 \geq 0$ such that

$$\|C(t)\| \leq C_0\, e^{\omega\,|t|}.$$

For every $\mathrm{Re}\,\lambda > \omega$, we have $\lambda^2 \in \rho(A)$ and

$$R(\lambda^2, A)u = \frac{1}{\lambda} \int_0^{+\infty} e^{-\lambda\, t}\, C(t)u\, dt\ , \qquad u \in E\ .$$

Chapter 1

Quantitative estimates

In this chapter we state our main results concerning with quantitative estimates of the convergence in Trotter's approximation theorem.

These estimates are related to the convergence of sequences of iterates of linear operators to an assigned semigroup, and to the resolvent operator of semigroups's generator.

We make no restriction on the general assumptions in the classical Trotter's Theorem and we only require a quantitative estimates of the voronovskaja-type formula in terms of suitable seminorms.

The results in this chapter are collected in [36, 37, 38].

1.1 Quantitative approximation of semigroups

We need a slight improvement of [64, Lemma III.5.1, p. 89]. On one hand this will provide us better constants in the subsequent main result and on the other hand it will be essential in order to avoid restrictions on the domain of the resolvent operator in the next section.

Lemma 1.1.1 *Let $L : E \to E$ be a linear operator on a Banach space E and assume that there exist $M > 0$ and $N \geq 1$ such that, for every $k \geq 1$,*

$$\|L^k\| \leq MN^k .$$

Then for every $u \in E$ and $k \geq 1$ we have

$$\|e^{k(L-I)}u - L^k u\| \leq M \left(N^{k-1}\sqrt{\frac{2k}{\pi}} + \frac{e^{k(N-1)} - N^k}{N-1} \right) \|Lu - u\| \quad (1.1.1)$$

if $N > 1$ and

$$\|e^{k(L-I)}u - L^k u\| \leq M \sqrt{\frac{2k}{\pi}} \|Lu - u\| \quad (1.1.2)$$

if $N = 1$.

Proof. Let $N > 1$; if $0 \le i < k$, we have

$$\begin{aligned}
\|L^k u - L^i u\| &= \sum_{j=i}^{k-1} \|L^{j+1}u - L^j u\| = \sum_{j=i}^{k-1} \|L^j(Lu-u)\| \\
&\le M\,\|Lu-u\| \sum_{j=i}^{k-1} N^j \\
&= M\,\|Lu-u\| \left(\frac{1-N^k}{1-N} - \frac{1-N^i}{1-N}\right) \\
&= M\,\frac{N^k - N^i}{N-1}\,\|Lu-u\|
\end{aligned}$$

and further

$$\frac{N^k - N^i}{N-1} = \sum_{j=i}^{k-1} N^j \le (k-i)N^{k-1}\ .$$

Similarly, if $0 \le k < i$,

$$\|L^k u - L^i u\| \le M\,\frac{N^i - N^k}{N-1}\,\|Lu-u\|\ .$$

Therefore we have

$$\begin{aligned}
&\|e^{k(L-I)}u - L^k u\| = \|e^{-k}\sum_{i=0}^{\infty}\frac{k^i}{i!}(L^i u - L^k u)\| \\
&\le e^{-k}\sum_{i=0}^{\infty}\frac{k^i}{i!}\,\|L^i u - L^k u\| \\
&\le e^{-k}\left(\sum_{i=0}^{k-1}\frac{k^i}{i!}\,\|L^i u - L^k u\| + \sum_{i=k+1}^{\infty}\frac{k^i}{i!}\,\|L^i u - L^k u\|\right) \\
&\le M\,\frac{e^{-k}}{N-1}\left(\sum_{i=0}^{k-1}\frac{k^i}{i!}\,(N^k - N^i) + \sum_{i=k+1}^{\infty}\frac{k^i}{i!}\,(N^i - N^k)\right)\|Lu-u\| \\
&= M\,\frac{e^{-k}}{N-1}\left(2\sum_{i=0}^{k-1}\frac{k^i}{i!}\,(N^k - N^i) + \sum_{i=0}^{\infty}\frac{k^i}{i!}\,(N^i - N^k)\right)\|Lu-u\| \\
&= M\,e^{-k}\left(2\sum_{i=0}^{k-1}\frac{k^i}{i!}\,\frac{N^k - N^i}{N-1} + \frac{1}{N-1}(e^{Nk} - N^k e^k)\right)\|Lu-u\| \\
&\le M\,e^{-k}\left(2\sum_{i=0}^{k-1}\frac{k^i}{i!}(k-i)N^{k-1} + \frac{e^{Nk} - N^k e^k}{N-1}\right)\|Lu-u\| \\
&= M\,e^{-k}\left(2N^{k-1}\left(k\sum_{i=0}^{k-1}\frac{k^i}{i!} - \sum_{i=1}^{k-1}\frac{k^i}{(i-1)!}\right) + \frac{e^{Nk} - N^k e^k}{N-1}\right) \\
&\quad \times \|Lu-u\|
\end{aligned}$$

$$= M\,e^{-k}\left(2N^{k-1}\left(k\sum_{i=0}^{k-1}\frac{k^i}{i!}-k\sum_{i=1}^{k-1}\frac{k^{i-1}}{(i-1)!}\right)+\frac{e^{Nk}-N^k e^k}{N-1}\right)$$
$$\times\|Lu-u\|$$
$$= M\,e^{-k}\left(2N^{k-1}k\frac{k^k}{k!}+\frac{e^{Nk}-N^k e^k}{N-1}\right)\|Lu-u\|\,.$$

Applying Stirling formula $k! = \sqrt{2\pi k}\,k^k\,e^{-k}\,e^{\theta_k/(12k)}$, $0 \le \theta_k \le 1$, we obtain

$$\|e^{k(L-I)}u - L^k u\|$$
$$\le M\,e^{-k}\left(2N^{k-1}k\frac{k^k}{\sqrt{2\pi k}\,k^k e^{-k}}+\frac{e^{Nk}-N^k e^k}{N-1}\right)\|Lu-u\|$$
$$= M\left(N^{k-1}\sqrt{\frac{2k}{\pi}}+\frac{e^{k(N-1)}-N^k}{N-1}\right)\|Lu-u\|$$

and this yields (1.1.1). Finally (1.1.2) can be similarly shown using the inequality $\|L^k u - L^i u\| \le M(k-i)\|Lu-u\|$ whenever $0 \le i < k$. □

Observe that (1.1.2) can be obtained from (1.1.1) taking the limit as $N \to 1^+$.

Now we can state one of the main results. Starting with a sequence of linear operators and the generator of a C_0-semigroup satisfying a quantitative Voronovskaja-type formula, we can evaluate the norm difference between the $k(n)$-iterate of the n-th linear operator and the semigroup. Suitable choices of the sequence $(k(n))_{n\in\mathbb{N}}$ ensure the convergence of the iterates to the semigroup.

Namely, let $(L_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators on a Banach space E and $A : \mathcal{D} \to E$ a linear operator satisfying the hypotheses of Trotter's Theorem, i.e. the stability condition

$$\|L_n^k\| \le M e^{\omega\,k/n}\,, \tag{1.1.3}$$

and the Voronovskaja-type formula

$$Af = \lim_{n\to\infty} n(L_n f - f)\,.$$

Moreover, assume that $\mathcal{D}$ is a dense subspace of E and $(\lambda - A)(\mathcal{D})$ is dense in E for some $\lambda > \omega$. Then from the classical Trotter's theorem (see [70, Theorem 5.1] or Theorem II.1.1) the closure of A generates a C_0-semigroup $(T(t))_{t\ge 0}$ which can be represented as limit of iterates of L_n, i.e. $T(t)f = \lim_{n\to\infty} L_n^{k(n)}(f)$ for every $f \in E$ and every sequence $(k(n))_{n\in\mathbb{N}}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$. Moreover, we have the following result.

Theorem 1.1.2 *Under the above assumptions, assume that D is a subspace of $\mathcal{D}$ such that for every $u \in D$ and $n \in \mathbb{N}$, we have*

$$\|n(L_n u - u)\| \le \varphi_n(u) \ , \tag{1.1.4}$$

and the following estimate of the Voronovskaja-type formula holds

$$\|n(L_n u - u) - Au\| \le \psi_n(u) \ , \tag{1.1.5}$$

where $\varphi_n, \psi_n : D \to [0, +\infty[$ are seminorms on the subspace D such that $\lim_{n\to\infty} \psi_n(u) = 0$ *for every $u \in D$.*

Then for every $t \ge 0$ and for every sequence $(k(n))_{n\ge1}$ of positive integers and $u \in D$, we have

$$\begin{aligned}\left\|T(t)u - L_n^{k(n)}u\right\| &\le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u) \\ &+M\left(\exp(\omega\, e^{\omega/n}\, t_n)\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\, e^{\omega\, k(n)/n}\, \frac{\sqrt{k(n)}}{n}\right. \\ &\left.+\frac{\omega}{n}\, \frac{k(n)}{n}\, \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n}\right)\right)\varphi_n(u)\end{aligned} \tag{1.1.6}$$

where $t_n := \sup\{t, k(n)/n\}$.

Proof. Let $n \ge 1$ and consider the linear bounded operator $A_n := n(L_n - I)$. It generates a uniformly continuous C_0-semigroup $(S_n(t))_{t\ge0}$ on E given by

$$S_n(t) = e^{t\, A_n} = e^{-nt}\, e^{nt\, L_n} = e^{-nt} \sum_{k=0}^{+\infty} \frac{(n\, t)^k}{k!}\, L_n^k \ , \qquad t \ge 0 \ .$$

Observe that $\mathcal{D}$ is a core for the closure of $(A, \mathcal{D})$ and consequently, from the first Trotter-Kato approximation theorem (see e.g. [48, Theorem 4.8, p. 209] and [64, Theorem III.4.4, p. 87]), we have that $(S_n(t))_{t\ge0}$ converges strongly to the C_0-semigroup $(T(t))_{t\ge0}$.

Consequently

$$\begin{aligned}\|S_n(t)\| &\le e^{-nt} \sum_{k=0}^{+\infty} \frac{(n\, t)^k}{k!}\, \|L_n^k\| \le M e^{-nt} \sum_{k=0}^{+\infty} \frac{(n\, t)^k}{k!}\, e^{\omega k/n} \\ &= M\, \exp\left(nt\left(e^{\omega/n} - 1\right)\right) \ , \qquad t \ge 0 \ .\end{aligned}$$

Now, let $(k(n))_{n\ge1}$ be an increasing sequence of positive integers and let $u \in D$. We have

$$\begin{aligned}\left\|T(t)u - L_n^{k(n)}u\right\| &\le \|T(t)u - S_n(t)u\| + \left\|S_n(t)u - S_n\left(\frac{k(n)}{n}\right)u\right\| \\ &+ \left\|S_n\left(\frac{k(n)}{n}\right)u - L_n^{k(n)}u\right\| \ ,\end{aligned} \tag{1.1.7}$$

and therefore we can get (1.1.6) by estimating each term in (1.1.7).

As regards the first term we observe that, for every $n, m \geq 1$, we have

$$\begin{aligned}\|S_n(t)u - S_m(t)u\| &= \left\| \int_0^1 \frac{d}{ds}\left(e^{snt(L_n-I)}\, e^{(1-s)mt(L_m-I)}u\right) ds \right\| \\ &\leq \int_0^1 \left\| (nt(L_n-I) - mt(L_m-I))\; e^{snt(L_n-I)}\, e^{(1-s)mt(L_m-I)}u \right\| ds \\ &\leq t\, \|n(L_n-I)u - m(L_m-I)u\| \int_0^1 \|S_n(st)\|\,\|S_m((1-s)t)\|\, ds \,.\end{aligned}$$

If $\omega > 0$ we have

$$\int_0^1 \|S_n(st)\|\,\|S_m((1-s)t)\|\, ds \leq M^2\, \frac{e^{nt\left(e^{\omega/n}-1\right)} - e^{mt\left(e^{\omega/m}-1\right)}}{nt\left(e^{\omega/n}-1\right) - mt\left(e^{\omega/m}-1\right)}$$

and hence

$$\begin{aligned}&\|S_n(t)u - S_m(t)u\| \\ &\quad\leq M^2\, \frac{e^{nt\left(e^{\omega/n}-1\right)} - e^{mt\left(e^{\omega/m}-1\right)}}{nt\left(e^{\omega/n}-1\right) - mt\left(e^{\omega/m}-1\right)}\, t\, \|n(L_n-I)u - m(L_m-I)u\| \;.\end{aligned}$$

Taking the limit as $m \to +\infty$ and using the inequality

$$e^x - 1 \leq x\, e^x \,, \qquad x \geq 0 \,, \tag{1.1.8}$$

we get

$$\begin{aligned}\|S_n(t)u - T(t)u\| &\leq M^2\, \frac{e^{nt\left(e^{\omega/n}-1\right)} - e^{\omega t}}{n\left(e^{\omega/n}-1\right) - \omega}\, \|n(L_n-I)u - Au\| \\ &\leq M^2\, \frac{e^{nt\left(e^{\omega/n}-1\right)} - e^{\omega t}}{n\left(e^{\omega/n}-1\right) - \omega}\, \psi_n(u) \\ &\leq M^2\, \frac{e^{\omega t}\left(e^{nt\left(e^{\omega/n}-1\right)-\omega t} - 1\right)}{n\left(e^{\omega/n}-1\right) - \omega}\, \psi_n(u) \\ &\leq M^2\, \frac{e^{\omega t} t\left(n\left(e^{\omega/n}-1\right) - \omega\right) e^{nt\left(e^{\omega/n}-1\right)-\omega t}}{n\left(e^{\omega/n}-1\right) - \omega}\, \psi_n(u) \\ &= M^2\, t e^{nt\left(e^{\omega/n}-1\right)}\, \psi_n(u) \\ &\leq M^2\, t e^{\omega t e^{\omega/n}}\, \psi_n(u) \,.\end{aligned}$$

Similarly, if $\omega = 0$ we obtain $\|S_n(t)u - T(t)u\| \leq M^2\, t\, \psi_n(u)$.

In regard to the second term in (1.1.7), using [64, Theorem I.2.4, d), p. 5] and (1.1.8), we have

$$\begin{aligned}\left\|S_n(t)u - S_n\left(\frac{k(n)}{n}\right)u\right\| &= \left\|\int_{k(n)/n}^{t} S_n(s)\,(n(L_n-I))u\,ds\right\| \\ &\le M\,\exp\left(n\,t_n\left(e^{\omega/n}-1\right)\right)\left|\frac{k(n)}{n}-t\right|\,\|n(L_n-I)u\| \\ &\le M\,e^{\omega\,t_n\,e^{\omega/n}}\,\left|\frac{k(n)}{n}-t\right|\,\varphi_n(u)\,,\end{aligned}$$

where $t_n := \max\{t, k(n)/n\}$.

Finally, from Lemma 1.1.1 with $N = e^{\omega/n}$, $k = k(n)$ and $L = L_n$, we obtain

$$\begin{aligned}&\left\|S_n\left(\frac{k(n)}{n}\right)u - L_n^{k(n)}u\right\| = \left\|e^{k(n)(L_n-I)}u - L_n^{k(n)}u\right\| \\ &\le M\left(e^{\omega(k(n)-1)/n}\sqrt{\frac{2k(n)}{\pi}} + \frac{e^{k(n)(e^{\omega/n}-1)} - e^{\omega\,k(n)/n}}{e^{\omega/n}-1}\right)\frac{\|n(Lu-u)\|}{n}\,.\end{aligned}$$

Using (1.1.8), we have

$$\begin{aligned}\frac{e^{k(n)(e^{\omega/n}-1)} - e^{\omega\,k(n)/n}}{e^{\omega/n}-1} &\le \frac{e^{k(n)\,(\omega/n)\,e^{\omega/n}} - e^{\omega\,k(n)/n}}{e^{\omega/n}-1} \\ &= \frac{e^{\omega\,k(n)/n}(e^{\omega\,(e^{\omega/n}-1)\,k(n)/n}-1)}{e^{\omega/n}-1} \\ &\le \omega\,(e^{\omega/n}-1)\,\frac{k(n)}{n}\,\frac{e^{\omega\,k(n)/n}\,e^{\omega\,(e^{\omega/n}-1)k(n)/n}}{e^{\omega/n}-1} \\ &= \omega\,\frac{k(n)}{n}\,e^{\omega\,e^{\omega/n}k(n)/n}\end{aligned}$$

and consequently

$$\begin{aligned}&\left\|S_n\left(\frac{k(n)}{n}\right)u - L_n^{k(n)}u\right\| \\ &\qquad\le M\left(e^{\omega\,k(n)/n}\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n} + \frac{\omega}{n}\,\frac{k(n)}{n}\,e^{\omega\,e^{\omega/n}k(n)/n}\right)\varphi_n(u)\,.\end{aligned}$$

If $\omega = 0$, from (1.1.2) we get $\|S_n(k(n)/n)u - L_n^{k(n)}u\| \le M\,\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\,\varphi_n(u)$.

Hence, collecting the above inequalities, from (1.1.7) we get

$$\begin{aligned}\left\|T(t)u - L_n^{k(n)}u\right\| &\le M^2\,t\,e^{\omega\,t\,e^{\omega/n}}\,\psi_n(u) + M\,e^{\omega\,t_n\,e^{\omega/n}}\,\left|\frac{k(n)}{n}-t\right|\,\varphi_n(u) \\ &+M\left(e^{\omega\,k(n)/n}\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n} + \frac{\omega}{n}\,\frac{k(n)}{n}\,e^{\omega\,e^{\omega/n}k(n)/n}\right)\varphi_n(u)\end{aligned}$$

if $\omega > 0$ and

$$\left\| T(t)u - L_n^{k(n)} u \right\| \le M^2\, t\, \psi_n(u) + M \left| \frac{k(n)}{n} - t \right| \varphi_n(u) + M \sqrt{\frac{2}{\pi}} \frac{\sqrt{k(n)}}{n} \varphi_n(u)$$

if $\omega = 0$, as required. □

Remarks 1.1.3

1. From the preceding result, the order of convergence of $(L_n^{k(n)})_{n\ge1}$ to $T(t)$ on the subspace D depends on the order of convergence of $(\psi_n)_{n\ge1}$ (and eventually of $(\varphi_n)_{n\ge1}$) to 0, on the order of convergence of $|t - k(n)/n|_{n\ge1}$ to 0 (the best choice is $k(n) = [nt]$ which gives an order of convergence of $1/n$) and on $\sqrt{k(n)}/n$ which behaves like $\sqrt{t/n}$ as $n \to +\infty$. In most applications, an asymptotic behavior like $\sqrt{t/n}$ as $n \to +\infty$ can be obtained.

2. We can always take $\varphi_n(u) := \|Au\| + \psi_n(u)$ but we have preferred to introduce an independent estimate of $\|n(L_n - I)u\|$ since some applications can be more precise, as in the case of Stancu-Schnabl operators considered in the next section.

3. If the operators L_n are linear contractions, then the stability condition (1.1.3) on its iterates is automatically satisfied and from the representation (II.1.2) we obtain that the semigroup $(T(t))_{t\ge0}$ is itself of linear contractions.

4. If $u \in D$ and $L_n u = u$ for every $n \ge 1$, the proof of Theorem 1.1.2 also gives $T(t)u = u$ for every $t \ge 0$. Hence, the semigroup preserves every function which is preserved by all approximating operators.

5. Observe that estimate (1.1.6) holds uniformly with respect to t in compact intervals. □

In concrete applications, we often take $k(n) = [nt]$. In this case we obviously have $t_n = t$ and

$$\left| \frac{[nt]}{n} - t \right| = \frac{nt - [nt]}{n} \le \frac{1}{n} .$$

Hence estimate (1.1.6) yields

$$\left\| T(t)u - L_n^{[nt]} u \right\| \le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u) \tag{1.1.9}$$
$$+ \frac{M}{\sqrt{n}} \left(\frac{\exp(\omega\, e^{\omega/n}\, t)}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\, e^{\omega\, t} + \frac{\omega\, t}{\sqrt{n}} \exp\left(\omega\, e^{\omega/n}\, t\right) \right) \varphi_n(u) .$$

In most applications the growth bound ω of the semigroup will be equal to 0; in this particular case (1.1.6) and (1.1.9) become respectively

$$\left\| T(t)u - L_n^{k(n)} u \right\| \le M^2\, t\, \psi_n(u) + M \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}} \, \frac{\sqrt{k(n)}}{n} \right) \varphi_n(u) \tag{1.1.10}$$

and

$$\left\| T(t)u - L_n^{[nt]} u \right\| \le M^2\, t\, \psi_n(u) + \frac{M}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right) \varphi_n(u)\, . \tag{1.1.11}$$

1.2 Estimate of the resolvent operator

1.2.1 Quantitative estimate of the convergence to the resolvent operator

The next result is concerned with the approximation of the resolvent operator of the closure of $(A, \mathcal{D})$.

If E is a real Banach space we consider the complexification $E_c \sim E \times E$ defined as usual by defining $(\alpha + i\beta)u := (\alpha u, \beta u)$ for every $\alpha, \beta \in \mathbb{R}$ and $u \in E$. An operator $L : E \to E$ can be regarded as acting on E_c by setting $L(u, v) = (L(u), L(v))$ for every $u, v \in E$.

At this point, for every $n \geq 1$ we define the linear operator $M_{\lambda,n} : E_c \to E_c$ as follows

$$M_{\lambda,n}u := \int_0^{+\infty} e^{-\lambda t} L_n^{[n\,t]} u \, dt \ , \qquad u \in E_c \ .$$

Theorem 1.2.1 *Consider the same assumptions of Theorem 1.1.2. If* $\omega \geq 0$*, then for every* $n \geq 1$*,* $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}\lambda > \omega\, e^{\omega/n}$ *and* $u \in D$*, we have*

$$\|R(\lambda, A)u - M_{\lambda,n}u\| \leq \frac{M^2}{(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2}\,\psi_n(u) + \frac{M}{\sqrt{n}} \left(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})} + \frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda - \omega)^{3/2}} + \frac{\omega}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2} \right) \varphi_n(u) \ . \tag{1.2.1}$$

In particular, we have that the sequence $(M_{\lambda,n})_{n\geq 1}$ *strongly converges to* $R(\lambda, A)$*.*

Proof. Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega\, e^{\omega/n}$. Using the integral representation of the resolvent operator (see, e.g., [48, Theorem II.1.10, (i), p. 55] and Chapter II) and taking into account (1.1.9) and the elementary properties of the gamma function, for every $n \geq 1$ and $u \in D$ we have

$$\|R(\lambda, A)u - M_{\lambda,n}u\| \leq \int_0^{+\infty} e^{-\operatorname{Re}\lambda\, t} \|T(t)u - L_n^{[nt]}u\| \, dt, \tag{1.2.2}$$

then

$$\begin{aligned}
\|R(\lambda,A)u - M_{\lambda,n}u\| &\le M^2\,\psi_n(u)\int_0^{+\infty} t\,\exp\Big(\big(\omega e^{\omega/n}-\operatorname{Re}\lambda\big)\,t\Big)\,dt\\
&\quad+\frac{M}{n}\,\varphi_n(u)\int_0^{+\infty}\exp\Big(\big(\omega e^{\omega/n}-\operatorname{Re}\lambda\big)\,t\Big)\,dt\\
&\quad+M\,\varphi_n(u)\sqrt{\frac{2}{\pi\,n}}\int_0^{+\infty}\sqrt{t}\,\exp\left((\omega-\operatorname{Re}\lambda)t\right)\,dt\\
&\quad+M\,\varphi_n(u)\,\frac{\omega}{n}\int_0^{+\infty} t\,\exp\Big(\big(\omega e^{\omega/n}-\operatorname{Re}\lambda\big)\,t\Big)\,dt\\
&=\frac{M^2}{(\operatorname{Re}\lambda-\omega\,e^{\omega/n})^2}\,\psi_n(u)+\frac{M}{n(\operatorname{Re}\lambda-\omega\,e^{\omega/n})}\,\varphi_n(u)\\
&\quad+\frac{M}{\sqrt{2n}\,(\operatorname{Re}\lambda-\omega)^{3/2}}\,\varphi_n(u)+\frac{M\omega}{n(\operatorname{Re}\lambda-\omega\,e^{\omega/n})^2}\,\varphi_n(u)\;.
\end{aligned}$$

Finally, the last part is a consequence of the density of $\mathcal{D}$ and the fact that estimate (1.2.2) imply that the sequence $(M_{\lambda,n}u)_{n\ge1}$ converges to $R(\lambda,A)u$ for every $u\in\mathcal{D}$. □

Remark 1.2.2 We explicitly point out that estimate (1.2.1) holds whenever $\operatorname{Re}\lambda>\omega$ if n is large enough (namely $n>\omega/\log(\operatorname{Re}\lambda/\omega)$), since $\lim_{n\to+\infty}e^{\omega/n}=1$. □

In the particular case $\omega=0$ we get, for every $n\ge1$, $\lambda\in\mathbb{C}$ such that $\operatorname{Re}\lambda>0$ and $u\in D$,

$$\|R(\lambda,A)u-M_{\lambda,n}u\|\le\frac{M^2}{(\operatorname{Re}\lambda)^2}\,\psi_n(u)+\frac{M}{\sqrt{n}\operatorname{Re}\lambda}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2\operatorname{Re}\lambda}}\right)\varphi_n(u)\;. \tag{1.2.3}$$

1.2.2 Approximation processes for resolvent operators

In this section we introduce some general sequences of linear operators, obtained from classical approximation processes, which approximate the resolvent operators of the generator of the C_0-semigroups.

The main aim is the possibility of representing the resolvent operators in terms of classical approximation operators.

First, we consider a sequence of $(L_n)_{n\ge1}$ of linear operators on a complex Banach space E (we consider its complexification if E is real) and assume that the hypotheses of Trotter's Theorem ore satisfied.

Now, let $(a_n)_{n\ge1}$ be a sequence of positive integers tending to $+\infty$. For every $n\ge1$, we consider the linear operator $P_{\lambda,a_n,n}:E\to E$ defined by

$$P_{\lambda,a_n,n}u:=\frac{1}{n}\sum_{k=0}^{a_n}e^{-\lambda k/n}L_n^k u\,,\qquad u\in E\;. \tag{1.2.4}$$

Theorem 1.2.3 *If the sequence* $(a_n)_{n\geq 1}$ *satisfies*

$$\lim_{n\to+\infty} \frac{a_n}{n} = +\infty \,, \tag{1.2.5}$$

then $\lim_{n\to+\infty} P_{\lambda,a_n,n}u = R(\lambda,A)u$ *for every* $u \in E$.

Proof. Since $\operatorname{Re}\lambda > \omega$ we have $\left\|e^{-\lambda/n}L_n\right\| \leq Me^{-(\operatorname{Re}\lambda-\omega)/n}$ and consequently $\|P_{\lambda,a_n,n}\| \leq M/(1-e^{-(\operatorname{Re}\lambda-\omega)})$. Hence the sequence $(P_{\lambda,a_n,n})_{n\geq 1}$ is equibounded and we can show the convergence property on the dense subspace $\mathcal{D}$. Let $u \in \mathcal{D}$; from [1, (1.3)], we have

$$R(\lambda,A)u = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{\infty} e^{-\lambda k/n} \frac{1-e^{-\lambda/n}}{\lambda/n} L_n^k u$$

and consequently

$$\begin{aligned}\|P_{\lambda,a_n,n}u - R(\lambda,A)u\| \leq & \frac{1}{n}\left\|\sum_{k=0}^{a_n} e^{-\lambda k/n}L_n^k u - \sum_{k=0}^{\infty} e^{-\lambda k/n}\frac{1-e^{-\lambda/n}}{\lambda/n}L_n^k u\right\| \\ &+ \left\|\frac{1}{n}\sum_{k=0}^{\infty} e^{-\lambda k/n}\frac{1-e^{-\lambda/n}}{\lambda/n}L_n^k u - R(\lambda,A)u\right\| \,.\end{aligned} \tag{1.2.6}$$

The second term in (1.2.6) tends to 0 by [1, (1.3)]. As regards to the first term, it is majored by

$$\begin{aligned}&\frac{M}{n}\|u\| \sum_{k=a_n+1}^{+\infty} e^{-(\operatorname{Re}\lambda-\omega)k/n} + \frac{1}{n}\left\|\sum_{k=0}^{+\infty} e^{-\lambda k/n}\left(1-\frac{1-e^{-\lambda/n}}{\lambda/n}\right)L_n^k u\right\| \\ &\leq \frac{M}{n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right)}\|u\| \left(e^{-(\operatorname{Re}\lambda-\omega)(a_n+1)/n} + \left|1-\frac{1-e^{-\lambda/n}}{\lambda/n}\right|\right) \,.\end{aligned} \tag{1.2.7}$$

Since $\lim_{n\to+\infty} n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right) = \operatorname{Re}\lambda - \omega$, the assumption (1.2.5) ensures that the first term in (1.2.6) tends to 0 and this completes the proof. □

Our next aim is to provide a quantitative estimate in the above Theorem 1.2.3. Now we assume that there exist the seminorms $\varphi_n, \psi_n : D \to [0,+\infty[$ on a subspace D of $\mathcal{D}$ such that $\lim_{n\to\infty}\psi_n(u) = 0$ for every $u \in D$ and

$$\|n(L_n u - u)\| \leq \varphi_n(u) \,, \quad \|n(L_n u - u) - Au\| \leq \psi_n(u) \,. \tag{1.2.8}$$

Theorem 1.2.4 *Under assumptions (1.2.5) and (1.2.8), for every* $n > \omega/\log(\mathrm{Re}\,\lambda/\omega)$ *(or* $n \geq 1$ *if* $\omega = 0$*) and* $u \in D$*, we have*

$$\|P_{\lambda,a_n,n}u - R(\lambda,A)u\| \leq \frac{M^2}{(\mathrm{Re}\,\lambda - \omega\, e^{\omega/n})^2}\,\psi_n(u) + \frac{M}{\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\mathrm{Re}\,\lambda - \omega\, e^{\omega/n})} + \frac{1}{\sqrt{2}\,(\mathrm{Re}\,\lambda-\omega)^{3/2}} + \frac{\omega}{\sqrt{n}\,(\mathrm{Re}\,\lambda - \omega\, e^{\omega/n})^2}\right)\varphi_n(u) + \frac{M\left(e^{-(\mathrm{Re}\,\lambda-\omega)a_n/n} + \frac{|\lambda|^{3/2}}{n\,|\mathrm{Re}\sqrt{\lambda}|}\right)}{(\mathrm{Re}\,\lambda-\omega)\left(1-\frac{\mathrm{Re}\,\lambda-\omega}{n}\right)}\,\|u\|\,. \tag{1.2.9}$$

Proof. We estimate the two terms at the righthand side of (1.2.6). Set $\lambda = |\lambda|\,e^{i\theta}$; from the series expansion of $e^{-\lambda/n}$, we get

$$1 - \frac{1-e^{-\lambda/n}}{\lambda/n} = \sum_{k=2}^{+\infty}\frac{(-1)^k\,\lambda^{k-1}}{n^{k-1}k!} = -\sum_{k=1}^{+\infty}\frac{|\lambda|^k e^{ik(\theta+\pi)}}{n^k(k+1)!}\,. \tag{1.2.10}$$

Set for simplicity for every $k \in \mathbb{N}$, $b_k := \sum_{j=0}^{k} e^{ij(\theta+\pi)}$ and $c_k := |\lambda|^k/(n^k\cdot (k+1)!)$. For every $r \in \mathbb{N}$ we obtain

$$\sum_{k=1}^{r} c_k e^{ik(\theta+\pi)} = \sum_{k=1}^{r} c_k\, b_k - \sum_{k=1}^{r} c_k\, b_{k-1} = \sum_{k=1}^{r}(c_k - c_{k-1})\, b_k + c_{r+1}\, b_r - c_1\, b_0$$

and, letting $r \to +\infty$, $\left|\sum_{k=1}^{+\infty} c_k e^{ik(\theta+\pi)}\right| \leq \sum_{k=1}^{+\infty}(c_k - c_{k-1})\,|b_k| + c_1\,|b_0|$. Since

$$|b_k| = \left|\frac{1-e^{i(k+1)(\theta+\pi)}}{1-e^{i(\theta+\pi)}}\right| \leq \frac{2}{\sqrt{(1+\cos\theta)^2+\sin^2\theta}} = \frac{2}{\sqrt{2(1+\cos\theta)}} = \frac{1}{|\cos(\theta/2)|}$$

we conclude

$$\left|\sum_{k=1}^{+\infty} c_k e^{ik(\theta+\pi)}\right| \leq \frac{1}{|\cos(\theta/2)|}\left(\sum_{k=1}^{+\infty}(c_k - c_{k-1}) + c_1\right) = \frac{2c_1}{|\cos(\theta/2)|}\,.$$

Hence, from (1.2.10) we obtain

$$\left|1 - \frac{1-e^{-\lambda/n}}{\lambda/n}\right| \leq \frac{|\lambda|}{n\,|\sin((\theta+\pi)/2)|} = \frac{|\lambda|}{n\,|\cos(\theta/2)|} = \frac{|\lambda|^{3/2}}{n\,\left|\mathrm{Re}\sqrt{\lambda}\right|}\,. \tag{1.2.11}$$

Since $1 - e^x \geq -x(1+x)$ whenever $x \leq 0$, we have

$$\frac{1}{n\left(1-e^{-(\mathrm{Re}\,\lambda-\omega)/n}\right)} \leq \frac{1}{(\mathrm{Re}\,\lambda-\omega)\,(1-(\mathrm{Re}\,\lambda-\omega)/n)} \tag{1.2.12}$$

and consequently, from (1.2.7) we get the following estimate of the first term in (1.2.6)

$$\frac{1}{n}\left\|\sum_{k=0}^{a_n} e^{-\lambda k/n} L_n^k u - \sum_{k=0}^{\infty} e^{-\lambda k/n}\,\frac{1-e^{-\lambda/n}}{\lambda/n} L_n^k u\right\| \le \frac{M\left(e^{-(\mathrm{Re}\,\lambda-\omega)a_n/n} + \frac{|\lambda|^{3/2}}{n|\mathrm{Re}\,\sqrt{\lambda}|}\right)}{(\mathrm{Re}\,\lambda-\omega)\,(1-(\mathrm{Re}\,\lambda-\omega)/n)}\|u\| \, .$$

In order to estimate the second term in (1.2.6), we consider the operator $M_{\lambda,n}$ introduced in the previous section and observe that

$$M_{\lambda,n}u := \int_0^{+\infty} e^{-\lambda\, t} L_n^{[n\,t]} u\, dt = \sum_{k=0}^{\infty} e^{-\lambda k/n}\,\frac{1-e^{-\lambda/n}}{\lambda}\, L_n^k u \, , \qquad u \in E$$

the last equality is valid since $L_n^{[nt]}$ is independent of t on each interval $[k/n, (k+1)/n[$. Then we can estimate the second term in (1.2.6) by Theorem 1.2.1

$$\|R(\lambda, A)u - M_{\lambda,n}u\| \le \frac{M^2}{(\mathrm{Re}\,\lambda-\omega\, e^{\omega/n})^2}\,\psi_n(u) + \frac{M}{\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\mathrm{Re}\,\lambda-\omega\, e^{\omega/n})} + \frac{1}{\sqrt{2}\,(\mathrm{Re}\,\lambda-\omega)^{3/2}} + \frac{\omega}{\sqrt{n}\,(\mathrm{Re}\,\lambda-\omega\, e^{\omega/n})^2}\right)\varphi_n(u) \, .$$

Using the above inequalities the proof is complete. □

Taking $a_n \ge [n\log n/\mathrm{Re}\,\lambda]$, estimate (1.2.9) becomes

$$\|P_{\lambda,a_n,n}u - R(\lambda, A)u\| \le C_1(\lambda)\,\psi_n(u) + \frac{C_2(\lambda)}{\sqrt{n}}\,\varphi_n(u) + \frac{C_3(\lambda)}{n}\,\|u\| \, , \quad (1.2.13)$$

for every $\omega \ge 0$, $u \in D$ and $n > \omega/\log(\mathrm{Re}\,\lambda/\omega)$ (or $n \ge 1$ if $\omega = 0$), where $C_i(\lambda)$, $i = 1, 2, 3$, are suitable constants depending only on λ.

Chapter 2

Applications to classical sequences of operators

2.1 Infinite-dimensional setting

2.1.1 Application to Schnabl-type operators

In the preceding chapter we have stated a general result concerning with the approximation of a C_0-semigroup in an abstract setting (Theorem 1.1.2). The general setting is motivated by some recent applications in population genetics involving Bernstein-Schnabl operators in an infinite-dimensional setting (see, e.g., [2]). Moreover starting with this sequence of operators, other sequences such as Stancu and Lototsky operators were considered in the same setting.

In this section, we consider all these operators and in each case we study the consequences of the quantitative estimates in the preceding chapter.

We start with the Bernstein-Schnabl operators and we state the quantitative estimates of the convergence of their iterates to the associated semigroup. These operators have been introduced and studied in [2, 24] and a unified treatment of these operators can be found in [9, Chapter 6] together with supplementary references.

The result in this section are collected in [37].

Consider a metrizable convex compact subset K of some locally convex space and let $T : C(K) \to C(K)$ be a positive projection on $C(K)$ such that its range $H := T(C(K))$ contains the subspace $A(K)$ of $C(K)$ consisting of all affine continuous real functions on K and is invariant under convex translation, in the sense that the function $x \mapsto h(tx + (1-t)z)$ belongs to H whenever $h \in H$, $t \in [0,1]$ and $z \in K$.

Now, for every $x \in K$ consider the probability Radon measure $\mu_x^T \in \mathcal{M}^+(K)$ defined by $\mu_x^T(f) = Tf(x)$ for every $f \in C(K)$.

For every $n \geq 1$, the n-th Bernstein-Schnabl operator $B_n : C(K) \to$

$C(K)$ associated with the projection T is defined by setting, for every $f \in C(K)$ and $x \in K$,

$$B_n f(x) := \int_K \dots \int_K f\left(\frac{x_1 + \dots + x_n}{n}\right) d\mu_x^T(x_1) \dots d\mu_x^T(x_n) . \qquad (2.1.1)$$

Lototsky-Schnabl operators are defined by considering a strictly positive function $\gamma \in C(K)$ with values in the interval $]0,1]$ and by substituting the measures μ_x^T with the probability Radon measures $\nu_x^T := \gamma(x)\,\mu_x^T + (1 - \gamma(x))\,\varepsilon_x \in \mathcal{M}^+(K)$, where ε_x denotes the Dirac measure at $x \in K$. Hence, for every $n \geq 1$, the n-th Lototsky-Schnabl operator $L_{n,\gamma} : C(K) \to C(K)$ is defined by

$$L_{n,\gamma} f(x) := \int_K \dots \int_K f\left(\frac{x_1 + \dots + x_n}{n}\right) d\nu_x^T(x_1) \dots d\nu_x^T(x_n) \qquad (2.1.2)$$

for every $f \in C(K)$ and $x \in K$.

Finally, in order to introduce the Stancu-Schnabl operators, we first define the polynomial

$$p_n(a) := \prod_{j=0}^{n-1} (1 + j\,a) , \qquad a \in \mathbb{R} ;$$

moreover, we use the convention to write $|v|_k = n$ for $v = (v_1, \dots, v_k) \in \mathbb{N}^k$ satisfying $v_1, \dots, v_k \geq 1$ and $\sum_{i=1}^k v_i = n$.

Now, we fix a sequence $(a_n)_{n\geq 1}$ of positive functions in $C(K)$ such that $(n\,a_n)_{n\geq 1}$ uniformly converges to $b \in C(K)$; as observed in [24] the result concerning a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers in [22] and [9] remain unchanged in the case where $(a_n)_{n\in\mathbb{N}}$ is a sequence of real continuous functions.

The n-th Stancu-Schnabl operator $S_{n,a_n} : C(K) \to C(K)$ is defined by setting, for every $f \in C(K)$ and $x \in K$,

$$\begin{aligned} S_{n,a_n} f(x) \quad := \quad & \frac{1}{p_n(a_n(x))} \sum_{k=1}^{n} \frac{n!}{k!} a_n^{n-k}(x) \sum_{|v|_k = n} \frac{1}{v_1 \cdots v_k} \qquad (2.1.3) \\ & \times \int_K \dots \int_K f\left(\frac{v_1\,x_1 + \dots + v_k\,x_k}{n}\right) d\mu_x^T(x_1) \dots d\mu_x^T(x_k) . \end{aligned}$$

Observe that the Bernstein-Schnabl operators can be obtained as a particular case of both Lototsky-Schnabl operators taking $\gamma = 1$ and of Stancu-Schnabl operators taking $a_n = 0$ for every $n \geq 1$.

Now, denote by $A_\infty(K)$ the subalgebra of $C(K)$ consisting of all functions in $C(K)$ which are finite products of elements of $A(K)$ and define the operator $L_T : A_\infty(K) \to C(K)$ by setting, for every $f = h_1 \cdots h_m \in A_\infty(K)$,

$$L_T(h_1 \cdots h_m) := \begin{cases} 0 \,, & m = 1 \,, \\ T(h_1\, h_2) - h_1\, h_2 \,, & m = 2 \,, \\ \displaystyle\sum_{1 \le i < j \le m} (T(h_i\, h_j) - h_i\, h_j) \prod_{\substack{r=1 \\ r \ne i,j}}^{m} h_r \,, & m \ge 3 \,. \end{cases} \tag{2.1.4}$$

Observe that $A_\infty(K)$ is dense in $C(K)$ by the Stone-Weierstrass theorem.

Moreover, if K is a compact convex subset of $\mathbb{R}^d$ then $A_\infty(K) \subset C^2(K)$ and for every $f \in A_\infty(K)$ we have (see [9, Theorem 6.2.5, p. 433])

$$L_T(f) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}\, \frac{\partial^2 f}{\partial x_i\, \partial x_j} \,,$$

where $a_{ij}(x) := T((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x) = T(\mathrm{pr}_i\, \mathrm{pr}_j)(x) - x_i\, x_j$ and pr_i denotes the canonical i-th projection.

In order to apply the results in Chapter 1, we recall that

$$\|L_{n,\gamma}\| \le 1 \,, \quad \|S_{n,a_n}\| \le 1 \,, \qquad n \ge 1 \,;$$

moreover, for every $f \in A_\infty(K)$, from [9, Section 6.2, pp. 427–429] we easily obtain

$$\|n(L_{n,\gamma} f - f) - \gamma\, L_T(f)\| \le \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \tag{2.1.5}$$

$$\|n(L_{n,\gamma} f - f)\| \le \|\gamma\, L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \,, \tag{2.1.6}$$

and further

$$\|n(S_{n,a_n} f - f) - (1+b)\, L_T(f)\| \le \left\| \frac{n a_n - b}{1 + a_n} \right\| \, \|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \,, \tag{2.1.7}$$

$$\|n(S_{n,a_n} f - f)\| \le \left\| \frac{1 + n a_n}{1 + a_n} \right\| \, \|L_T(f)\| + \frac{1}{n} \sum_{i \in I} \|L_i(f)\| \,, \tag{2.1.8}$$

where I is a set of indices and, for every $i \in I$, $L_i : A_\infty(K) \to C(K)$ is a linear map such that, for every $f = h_1 \cdots h_m \in A_\infty(K)$, $L_i(h_1 \cdots h_m)$ belongs to the linear subspace generated by

$$\{h_1 \cdots h_m, T(h_1 h_2) h_3 \cdots h_m, \ldots, T(h_1 h_2 h_3) h_4 \cdots h_m, \ldots, T(h_1 \cdots h_m)\}$$

and is different from 0 only for a finite set of indices.

We have the following result.

Theorem 2.1.1 *Assume that $T(h_1 h_2) \in A(K)$ for every $h_1, h_2 \in A(K)$. Then*

1) **Lototsky operators**
The closure of the operator $(\gamma L_T, A_\infty(K))$ *generates a* C_0*-semigroup* $(T_\gamma(t))_{t\geq 0}$ *of positive contractions on* $C(K)$ *and, for every* $t \geq 0$, $(k(n))_{n\geq 1}$ *sequence of positive integers and* $f \in A_\infty(K)$, *we have*

$$\|T_\gamma(t)f - L_{n,\gamma}^{k(n)} f\| \leq \frac{t}{n} \sum_{i\in I} \|L_i(f)\| \tag{2.1.9}$$
$$+ \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}} \frac{\sqrt{k(n)}}{n} \right) \left(\|\gamma\, L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right)$$

and in particular, taking $k(n) = [nt]$

$$\|T_\gamma(t)f - L_{n,\gamma}^{[nt]} f\| \leq \frac{t}{n} \sum_{i\in I} \|L_i(f)\| \tag{2.1.10}$$
$$+\frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right) \left(\|\gamma\, L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) .$$

Moreover, for every $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}\lambda > 0$ *and* $n \geq 1$, *consider the operator* $L_{\lambda,n,\gamma} : C(K) \to C(K)$ *defined by*

$$L_{\lambda,n,\gamma} f := \int_0^{+\infty} e^{-\lambda\, t} L_{n,\gamma}^{[n\,t]} f \, dt \,, \qquad f \in C(K)$$

and let $R(\lambda, \gamma\, L_T)$ *be the resolvent operator of the closure of* $(\gamma\, L_T, A_\infty(K))$. *Then, for every* $n \geq 1$ *and* $f \in A_\infty(K)$ *we have*

$$\|R(\lambda, \gamma\, L_T)f - L_{\lambda,n,\gamma} f\| \leq \frac{1}{n(\operatorname{Re}\lambda)^2} \sum_{i\in I} \|L_i(f)\| \tag{2.1.11}$$
$$+\frac{1}{\sqrt{n}\operatorname{Re}\lambda} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2\operatorname{Re}\lambda}} \right) \left(\|\gamma\, L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) .$$

2) **Stancu operators**
The closure of the operator $((1{+}b)\, L_T, A_\infty(K))$ *generates a* C_0*-semigroup* $(T_{1+b}(t))_{t\geq 0}$ *of positive contractions on* $C(K)$ *and, for every* $t \geq 0$, $(k(n))_{n\geq 1}$ *sequence of positive integers and* $f \in A_\infty(K)$, *we have*

$$\|T_{1+b}(t)f - S_{n,a_n}^{k(n)} f\| \leq t \left(\left\| \frac{na_n - b}{1 + a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right)$$
$$+ \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}} \frac{\sqrt{k(n)}}{n} \right)$$
$$\times \left(\left\| \frac{1 + na_n}{1 + a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) \tag{2.1.12}$$

and in particular, taking $k(n) = [nt]$,

$$\|T_{1+b}(t)f - S_{n,a_n}^{[nt]} f\| \le t \left(\left\| \frac{na_n - b}{1+a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right)$$
$$+ \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right)$$
$$\times \left(\left\| \frac{1+na_n}{1+a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) . \tag{2.1.13}$$

Moreover, for every $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}\lambda > 0$ *and* $n \ge 1$*, consider the operator* $S_{\lambda,n,a_n} : C(K) \to C(K)$ *defined by*

$$S_{\lambda,n,a_n} f := \int_0^{+\infty} e^{-\lambda t} S_{n,a_n}^{[n\,t]} f\, dt , \qquad f \in C(K) .$$

If we denote by $R(\lambda, (1+b)\, L_T)$ *the resolvent operator of the closure of* $((1+b)\, L_T, A_\infty(K))$*, for every* $n \ge 1$ *and* $f \in A_\infty(K)$ *we have*

$$\|R(\lambda, (1+b)\, L_T)f - S_{\lambda,n,a_n} f\| \tag{2.1.14}$$
$$\le \frac{1}{(\operatorname{Re}\lambda)^2} \left(\left\| \frac{na_n - b}{1+a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right)$$
$$+ \frac{1}{\sqrt{n}\operatorname{Re}\lambda} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2\operatorname{Re}\lambda}} \right) \tag{2.1.15}$$
$$\times \left(\left\| \frac{1+na_n}{1+a_n} \right\| \|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right) .$$

3) **Bernstein operators**
In the particular case of Bernstein-Schnabl operators the preceding estimates become

$$\|T(t)f - B_n^{k(n)} f\| \le \frac{t}{n} \sum_{i\in I} \|L_i(f)\| \tag{2.1.16}$$
$$+ \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}} \frac{\sqrt{k(n)}}{n} \right) \left(\|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right)$$

and, taking $k(n) = [nt]$,

$$\|T(t)f - B_n^{[nt]} f\| \le \frac{t}{n} \sum_{i\in I} \|L_i(f)\| \tag{2.1.17}$$
$$+ \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right) \left(\|L_T(f)\| + \frac{1}{n} \sum_{i\in I} \|L_i(f)\| \right)$$

and further

$$\|R(\lambda, L_T)f - B_{\lambda,n}f\| \le \frac{1}{n(\operatorname{Re}\lambda)^2}\sum_{i\in I}\|L_i(f)\| \tag{2.1.18}$$
$$+\frac{1}{\sqrt{n}\operatorname{Re}\lambda}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2\operatorname{Re}\lambda}}\right)\left(\|L_T(f)\|+\frac{1}{n}\sum_{i\in I}\|L_i(f)\|\right),$$

where $(T(t))_{t\ge 0}$ *is the* C_0*-semigroup generated by the closure of* $(L_T, A_\infty(K))$*,* $B_{\lambda,n} : C(K) \to C(K)$ *is defined by*

$$B_{\lambda,n}f := \int_0^{+\infty} e^{-\lambda t} B_n^{[n\,t]} f\, dt\,, \qquad f \in C(K)\,,$$

and $R(\lambda, L_T)$ *is the resolvent operator of the closure of* $(L_T, A_\infty(K))$*.*

Proof. The existence of the C_0-semigroups generated by the closures of the operators $(\gamma\, L_T, A_\infty(K))$ and $((1+b)\, L_T, A_\infty(K))$ is a consequence of [9, Theorem 6.2.6, p. 436]. Moreover, from (2.1.6)–(2.1.5) we can apply Theorem 1.1.2 and Theorem 1.2.1 considering the seminorms $\varphi_n : A_\infty(K) \to \mathbb{R}$ and $\psi_n : A_\infty(K) \to \mathbb{R}$ defined by

$$\varphi_n(f) := \|\gamma\, L_T(f)\| + \frac{1}{n}\sum_{i\in I}\|L_i(f)\|\,,\ \psi_n(f) := \frac{1}{n}\sum_{i\in I}\|L_i(f)\|\,, \quad f \in A_\infty(K)\,,$$

and we get (2.1.9) and (2.1.11). The particular case $k(n) = [nt]$ follows from (1.1.11).

Analogously, if we define

$$\varphi_n(f) := \left\|\frac{1+na_n}{1+a_n}\right\| \|L_T(f)\| + \frac{1}{n}\sum_{i\in I}\|L_i(f)\|\,,$$
$$\psi_n(f) := \left\|\frac{na_n - b}{1+a_n}\right\| \|L_T(f)\| + \frac{1}{n}\sum_{i\in I}\|L_i(f)\|\,,$$

from (2.1.8)–(2.1.7), Theorem 1.1.2, Theorem 1.2.1 and (1.1.11) we get (2.1.12), (2.1.13) and (2.1.14).

Finally, the case of Bernstein-Schnabl operators is obtained taking $\gamma = 1$ in (2.1.9)–(2.1.11) (or $a_n = 0$ in (2.1.12)–(2.1.14)). □

2.2 Finite dimensional setting

2.2.1 Best order of convergence in $C^{2,\alpha}(K)$

In this section we consider a domain K of $\mathbb{R}^d$ and apply Theorem 1.1.2 and 1.2.1 to some classical sequences of linear operators connected with some second-order differential operators. In order to describe the rate of convergence in the Voronovskaja-type formula we restrict our attention to the class $C^{2,\alpha}(K)$ of twice differentiable functions with α-Hölder continuous second-order derivative, and give a general quantitative estimate in terms of the α-Hölder constant defined by

$$L_{f''} := \sup_{\substack{x,y\in K\\ x\neq y}} \frac{1}{|x-y|^\alpha}|f''(x)-f''(y)| \,. \qquad (2.2.1)$$

This result can be easily applied to a wide range of linear operators, by simply evaluating them at the functions $(\mathrm{pr}_i - x_i)$, $(\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)$, this will determinate the coefficient of the differential operator associated with the Voronovskaja formula, and at the function $(\mathrm{pr}_i - x_i)^2(\mathrm{pr}_j - x_j)^2$ which affects the rate of convergence.

In the next theorem we consider a linear operator L on $C(K)$, and its associated differential operator

$$\begin{aligned}\mathcal{A}_L f(x) := \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\, L\left((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x) \qquad (2.2.2)\\ + \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x)\, L(\mathrm{pr}_i - x_i)(x)\end{aligned}$$

defined for every $f \in C^2(K)$ and $x \in K$. This operator is strictly related to the differential operator associated with the Voronovskaja-type formula when we shall consider a sequence of linear operators.

Theorem 2.2.1 *Let $K \subset \mathbb{R}^d$, and $L : C(K) \to C(K)$ a linear positive operator. For every $x \in K$ denote by $\psi_x \colon K \to \mathbb{R}$ the real function defined by $\psi_x(y) := |y-x|$ for every $y \in K$. Then for every $f \in C^{2,\alpha}(K)$ we have*

$$\begin{aligned}|L(f)(x) - f(x) - \mathcal{A}_L(x)| \le |f(x)||L\mathbf{1}(x) - 1| \qquad (2.2.3)\\ + \frac{L_{f''}}{2}\left(L(\psi_x^2)(x)\right)^{\alpha/2}\left((L(\psi_x^2)(x))^2 L(\mathbf{1})(x) + L(\psi_x^4)(x)\right)^{1/2} \,.\end{aligned}$$

Proof. Let $f \in C^{2,\alpha}(K)$ and $x = (x_1,\dots,x_d) \in K$. For every $y = (y_1,\dots,y_d) \in K$, there exists $\xi(y)$ in the segment joining x and y such that

$$\begin{aligned}
&f(y) - f(x) \\
&\quad = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\,(y_i - x_i) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(y))\,(y_i - x_i)(y_j - x_j) \\
&\quad = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\,(y_i - x_i) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\,(y_i - x_i)(y_j - x_j) \\
&\qquad + \frac{1}{2}\sum_{i,j=1}^{d} \left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(y)) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right)(y_i - x_i)(y_j - x_j)\ .
\end{aligned}$$

Hence

$$\begin{aligned}
&f - f(x)\cdot \mathbf{1} \\
&\quad = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\,(\mathrm{pr}_i - x_i) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\,(\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d} \left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}\circ\xi - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right)(\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\ ,
\end{aligned}$$

and evaluating L of both sides at the point x we get

$$\begin{aligned}
L(f)(x) - f(x) + f(x) - f(x)\cdot L(\mathbf{1})(x) &= \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\,L(\mathrm{pr}_i - x_i) \\
&+ \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\,L\big((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\big)(x) \\
&+ \frac{1}{2}\sum_{i,j=1}^{d} L\left(\left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}\circ\xi - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right)(\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x)\ .
\end{aligned}$$

Taking into account that L is positive we can write

$$\begin{aligned}
|L(f)(x) - f(x) - \mathcal{A}f(x)| &\le |f(x)||L\mathbf{1}(x) - 1| \\
&+ \frac{1}{2}\sum_{i,j=1}^{d} L\left(\left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}\circ\xi - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right)(\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x)\ .
\end{aligned} \tag{2.2.4}$$

Since $f \in C^{2,\alpha}(K_d)$ we can estimate the last term as follows

$$\sum_{i,j=1}^{d} \left|\frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(y)) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right| \le L_{f''}|y - x|^{\alpha}\ ,$$

where $L_{f''}$ is the Lipschitz constant of f given by (2.2.1). Moreover, using the inequalities $|(y_i - x_i)(y_j - x_j)| \leq |y - x|^2$ from (2.2.4) we get

$$|L(f)(x) - f(x) - \mathcal{A}f(x)| \leq |f(x)||L\mathbf{1}(x) - 1| + \frac{L_{f''}}{2} L\left(\psi_x^{2+\alpha}\right)(x) .$$

At this point using the Cauchy-Schwartz inequality (see, e.g., [9, Section 1.2, p. 21]) we obtain

$$|L(f)(x) - f(x) - \mathcal{A}f(x)| \leq |f(x)||L\mathbf{1}(x) - 1| + \frac{L_{f''}}{2} \sqrt{L\left(\psi_x^2\right)(x)} \sqrt{L\left(\psi_x^{2+2\alpha}\right)(x)} .$$

Observe that for every $\delta > 0$, we have

$$\psi_x(y)^{2+2\alpha} \leq \left(\delta^2 + \frac{\psi_x(y)^4}{\delta^2}\right) \delta^{2\alpha} ;$$

indeed if $|y - x| \leq \delta$ we obviously have $|y - x|^{2+2\alpha} \leq \delta^{2+2\alpha}$ and otherwise if $|y - x| > \delta$ then $1 \leq \left(\frac{|y-x|}{\delta}\right)^{2-2\alpha}$ and $|y - x|^{2+2\alpha} \leq |y - x|^{2+2\alpha} \left(\frac{|y-x|}{\delta}\right)^{2-2\alpha} = \frac{|y-x|^4}{\delta^2} \delta^{2\alpha}$. Therefore

$$\begin{aligned}
|L(f)(x) - f(x) - \mathcal{A}f(x)| &\leq |f(x)||L\mathbf{1}(x) - 1| \\
&\quad + \frac{L_{f''}}{2} \sqrt{L\left(\psi_x^2\right)(x)} \sqrt{\delta^{2\alpha}\left(\delta^2 L(\mathbf{1})(x) + \frac{1}{\delta^2} L(\psi_x^4)(x)\right)} \\
&= |f(x)||L\mathbf{1}(x) - 1| + \frac{L_{f''}}{2} \delta^\alpha \sqrt{\delta^2 L(\psi_x^2)(x) L(\mathbf{1})(x) + \frac{L(\psi_x^2)(x)}{\delta^2} L(\psi_x^4)(x)} ,
\end{aligned}$$

and choosing $\delta^2 = L(\psi_x^2)(x)$ we obtain

$$\begin{aligned}
&|L(f)(x) - f(x) - \mathcal{A}f(x)| \leq |f(x)||L\mathbf{1}(x) - 1| \\
&\quad + \frac{L_{f''}}{2} \left(L(\psi_x^2)(x)\right)^{\alpha/2} \left((L(\psi_x^2)(x))^2 L(\mathbf{1})(x) + L(\psi_x^4)(x)\right)^{1/2} .
\end{aligned}$$

□

In concrete applications we have a sequence of linear operators $(L_n)_{n\in\mathbb{N}}$, and a sequence of positive real numbers $(h_n)_{n\in\mathbb{N}}$ converging to zero, such that the operator

$$\frac{\mathcal{A}_{L_n}}{h_n} := \mathcal{A}_n$$

converges to a second-order differential operator

$$Af(x) := \sum_{i,j}^{d} a_{i,j}(x) D_{i,j} f(x) + \sum_{i=1}^{d} b_i(x) D_i f(x)$$

where $a_{i,j}, b_i$ are bounded, positive, continuous functions on $\overset{\circ}{K}$. The link between the linear operators L_n and A is given by a Voronovskaja-type formula:

Theorem 2.2.2 *Let K be a set of $\mathbb{R}^d$, let $(h_n)_{n\in\mathbb{N}}$ be a sequence of positive real number converging to 0 such that for every $x \in K$ and $f \in C^2(K)$, with uniformly continuous and bounded second-order partial derivatives,*

1. $\displaystyle\lim_{n\to\infty} \mathcal{A}_n f(x) = Af(x)$
2. $\displaystyle\lim_{n\to\infty} \frac{L_n\mathbf{1}(x) - 1}{h_n} = 0$
3. $\displaystyle\lim_{n\to\infty} \frac{L_n(\psi_x^4)(x)}{h_n} = 0.$

Then

$$\lim_{n\to\infty} \frac{L_n f(x) - f(x)}{h_n} = Af(x)\ , \quad \text{for every } x \in K \text{ and } f \in C^2(K)\ . \tag{2.2.5}$$

Proof. Let $f \in C^2(K)$ with uniformly continuous and bounded second-order partial derivatives. We can still apply (2.2.4) to the operator L_n and dividing by h_n we get, for every $x \in K$,

$$\begin{aligned}
&\left|\frac{L_n f(x) - f(x)}{h_n} - Af(x)\right| \le \frac{1}{h_n}|L_n f(x) - f(x) - \mathcal{A}f(x)| + |\mathcal{A}_n f(x) - Af(x)| \\
&\quad \le |f(x)|\frac{|L_n\mathbf{1}(x) - 1|}{h_n} \\
&\quad\quad + \left|\frac{1}{2h_n}\sum_{i,j=1}^{d} L_n\left(\left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}\circ\xi - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right)(\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x)\right| \\
&\quad\quad + |\mathcal{A}_n f(x) - Af(x)|\ .
\end{aligned}$$

Since f'' is uniformly continuous for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t - x| \le \delta$ implies that $\left|\frac{\partial^2 f}{\partial x_i\,\partial x_j}(t) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right| \le \varepsilon$. Then

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(t)) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right| |(t_i - x_i)(t_j - x_j)| \\
&\quad \le \varepsilon|(t_i - x_i)(t_j - x_j)| + 2\left\|\frac{\partial^2 f}{\partial x_i\,\partial x_j}\right\| \frac{(t-x)^4}{\delta^2}\ ;
\end{aligned}$$

indeed if $|t - x| \le \delta$ we have $|\xi(t) - x| \le |t - x|$, and

$$\left|\frac{\partial^2 f}{\partial x_i\,\partial x_j}(\xi(t)) - \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\right| \le \varepsilon\ ; \tag{2.2.6}$$

conversely if $|t-x|>\delta$ we have that $1<\frac{|t-x|}{\delta}$ and using the inequality $|(t_i-x_i)(t_j-x_j)|\le|t-x|^2$ we have again the validity of (2.2.6). Finally

$$\begin{aligned}
&\left|\frac{L_nf(x)-f(x)}{h_n}-Af(x)\right| \\
&\quad\le |f(x)|\frac{|L_n\mathbf{1}(x)-1|}{h_n}+\varepsilon\frac{1}{2}\sum_{i,j=1}^{d}\frac{L_n((\mathrm{pr}_i-x_i)(\mathrm{pr}_j-x_j))(x)}{h_n} \\
&\quad+\|D^2f\|\frac{L_n(\psi_x^4)(x)}{\delta^2h_n}+|\mathcal{A}_nf(x)-Af(x)|\ ,
\end{aligned}$$

which converges to $\varepsilon\sum_{i,j=1}^{d}a_{i,j}(x)$ as $n\to\infty$. Since ε is arbitrary the proof is complete. □

Remark 2.2.3 If the hypotheses 1., 2. and 3. in Theorem 2.2.2 hold with respect a uniform norm (or with respect a weighted uniform norm) then (2.2.5) holds uniformly (or with respect to the weighted uniform norm) as well. □

2.2.2 Application to Bernstein operators

In this section we consider the particular case of the standard simplex of $\mathbb{R}^d$

$$K_d:=\left\{x=(x_1,\dots,x_d)\in\mathbb{R}^d\mid x_1,\dots,x_d\ge 0\ ,\ \sum_{k=1}^{d}x_k\le 1\right\}\ ,$$

and the classical sequences of multi-dimensional Bernstein, Lototsky and Stancu operator on K_d. These operators coincide with the Bernstein-Schnabl, Lototsky-Schnabl and Stancu-Schnabl investigated in the preceding section considering the particular projection $T_d:C(K_d)\to C(K_d)$ defined by

$$T_df(x):=\sum_{i=0}^{d}x_if(\delta_{i1},\dots,\delta_{id})\ ,\qquad f\in C(K_d)\ ,\ x=(x_1,\dots,x_d)\in K_d\ ,$$

which maps any continuous function f into the affine functions which interpolates f at the vertices of K_d (see [9, Section 6.3.3, p. 450]).

Observe that the projection T_d satisfies all our general assumptions and also those in Theorem 2.1.1, and therefore we already have an estimate of the convergence of the iterates of our operators to the associated semigroup on the subspace $A_\infty(K_d)$. However, here, we want to point out some additional information on a larger subspace.

The results in this section have been published in [36] in a preliminary version and are stated in [37] in the definitive version.

We consider the case of Bernstein operators, which are explicitly given by

$$B_n f(x_1,\dots,x_d) := \sum_{h_1+\dots+h_d\le n} \frac{n!}{h_0!\,h_1!\dots h_d!}\, x_0^{h_0}\, x_1^{h_1}\dots x_d^{h_d}\, f\left(\frac{h_1}{n},\dots,\frac{h_d}{n}\right)$$

for every $f\in C(K_d)$ and $(x_1,\dots,x_d)\in K_d$, where $x_0 := 1-x_1-\dots-x_d$ and $h_0 := n-h_1-\dots-h_d$.

In our situation, the operator L_T, defined by (2.1.4), coincides on $A_\infty(K_d)$ with the differential operator $A : C^2(K_d)\to C(K_d)$ defined by

$$Af(x) = \sum_{i,j=1}^{d} \frac{x_i(\delta_{ij}-x_j)}{2}\frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) \tag{2.2.7}$$

whenever $f\in C^2(K_d)$ and $x=(x_1,\dots,x_d)\in K_d$.

It is well-known that the closure of $(A, C^2(K_d))$ generates a C_0-semigroup of positive contractions on $C(K_d)$ and that $C^2(K_d)$ is a core for this closure (see e.g. [9, Theorem 6.2.6, p. 436] or also [49]).

From Theorem 2.1.1 the estimates required in Theorems 1.1.2 and 1.2.1 are already available in $A_\infty(K_d)$. However our aim is to investigate the validity of similar estimates in the space $C^{2,\alpha}(K_d)$.

Taking $L=B_n$ in (2.2.2) we get the differential operator

$$\begin{aligned}\mathcal{A}_{B_n} f(x) := \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\, B_n\left((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x)\\ +\sum_{i=1}^{d}\frac{\partial f}{\partial x_i}(x)\, B_n(\mathrm{pr}_i - x_i)\,,\end{aligned}$$

and in order to apply Theorem 2.2.1 we need to evaluate the operator B_n at the function $(\mathrm{id}-x)^4$, where id denote the identity function defined by $\mathrm{id}(x) := x$ for every $x\in\mathbb{R}$.

Proposition 2.2.4 *For every $x\in K_d$, we have*

$$\begin{aligned}B_n((\mathrm{id}-x)^4)(x) = \frac{1}{n^2}\left(\psi(x)^2 + 2\sum_{i,j=1}^{d} x_i^2(\delta_{i,j}-x_j)^2\right) \qquad (2.2.8)\\ +\frac{1}{n^3}\left(\sum_{i,j=1}^{d} x_i x_j(x_i+x_j-3x_ix_j) + \sum_{i=1}^{d} x_i(1-2x_i)^2\right.\\ \left. -\psi(x)^2 - 2\sum_{i,j=1}^{d} x_i^2(\delta_{i,j}-x_j)^2\right)\,,\end{aligned}$$

where $\psi(x)=\sum_{i=1}^{d} x_i(1-x_i)$.

PROOF. We have

$$B_n((\mathrm{id}-x)^4)(x) = \sum_{i,j=1}^{d} B_n((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x)\ ;$$

in order to compute $B_n((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x)$ we use the fact that B_n coincides with the n-th Bernstein-Schnabl operator associated to the projection T_d, that is

$$B_n f(x) = \int_{K_d} \dots \int_{K_d} f\left(\frac{x_1+\cdots+x_n}{n}\right) d\mu_x^{T_d}(x_1)\ \dots\ d\mu_x^{T_d}(x_n)\ . \quad (2.2.9)$$

We fix $x \in K_d$ and consider two affine function h_1, h_2 such that $h_1(x) = h_2(x) = 0$. Consider the function $f := h_1^2 h_2^2$; then we have

$$h_1^2 h_2^2\left(\frac{x_1+\cdots+x_n}{n}\right) = \frac{1}{n^4}\sum_{i,j,k,l=1}^{n} h_1(x_i)h_1(x_j)h_2(x_k)h_2(x_l)\ ,$$

and consequently

$$\begin{aligned}
&B_n(h_1^2h_2^2)(x)\\
&\quad = \frac{1}{n^4}\sum_{i,j,k,l=1}^{n}\int_{K_d}\dots\int_{K_d} h_1(x_i)h_1(x_j)h_2(x_k)h_2(x_l)\, d\mu_x^{T_d}(x_1)\ \dots\ d\mu_x^{T_d}(x_n)\\
&\quad = \frac{1}{n^4}\sum_{\substack{i,j,k,l=1\\ i=j=k=l}}^{n} T_d(h_1^2h_2^2)(x) + \frac{1}{n^4}\sum_{\substack{i,j,k,l=1\\ i=j,k=l,i\neq l}}^{n} T_d(h_1^2)(x)T_d(h_2^2)(x)\\
&\qquad + \frac{1}{n^4}\sum_{\substack{i,j,k,l=1\\ i=k,j=l,i\neq j}}^{n} (T_d(h_1h_2)(x))^2 + \frac{1}{n^4}\sum_{\substack{i,j,k,l=1\\ i=l,j=k,i\neq j}}^{n} (T_d(h_1h_2)(x))^2\\
&\quad = \frac{1}{n^3}T_d(h_1^2h_2^2)(x) + \frac{n(n-1)}{n^4}T_d(h_1^2)(x)T_d(h_2^2)(x) + 2\frac{n(n-1)}{n^4}(T_d(h_1h_2)(x))^2\\
&\quad = \frac{1}{n^2}\left[T_d(h_1^2)(x)T_d(h_2^2)(x) + 2\,(T_d(h_1h_2)(x))\right]\\
&\qquad + \frac{1}{n^3}\left[T_d(h_1^2h_2^2)(x) - T_d(h_1^2)(x)T(h_2^2)(x) - 2(T_d(h_1h_2)(x))^2\right]\ .
\end{aligned}$$

Now let $1 \le i,j \le d$ and take $h_1 = \mathrm{pr}_i - x_i$ and $h_2 = \mathrm{pr}_j - x_j$; we obtain

$$T_d((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x) = x_i(\delta_{i,j} - x_j)$$

and

$$T_d((\mathrm{pr}_i - x_i)^2(\mathrm{pr}_j - x_j)^2)(x) = x_ix_j(x_i + x_j - 3x_ix_j) + \delta_{i,j}x_i(1-2x_i)^2\ .$$

Hence

$$\begin{aligned} &B_n((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x) \\ &= \frac{1}{n^2}(x_i(1-x_i)x_j(1-x_j) + 2x_i^2(\delta_{i,j} - x_j)^2) \\ &\quad + \frac{1}{n^3}\left(x_i x_j(x_i + x_j - 3x_i x_j) + \delta_{i,j} x_i (1-2x_i)^2 \right. \\ &\quad \left. - x_i(1-x_i)x_j(1-x_j) - 2x_i^2(\delta_{i,j} - x_j)^2\right) , \end{aligned}$$

and this completes the proof. □

Remark 2.2.5 We have

$$\left|B_n((\mathrm{id} - x)^4)(x)\right| \leq \frac{1}{n^2}\left(\psi(x)^2 + 2\sum_{i,j=1}^{d} x_i^2(\delta_{i,j} - x_j)^2\right) + \frac{3}{n^3} . \quad (2.2.10)$$

Indeed since $x \in K_d$, we have $0 \leq \sum_{i=1}^{d} x_i^2 \leq 1$ and $(1-2x_i)^2 \leq 1$ and therefore

$$\begin{aligned} &\sum_{i,j=1}^{d} x_i x_j (x_i + x_j - 3x_i x_j) + \sum_{i=1}^{d} x_i(1-2x_i)^2 - \psi(x)^2 - 2\sum_{i,j=1}^{d} x_i^2(\delta_{i,j} - x_j)^2 \\ &\quad \leq 2\sum_{i,j=1}^{d} x_i x_j^2 + \sum_{i=1}^{d} x_i(1-2x_i)^2 \leq 2\sum_{i=1}^{d} x_i \sum_{j=1}^{d} x_i^2 + \sum_{i=1}^{d} x_i \leq 3 . \end{aligned}$$

Using the above inequalities, (2.2.10) directly follows from (2.2.8). □

Now we can establish a quantitative version of the Voronovskaja-type formula for the Bernstein operators in the space $C^{2,\alpha}([0,1])$.

Theorem 2.2.6 (Quantitative Voronovskaja's formula for Bernstein operators) *Consider the Bernstein operators on $C(K_d)$ and the differential operator (2.2.7). For every $f \in C^{2,\alpha}(K_d)$ and $x \in K_d$ we have*

$$|n(B_n(f)(x) - f(x)) - Af(x)| \leq L_{f''}\left(\frac{1}{2} - \frac{1}{2d} + \frac{3}{4n}\right)^{1/2}\left(\frac{\psi(x)}{n}\right)^{\alpha/2}$$

if $d > 1$ and

$$|n(B_n(f)(x) - f(x)) - Af(x)| \leq L_{f''}\left(\frac{1}{16} + \frac{3}{4n}\right)^{1/2}\left(\frac{\psi(x)}{n}\right)^{\alpha/2}$$

if $d = 1$, where $\psi(x) = \sum_{i=1}^{d} x_i(1-x_i)$.

PROOF. Recalling that, for every $i, j = 1, \ldots, d$,

$$B_n\mathbf{1} = \mathbf{1}\,, \qquad B_n\mathrm{pr}_i = \mathrm{pr}_i\,, \qquad B_n(\mathrm{pr}_i\,\mathrm{pr}_j) = \mathrm{pr}_i\,\mathrm{pr}_j + \frac{1}{n}\,\mathrm{pr}_i\,(\delta_{ij} - \mathrm{pr}_j)\,,$$

we have

$$B_n(\mathrm{pr}_i - x_i)(x) = 0\,, \quad B_n\left((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j)\right)(x) = \frac{x_i(\delta_{ij} - x_j)}{n},$$

and therefore

$$n\mathcal{A}_{B_n}f = Af$$

and $B_n(\psi_x^2)(x) = \frac{1}{n}\sum_{i=1}^n x_i(1-x_i) = \frac{1}{n}\psi(x)$. From Theorem 2.2.1 and (2.2.10) we have

$$\begin{aligned}
&|n(B_n(f)(x) - f(x)) - Af(x)| \\
&\quad\leq \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2} n\left(\frac{\psi(x)^2}{n^2} + B_n((\mathrm{id}-x)^4)(x)\right)^{1/2} \\
&\quad= \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2} n\left(\frac{2}{n^2}\psi(x)^2 + \frac{2}{n^2}\sum_{i,j=1}^d x_i^2(\delta_{i,j} - x_j)^2 + \frac{3}{n^3}\right)^{1/2} \\
&\quad\leq \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2}\left[2\left(\psi(x)^2 + \sum_{i,j=1}^d x_i^2(\delta_{i,j} - x_j)^2\right) + \frac{3}{n}\right]^{1/2}.
\end{aligned}$$

The function $g(x) = \psi(x)^2 + \sum_{i,j=1}^d x_i^2(\delta_{i,j} - x_j)^2$ attains its maximum in K_d at the point $\overline{x} = (1/d, \ldots, 1/d)$ if $d > 1$ and at $\overline{x} = 1/2$ if $d = 1$; then

$$\begin{aligned}
g(\overline{x}) &= \left(\sum_{i=1}^d \frac{1}{d}\left(1 - \frac{1}{d}\right)\right)^2 + \sum_{i=1}^d \left(\frac{1}{d}\right)^2\left(1 - \frac{1}{d}\right)^2 + \sum_{i\neq j}\left(\frac{1}{d}\right)^4 \\
&= \left(1 - \frac{1}{d}\right)^2 + \frac{1}{d}\left(1 - \frac{1}{d}\right)^2 + \frac{d(d-1)}{d^4} = 1 - \frac{1}{d}
\end{aligned}$$

if $d > 1$ and $g(\overline{x}) = 1/8$ if $d = 1$. Therefore

$$|n(B_n(f)(x) - f(x)) - Af(x)| \leq \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2}\left(2\left(1 - \frac{1}{d}\right) + \frac{3}{n}\right)^{1/2}$$

if $d > 1$ and

$$|n(B_n(f)(x) - f(x)) - Af(x)| \leq \frac{L_{f''}}{2}\frac{1}{n^{\alpha/2}}(\psi(x))^{\alpha/2}\left(\frac{1}{4} + \frac{3}{n}\right)^{1/2}$$

if $d = 1$. □

Remark 2.2.7 Taking into account that $\psi(x) \le 1-\frac{1}{d}$ if $d>1$ and $\psi(x) \le \frac{1}{4}$ if $d=1$, we have, for $n>1$

$$\|n(B_n(f)-f)-Af\| \le L_{f''}\frac{1}{n^{\alpha/2}} \;. \tag{2.2.11}$$

□

The following last result is a consequence of Theorem 1.1.2, 1.2.1 and (1.1.11) by means of (2.2.11)

Theorem 2.2.8 *Consider the Bernstein operators on $C(K_d)$ and the differential operator (2.2.7). Then, the closure of $(A, C^2(K_d))$ generates a C_0-semigroup $(T(t))_{t\ge 0}$ on $C(K_d)$ such that, for every $t \ge 0$, $(k(n))_{n\ge 1}$ sequence of positive integers and $f \in C^{2,\alpha}(K_d)$, we have*

$$\begin{aligned} &\|T(t)f - B_n^{k(n)}f\| \\ &\quad \le \frac{L_{f''}\,t}{n^{\alpha/2}} + \left(\left|\frac{k(n)}{n}-t\right| + \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\right)\left(\|Af\| + \frac{L_{f''}}{n^{\alpha/2}}\right) \;, \end{aligned} \tag{2.2.12}$$

and in particular, taking $k(n)=[nt]$,

$$\|T(t)f - B_n^{[nt]}f\| \le \frac{L_{f''}\,t}{n^{\alpha/2}} + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\right)\left(\|Af\| + \frac{L_{f''}}{n^{\alpha/2}}\right) \;. \tag{2.2.13}$$

Moreover, for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$ and $n \ge 1$, consider the operator $B_{\lambda,n} : C(K_d) \to C(K_d)$ defined by

$$B_{\lambda,n}f := \int_0^{+\infty} e^{-\lambda t} B_n^{[n\,t]} f\,dt \;, \qquad f \in C(K) \;.$$

If $R(\lambda, A)$ denotes the resolvent operator of the closure of $(A, C^2(K_d))$, for every $n>1$ and $f \in C^{2,\alpha}(K_d)$ we have

$$\begin{aligned} \|R(\lambda,A)f - B_{\lambda,n}f\| &\le \frac{1}{(\operatorname{Re}\lambda)^2}\,\frac{L_{f''}}{n^{\alpha/2}} \\ &+ \frac{1}{\sqrt{n}\operatorname{Re}\lambda}\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2\operatorname{Re}\lambda}}\right)\left(\|Af\| + \frac{L_{f''}}{n^{\alpha/2}}\right) \;. \end{aligned} \tag{2.2.14}$$

In the one dimensional case we have a partial converse result.

First we recall some needed notions related to the smoothness of a function f.

The *divided difference operator* $\Delta_h(f,x)$ is defined by

$$\Delta_h(f,x) := f(x+h) - f(x) \;, \qquad h \ge 0 \;.$$

If $r \geq 2$, the *r-th order divided difference* Δ_h^r is defined as the r-fold composition of Δ_h with itself.

If $f \in C([a,b])$, then the *modulus of continuity of* f is defined by

$$\omega(f,\delta) := \sup_{0\leq h\leq\delta} \|\Delta_h(f,\cdot)\|_{[a,b-h]} , \qquad 0 \leq \delta \leq b-a .$$

Accordingly, if $r \geq 2$ the *r-th order modulus of smoothness* is given by

$$\omega_r(f,\delta) := \sup_{0\leq h\leq\delta} \|\Delta_h^r(f,\cdot)\|_{[a,b-rh]} , \qquad 0 \leq \delta \leq \frac{b-a}{r} .$$

If $0 < \alpha \leq 1$, we e denote by $\mathrm{Lip}\,\alpha$ the set of all functions $f \in C([a,b])$ for which there exists $M > 0$ such that $\omega(f,\delta) \leq M\delta^\alpha$, i.e. $\mathrm{Lip}\,\alpha$ is the Lipschitz α-class. If $0 < \alpha \leq 2$ we denote by $\mathrm{Lip}^*\,\alpha$ the set of all functions $f \in C([a,b])$ for which there exists $M > 0$ such that $\omega_2(f,\delta) \leq M\delta^\alpha$. It is noteworthy that if $0 < \alpha < 1$ the class $\mathrm{Lip}^*\,\alpha$ and $\mathrm{Lip}\,\alpha$ coincide (see [45, p. 6 (1.3.5)]).

Proposition 2.2.9 *Consider the Bernstein operators on* $C([0,1])$ *and the differential operator* (2.2.7)*. Let* $f \in C^2([0,1])$ *and assume that there exist a constant* $C > 0$ *and* $\alpha \in]0,1[$ *such that*

$$\|n(B_nf - f) - Af\| \leq \frac{C}{n^{\alpha/2}} . \tag{2.2.15}$$

then $f \in C^{2,\alpha}_{\mathrm{loc}}(0,1)$.

PROOF. First we explicitly evaluate the differential operator (2.2.7) which, in our situation, becomes

$$Af(x) = \frac{x(1-x)}{2} f''(x) .$$

Now let $f \in C^{2,\alpha}_{\mathrm{loc}}([0,1])$; since $f \in C^2([0,1])$, for every $x \in [0,1]$ and $y \in [0,1]$, there exists $\xi(y)$ in the segment joining x and y such that

$$f(y) - f(x) = f'(x)\,(y-x) + \frac{1}{2}\,f''(x)\,(y-x)^2 + \eta(y,x)\,(y-x)^2 , \tag{2.2.16}$$

where $\eta(t,x) := \frac{1}{2}\left(f''(\xi(t)) - f''(x)\right)$. Then we can write

$$\begin{aligned} B_n(f)(x) - f(x) &= B_n(\mathrm{id}-x)(x) + \frac{1}{2}f''(x)B_n(\mathrm{id}-x)^2(x) \\ &\quad + B_n((\mathrm{id}-x)^2\eta(\mathrm{id},x))(x) . \end{aligned}$$

Taking into account that

$$B_n(\mathrm{id}-x)(x) = 0 , \quad B_n(\mathrm{id}-x)^2(x) = \frac{x(1-x)}{n} ,$$

we have

$$B_n(f)(x) - f(x) = \frac{1}{n} Af(x) + B_n((\mathrm{id} - x)^2 \eta(\mathrm{id}, x))(x) \; ,$$

and from (2.2.15) it follows that

$$\left| B_n \left((\mathrm{id} - x)^2 \eta(\mathrm{id}, x) \right) (x) \right| \leq \frac{C}{n} \frac{1}{n^{\alpha/2}} \; . \qquad (2.2.17)$$

Now, we consider a linear combination of Bernstein polynomials introduced by Butzer [20] and recursively defined by

$$\begin{aligned} & B_{n,0} := B_n \\ & (2^r - 1) B_{n,r} = 2^r B_{2n,r-1} - B_{n,r-1} \; . \end{aligned}$$

A result of Ditzian [47] states that

$$\|B_{n,r}(f) - f\| = O\left(\frac{1}{n^{\beta/2}}\right) \iff \|\varphi^\beta \Delta_h^{2r} f\|_{[rh,1-rh]} = O(h^\beta) \qquad (2.2.18)$$

for $\beta < 2r$ and $\varphi^2(x) := x(1-x)$.
If we set $g_x(y) := (y-x)^2 \eta(y,x)$ and take $r = 2$ we get

$$B_{n,2} = \frac{8}{3} B_{4n} - 2B_{2n} + \frac{1}{3} B_n$$

and from (2.2.17) it follows that $\|B_{n,2}(g_x)\| = O\left(\frac{1}{n^{1+\alpha/2}}\right)$. Therefore we take (2.2.18) with $\beta = 2 + \alpha$, then for every $\delta > 2h$ exists $C = C(\delta) > 0$ such that

$$\left| \Delta_h^4 g_x(y) \right| \leq C h^{2+\alpha} \; , \qquad y \in [\delta, 1-\delta] \; . \qquad (2.2.19)$$

Now we evaluate $\Delta_h^4 g_x(y)$ at the point $y = x$; we have

$$\Delta_h^4 g_x(x) = g_x(x+4h) - 4g_x(x+3h) + 6g_x(x+2h) - 4g_x(x+h) + g_x(x) \; .$$

Taking into account that, from (2.2.16), for every $s \geq 0$

$$\begin{aligned} g_x(x+s) &= f(x+s) - f(x) - f'(x)(x+s-x) - \frac{1}{2} f''(x)(x+s-x)^2 \\ &= f(x+s) - f(x) - f'(x)s - \frac{1}{2} f''(x) s^2 \; , \end{aligned}$$

we have

$$\begin{aligned} \Delta_h^4 g_x(x) &= f(x+4h) - 4f(x+3h) + 6f(x+2h) - 4f(x+h) \\ &\quad -f(x)(1-4+6-4) \\ &\quad -f'(x)h(4-12+12-4) - f''(x)h^2(16-36+24-4) \\ &= \Delta_h^4 f(x) \; . \end{aligned}$$

Now we evaluate the 4-*th* order finite-difference of f in terms of the second-order derivative of f,

$$\begin{aligned}\Delta_h^4 f(x) &= \int_0^h \Delta_h^3 f'(x+t)dt = \int_0^h \int_0^h \Delta_h^2 f''(x+t+s)dsdt \\ &= h^2 \Delta_h^2 f''(x+\xi(x)) \qquad (2.2.20)\end{aligned}$$

where $\xi(x) \in [x, x+2h]$. Now we consider the function $z(x) := x + \xi(x)$ for $x \in [\delta, 1-\delta]$, we have that $z(\delta) \leq \delta + 2h$ and $1-\delta \leq z(1-\delta)$, then $z^{-1}([\delta + 2h, 1-\delta]) \subset [\delta, 1-\delta]$. Consequently, since the function z is continuous, for every $h \geq 0$ and $\overline{z} \in [\delta + 2h, 1-\delta]$ we can take $x \in [\delta, 1-\delta]$ such that $z(x) = x + \xi(x) = \overline{z}$, and from (2.2.19) and (2.2.20) we can write

$$\left|h^2 \Delta_h^2 f''(z)\right| = |\Delta_h^4 g_x(x)| \leq C h^{2+\alpha} \,, \qquad z \in [\delta + 2h, 1-\delta] \,,$$

and hence, since $\delta > 2h$,

$$\|\Delta_h^2 f''\|_{[2\delta, 1-\delta]} = O(h^\alpha) \,.$$

The last expression yields $\omega_2(f, h) \leq C h^\alpha$, i.e. $f'' \in \mathrm{Lip}^*(\alpha)$ locally in $(0,1)$. Since $0 < \alpha < 1$, we have that f is also in $\mathrm{Lip}(\alpha)$. □

For the sake of brevity we do not investigate the analogous results for Lototsky-Schnabl and Stancu-Schnabl operators in this setting. In the next section we give some details on a particular class of Stancu operators.

2.2.3 Application to Stancu operators

In this section we consider some quantitative estimates of the convergence of suitable combinations of iterates of Stancu operators to the associated C_0-semigroup and the resolvent operator of its generator in the context of spaces of continuous functions on the d-dimensional simplex.
Stancu operators were introduced by D. D. Stancu in [66, 67] in the context of spaces of continuous functions on the interval $[0,1]$; if $a \in \mathbb{R}$, the n-th Stancu operator $Q_{n,a} : C([0,1]) \to C([0,1])$ is defined by setting

$$Q_{n,a}f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) q_{nk}(x,a), \quad f \in C([0,1]),\ x \in [0,1],$$

where

$$q_{nk}(x,a) := \binom{n}{k} \frac{\Phi_k(x,a)\,\Phi_{n-k}(1-x,a)}{\Phi_n(1,a)}, \quad \Phi_k(x,a) := \prod_{j=0}^{k-1}(x+ja).$$

In this setting these operators have been studied by Mühlbach [60, 61]. Further generalizations were considered by Felbecker [51] and by Campiti [22, 23]; in these last papers also connections with the representation of a suitable C_0-semigroups have been considered.

The results in this section have been published in [39].

First, we consider the standard simplex K_d of $\mathbb{R}^d$ and the Stancu operators $S_{n,a_n} : C(K_d) \to C(K_d)$ on K_d, which are associated with a sequence $(a_n)_{n\geq 1}$ of positive real numbers and are defined by setting, for every $f \in C(K_d)$ and $x = (x_1,\dots,x_d) \in K_d$,

$$S_{n,a_n}f(x_1,\dots,x_d) := \frac{1}{p_n(a_n)} \sum_{h_1+\cdots+h_d\leq n} f\left(\frac{h_1}{n},\dots,\frac{h_d}{n}\right) \times \frac{n!}{h_0!\,h_1!\dots h_d!} \prod_{i=0}^{d} \Phi_{h_i}(x_i,a_n), \tag{2.2.21}$$

for every $f \in C(K_d)$ and $(x_1,\dots,x_d) \in K_d$, where as usual $x_0 := 1 - x_1 - \cdots - x_d$, $h_0 := n - h_1 - \cdots - h_d$ and

$$p_n(a) := \prod_{j=0}^{n-1}(1+j\,a), \qquad a \in \mathbb{R}.$$

In the sequel we assume that the sequence $(na_n)_{n\geq 1}$ converges to $b \geq 0$ and consider the differential operator $A : C^2(K_d) \to C(K_d)$ defined by

$$Af(x) = (1+b) \sum_{i,j=1}^{d} \frac{x_i(\delta_{ij}-x_j)}{2} \frac{\partial^2 f}{\partial x_i\,\partial x_j}(x)\,, \tag{2.2.22}$$

whenever $f \in C^2(K_d)$ and $x = (x_1, \ldots, x_d) \in K_d$.

It is well known that the closure of the operator $(A, C^2(K_d))$ generates a C_0-semigroup of positive contractions on $C(K_d)$, and that $C^2(K_d)$ is a core for this closure (see e.g. [9, Theorem 6.2.6, p. 436]).

Moreover, the operator A is connected with Stancu operators by means of the following Voronovskaja's formula established in [51, 23]:

$$\lim_{n\to+\infty} n(S_{n,a_n}(f) - f) = A(f), \qquad f \in C^2(K_d). \tag{2.2.23}$$

Now we establish a quantitative version of (2.2.23)

Proposition 2.2.10 *Consider the Stancu operators (2.2.21) on $C(K_d)$ and the differential operator (2.2.22). There exists a constant $C > 0$ such that for every $f \in C^{2,\alpha}(K_d)$ we have*

$$\|n(S_{n,a_n}(f) - f) - A(f)\| \le C\left(\frac{1}{n^{\alpha/2}} + |na_n - b|\right) M_f \ ,$$

where M_f is the seminorm on $C^{2,\alpha}(K_d)$ defined by

$$M_f := L_{f''} + \|D^2 f\| \ . \tag{2.2.24}$$

Proof. Let $f \in C^{2,\alpha}(K_d)$ we have

$$\begin{aligned} &|n(S_{n,a_n}(f) - f)(x) - Af(x)| \qquad (2.2.25)\\ &\quad \le |n(S_{n,a_n}f(x) - f(x) - \mathcal{A}_{S_{n,a_n}}f(x))| + |n\mathcal{A}_{S_{n,a_n}}f(x) - Af(x)| \ , \end{aligned}$$

where $\mathcal{A}_{S_{n,a_n}}$ is the operator (2.2.2) obtained by taking $L = S_{n,a_n}$. In order to write an explicit expression of S_{n,a_n}, we recall that, for every $i, j = 1, \ldots, d$,

$$S_{n,a_n}(\mathbf{1}) = \mathbf{1}, \qquad S_{n,a_n}(\mathrm{pr}_i) = \mathrm{pr}_i,$$

$$S_{n,a_n}(\mathrm{pr}_i\,\mathrm{pr}_j) = \mathrm{pr}_i\,\mathrm{pr}_j + \frac{1 + na_n}{n(1 + a_n)}\,\mathrm{pr}_i\,(\delta_{ij} - \mathrm{pr}_j) \ ,$$

and consequently

$$S_{n,a_n}((\mathrm{pr}_i - x_i)(\mathrm{pr}_j - x_j))(x) = \frac{1 + na_n}{n(1 + a_n)}\,x_i\,(\delta_{ij} - x_j) \tag{2.2.26}$$

and

$$S_{n,a_n}(\mathrm{pr}_i - x_i)(x) = 0 \ .$$

Hence the operator $\mathcal{A}_{S_{n,a_n}}$ becomes

$$\mathcal{A}_{S_{n,a_n}}f(x) = \frac{1 + na_n}{n(1 + a_n)} \sum_{i,j=1}^{d} \frac{x_i\,(\delta_{ij} - x_j)}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \ .$$

In regard to the first term in (2.2.25), we apply Theorem 2.2.1 with $L = S_{n,a_n}$, and we get

$$\begin{aligned}&|n(S_{n,a_n}f(x) - f(x) - \mathcal{A}_{S_{n,a_n}}f(x))|\\ &\qquad \le n\frac{L_{f''}}{2}\left(S_{n,a_n}(\psi_x^2)(x)\right)^{\alpha/2}\left((S_{n,a_n}(\psi_x^2)(x))^2 L(\mathbf{1})(x) + L(\psi_x^4)(x)\right)^{1/2} .\end{aligned}$$

From (2.2.26) we have

$$S_{n,a_n}(\psi_x^2)(x) = \frac{1+na_n}{n(1+a_n)}\psi(x)\left(\sum_{i=1}^{d} x_i(1-x_i)\right)$$

and therefore

$$\left|S_{n,a_n}(\psi_x^2)(x)\right| \le \frac{1+na_n}{n(1+a_n)}\left(1-\frac{1}{d}\right) \le \frac{1+na_n}{n(1+a_n)} .$$

Moreover from [9, Lemma 6.2.2, p. 429], we obtain the existence of a constant $C_1 > 0$ such that

$$\left|S_{n,a_n}\left((\mathrm{pr}_i - x_i)^2(\mathrm{pr}_j - x_j)^2\right)(x)\right| \le \frac{C_1}{n^2}$$

and hence

$$S_{n,a_n}(\psi_x^4)(x) \le \frac{d^2 C_1}{n^2} .$$

The first term in (2.2.25) can be estimated as follows

$$\begin{aligned}&|n(S_{n,a_n}f(x) - f(x) - \mathcal{A}_{S_{n,a_n}}f(x))|\\ &\qquad \le \frac{L_{f''}}{2}\left(\frac{1+na_n}{n(1+a_n)}\right)^{\alpha/2} n\left(\frac{1+na_n}{n(1+a_n)} + \frac{d\sqrt{C_1}}{n}\right) .\end{aligned}$$

In regard to the second term in (2.2.25) we have

$$\begin{aligned}|n\mathcal{A}_{S_{n,a_n}}f(x) - Af(x)| &\le \left|\frac{1+na_n}{1+a_n} - (b+1)\right| \left|\sum_{i,j=1}^{d}\frac{x_i(\delta_{ij}-x_j)}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right|\\ &\le \left|\frac{na_n - b - a_n(1+b)}{1+a_n}\right| \|D^2 f\|\\ &\le \left(|na_n - b| + \frac{1}{n}na_n(b+1)\right)\|D^2 f\|\\ &\le \left(|na_n - b| + \frac{C_2}{n}\right)\|D^2 f\| .\end{aligned}$$

Finally from the above inequalities it follows

$$\begin{aligned}
&|n(S_{n,a_n}(f)-f)(x)-Af(x)| \\
&\quad\le \frac{L_{f''}}{2}\left(\frac{1+na_n}{n(1+a_n)}\right)^{\alpha/2} n\left(\frac{1+na_n}{n(1+a_n)}+\frac{d\sqrt{C_1}}{n}\right) \\
&\qquad+\left(|na_n-b|+\frac{C_2}{n}\right)\|D^2 f\| \\
&\quad\le C_3\frac{1}{n^{\alpha/2}}L_{f''}+\left(|na_n-b|+\frac{C_2}{n}\right)\|D^2 f\| \\
&\quad\le C\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right)M_f\ ,
\end{aligned}$$

where M_f is the seminorm defined by (2.2.24). □

The preceding result allows us to get the quantitative estimate obtained in Theorem 1.1.2 in the particular case where the growth bound of the semigroup is equal to 0.

Theorem 2.2.11 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $(na_n)_{n\in\mathbb{N}}$ converges to $b\in\mathbb{R}$ and consider the Stancu operators on $C(K_d)$ and the differential operator (2.2.22). Then, the closure of $(A,C^2(K_d))$ generates a C_0-semigroup $(T(t))_{t\ge 0}$ on $C(K_d)$ such that, for every $t\ge 0$, $(k(n))_{n\in\mathbb{N}}$ sequence of positive integers and $f\in C^{2,\alpha}(K_d)$, we have*

$$\begin{aligned}
\|T(t)f-S_{n,a_n}^{k(n)}f\| &\le CM_f\,t\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right) \qquad (2.2.27)\\
&+\left(\left|\frac{k(n)}{n}-t\right|+\sqrt{\frac{2}{\pi}}\frac{\sqrt{k(n)}}{n}\right)\left(\|A(f)\|+CM_f\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right)\right)
\end{aligned}$$

and taking $k(n):=[nt]$,

$$\begin{aligned}
\|T(t)f-S_{n,a_n}^{[nt]}f\| &\le CM_f\,t\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right) \qquad (2.2.28)\\
&+\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}+\sqrt{\frac{2t}{\pi}}\right)\left(\|A(f)\|+CM_f\left(\frac{1}{n^{\alpha/2}}+|na_n-b|\right)\right).
\end{aligned}$$

In the particular case where $a_n:=b/n$, estimate (2.2.27) becomes

$$\|T(t)f-S_{n,b/n}^{k(n)}f\|\le\frac{CM_f\,t}{n^{\alpha/2}}+\left(\left|\frac{k(n)}{n}-t\right|+\sqrt{\frac{2}{\pi}}\frac{\sqrt{k(n)}}{n}\right)\left(\|A(f)\|+\frac{CM_f}{n^{\alpha/2}}\right)\ ,$$

and if $k(n):=[nt]$, from (2.2.28) we get

$$\|T(t)f-S_{n,b/n}^{[nt]}f\|\le\frac{CM_f\,t}{n^{\alpha/2}}+\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}+\sqrt{\frac{2t}{\pi}}\right)\left(\|A(f)\|+\frac{CM_f}{n^{\alpha/2}}\right).$$

In order to approximate the resolvent operator of the closure of $(A, C^2(K_d))$, Let $(s_n)_{n\geq 1}$ be a sequence of positive integers tending to $+\infty$ and for every $n \geq 1$, consider the linear operator $P_{\lambda,s_n,n,a_n} : C(K_d) \to C(K_d)$ defined by

$$P_{\lambda,s_n,n,a_n}(u) := \frac{1}{n}\sum_{k=0}^{s_n} e^{-\lambda k/n} S^k_{n,a_n}(u), \qquad u \in C(K_d).$$

We are now in a position to state the following result.

Theorem 2.2.12 *For every $n \geq 1$ and $f \in C^{2,\alpha}(K_d)$, we have*

$$\begin{aligned}
&\|P_{\lambda,s_n,n,a_n}(f) - R(\lambda, A)f\| \\
&\quad \leq \frac{1}{(\operatorname{Re}\lambda)^2}\left(\|A(f)\| + CM_f\left(\frac{1}{n^{\alpha/2}} + |na_n - b|\right)\right) \\
&\qquad + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}\operatorname{Re}\lambda} + \frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda)^{3/2}}\right)\left(CM_f\left(\frac{1}{n^{\alpha/2}} + |na_n - b|\right)\right) \\
&\qquad + \frac{e^{-(\operatorname{Re}\lambda)s_n/n} + \frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}}{\left(1 - \frac{\operatorname{Re}\lambda}{n}\right)\operatorname{Re}\lambda}\,\|f\|.
\end{aligned}$$

Hence, if we assume that

$$\lim_{n\to+\infty} \frac{s_n}{n} = +\infty,$$

then the sequence $(P_{\lambda,s_n,n,a_n})_{n\geq 1}$ strongly converges to $R(\lambda, A)$ on $C(K_d)$.

If we take in particular $a_n := b/n$, then

$$\begin{aligned}
\|P_{\lambda,s_n,n,b/n}(f) - R(\lambda, A)f\| &\leq \frac{1}{(\operatorname{Re}\lambda)^2}\left(\|A(u)\| + \frac{CM_f}{n^{\alpha/2}}\right) \\
&\quad + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}\operatorname{Re}\lambda} + \frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda)^{3/2}}\right)\frac{CM_f}{n^{\alpha/2}} \\
&\quad + \frac{e^{-(\operatorname{Re}\lambda)s_n/n} + \frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}}{\left(1 - \frac{\operatorname{Re}\lambda}{n}\right)\operatorname{Re}\lambda}\,\|f\|.
\end{aligned}$$

For the sake of simplicity, we have associated Stancu operators with a sequence $(a_n)_{n\geq 1}$ of real numbers; as observed in [24], all estimates concerning Stancu operators remain valid if we take a sequence $(a_n)_{n\in\mathbb{N}}$ of continuous functions on K_d such that $(na_n)_{n\in\mathbb{N}}$ uniformly converges to a function $b \in C(K_d)$ and consequently also the results in this section are true in this more general context. We explicitly observe that in this case the differential operator A is more general, as well as the estimates on the semigroup and the resolvent operators.

Chapter 3

Steklov operators

In this chapter we consider Steklov operators in spaces of continuous functions on the real line and on a bounded interval. We study the connections of these operators with some second-order degenerate parabolic problems establishing a general Voronovskaja-type formula. We also need a quantitative version of Voronovskaja's formula in order to apply the quantitative estimates in Chapter 1.

The choice of Steklov operators is motivated by the fact that these operators can be used in different setting, such as spaces of continuous functions or weighted spaces of continuous functions both on bounded than unbounded real intervals.

The results in this chapter have been obtained in collaboration with I. Rasa (Cluj-Napoca, Romania) and published in [33], [34].

3.1 Steklov operators on the real line

In this section we point out some general properties of Steklov operators and we construct a sequence which can be canonically associated with an assigned second-order differential operator A.

Let $L^1_{\mathrm{loc}}(\mathbb{R})$ be the space of all locally integrable real functions and for every $b > 0$ define the integral mean operator $M_b : L^1_{\mathrm{loc}}(\mathbb{R}) \to C(\mathbb{R})$ by putting

$$M_b f(x) := \frac{1}{2b} \int_{x-b}^{x+b} f(t)\, dt \,, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}) \,, \quad x \in \mathbb{R} \,. \tag{3.1.1}$$

Then, for every $n \geq 1$, the n-th Steklov operator $S_{n,b} : L^1_{\mathrm{loc}}(\mathbb{R}) \to C(\mathbb{R})$ is defined by setting

$$S_{n,b} := M_b^n \,, \tag{3.1.2}$$

where, as usual, M_b^0 is the identity operator and $M_b^n = M_b \circ M_b^{n-1}$ if $n \geq 1$.

Observe that, for every $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\begin{aligned} S_{0,b}f(x) &= f(x)\,, \\ S_{n,b}f(x) &= M_b\left(S_{n-1,b}f\right)(x) = \frac{1}{2b}\int_{x-b}^{x+b} S_{n-1,b}f(t)dt\,. \end{aligned}$$

The preceding definition is meaningful also in the case where b is a bounded continuous strictly positive real function on $\mathbb{R}$; in this case the integration is extended over the interval $[x-b(x), x+b(x)]$ and (3.1.1) becomes

$$M_b f(x) := \frac{1}{2b(x)}\int_{x-b(x)}^{x+b(x)} f(t)\,dt\,, \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R})\,, \quad x \in \mathbb{R}\,. \tag{3.1.3}$$

In this section we shall be mainly interested in studying some properties of Steklov operators in the space $C(\overline{\mathbb{R}})$ of all continuous real functions on $\mathbb{R}$ which admit finite limits at the points $\pm\infty$. The space $C(\overline{\mathbb{R}})$ is endowed with the uniform norm and obviously every function in $C(\overline{\mathbb{R}})$ is bounded and uniformly continuous.

Moreover as usual, we shall denote by $C^2(\overline{\mathbb{R}})$ the space of all functions $f \in C(\overline{\mathbb{R}})$ which are twice differentiable and such that $f'' \in C(\overline{\mathbb{R}})$. Observe that $C^2(\overline{\mathbb{R}})$ is obviously dense in $C(\overline{\mathbb{R}})$ with respect to the uniform norm.

Observe that if $f \in C(\mathbb{R})$ and $\lim_{x\to+\infty} f(x) = \ell \in \mathbb{R}$, then we also have $\lim_{x\to+\infty} S_{n,b}f(x) = \ell$. Indeed, this easily follows from an inductive argument on the integer $n \geq 0$ using the equality $S_{n,b}f(x) = S_{n-1,b}f(\xi)$ which holds for some $\xi \in [x-b(x), x+b(x)]$ and using the boundedness of the function b which implies that $\lim_{x\to+\infty}(x-b(x)) = +\infty$.

Hence Steklov operators may be regarded as linear operators from $C(\overline{\mathbb{R}})$ into $C(\overline{\mathbb{R}})$ and in this case they become positive linear contractions with respect to the uniform norm.

Our aim is to use these operators for the investigation of some degenerate second-order differential operators.

Namely let $a \in C(\overline{\mathbb{R}})$ be a strictly positive function such that

$$\mathrm{id}\cdot a \in C(\overline{\mathbb{R}})\,. \tag{3.1.4}$$

Consider the differential operator $A : D(A) \to C(\overline{\mathbb{R}})$ defined by

$$Au(x) := \frac{1}{6}\,a(x)^2\,u''(x)\,, \qquad u \in D(A)\,, \quad x \in \mathbb{R}\,, \tag{3.1.5}$$

where

$$D(A) := \left\{u \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid a^2\,u'' \in C(\overline{\mathbb{R}})\right\}\,. \tag{3.1.6}$$

Remark 3.1.1 Observe that A is a second-order elliptic differential operator, which is degenerate since a vanishes at the endpoints of the real line.

Moreover, the endpoints $\pm\infty$ are natural endpoints and hence the operator $(A, D(A))$ generates a C_0-semigroup of contractions on $C(\overline{\mathbb{R}})$ on the maximal domain $D(A)$.

Furthermore, every function $u \in D(A)$ also satisfies Ventcel's boundary conditions

$$\lim_{x\to\pm\infty} a(x)^2\, u''(x) = 0\ .$$

Indeed if $u \in D(A) \setminus D_V(A)$, we should have for example $\lim\limits_{x\to+\infty} Au(x) = \ell \neq 0$. Then $\lim\limits_{x\to+\infty} \dfrac{1}{6x^2}\, a^2(x)x^2u''(x) = \ell$ and therefore $\lim\limits_{x\to+\infty} u''(x) \neq 0$; consequently $\lim\limits_{x\to+\infty} \dfrac{1}{x^2}u(x) \neq 0$, contradicting the condition $u \in C(\overline{\mathbb{R}})$. At the point $-\infty$ we can reason similarly.

Hence we conclude that in this case the maximal domain coincides with Ventcel's domain. For these and further results we also refer to the Chapter II, [48, Chapter VI, Section 4] and [69]. □

We shall be interested in studying the connections between the differential operator A and sequences of Steklov type operators.

Namely, we define the functions $b_n := a/n$ and consider the operators $L_n : C(\overline{\mathbb{R}}) \to C(\overline{\mathbb{R}})$ defined by

$$L_n f(x) := S_{n,b_n} f(x)\ , \qquad f \in C(\overline{\mathbb{R}})\ , \quad x \in \mathbb{R}\ . \tag{3.1.7}$$

Since b_n is bounded, the operator L_n is well-defined as an operator acting on $C(\overline{\mathbb{R}})$; furthermore, since a vanishes at $\pm\infty$, we have that L_n interpolates every function $f \in C(\overline{\mathbb{R}})$ at $\pm\infty$, in the sense that $\lim_{x\to\pm\infty} L_n f(x) = \lim_{x\to\pm\infty} f(x)$.

Moreover, observe that $L_n = M_{b_n}^n$ for every $n \in \mathbb{N}$ and consequently, for every $k \geq 1$, we also have

$$L_n^k = M_{b_n}^{kn} = S_{kn,b_n}\ . \tag{3.1.8}$$

Using (3.1.7), the operators L_n can be extended in a natural way to the space of all continuous functions on $\mathbb{R}$, and for this extension we can state the following properties, for every $x \in \mathbb{R}$:

i) $L_n\mathbf{1}(x) = S_{n,b_n}\mathbf{1}(x) = 1\ ,$

ii) $L_n\mathrm{id}(x) = S_{n,b_n}\mathrm{id}(x) = x\ ,$

iii) $L_n(\mathrm{id}^2)(x) = S_{n,b_n}(\mathrm{id}^2)(x)$
$= x^2 + \frac{1}{3}\sum_{i=0}^{n-1} S_{i,b_n}(b_n^2)(x) = x^2 + \frac{1}{3n}\left(\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right)\ .$

Since S_{i,b_n} is a linear contraction, we can conclude that

$$\lim_{n\to+\infty} L_n(\mathrm{id}^2)(x) = \lim_{n\to+\infty} S_{n,b_n}(\mathrm{id}^2)(x) = x^2$$

uniformly with respect to $x \in \mathbb{R}$.

We need some further preliminary results in order to state some deeper properties of the operators L_n.

Proposition 3.1.2 *For every $f \in C(\overline{\mathbb{R}})$, $k \geq 1$ and $x \in \mathbb{R}$,*

$$\|S_{k,b_n}(f) - f\| \leq k\|S_{1,b_n}(f) - f\| \,. \tag{3.1.9}$$

Proof. We argue by induction on the integer $k \geq 1$. If $k = 1$ then (3.1.9) is obviously true. Now, assume that (3.1.9) holds for $k \geq 1$. We have

$$\begin{aligned} |S_{k+1,b_n}f(x) - f(x)| &\leq |S_{1,b_n}(S_{k,b_n}f - f)(x)| + |S_{1,b_n}f(x) - f(x)| \\ &\leq \|S_{k,b_n}(f) - f\| + \|S_{1,b_n}(f) - f\| \\ &\leq k\|S_{1,b_n}(f) - f\| + \|S_{1,b_n}(f) - f\| \\ &\leq (k+1)\|S_{1,b_n}(f) - f\| \end{aligned}$$

and this completes the induction argument. □

As a consequence of the above result, we can state the following estimate.

Proposition 3.1.3 *For every $f \in C^2(\overline{\mathbb{R}})$ we have*

$$\|S_{k,b_n}f - f\| \leq \frac{\|a^2\|\|f''\|}{6}\frac{k}{n^2} \,. \tag{3.1.10}$$

Proof. Let $f \in C^2(\overline{\mathbb{R}})$; for every $x, t \in \mathbb{R}$ we can write

$$f(t) - f(x) = f'(x)\,(t-x) + \frac{1}{2}f''(\xi_t)(t-x)^2$$

with ξ_t between t and x.

Then

$$\begin{aligned} S_{1,b}f(x) - f(x) &= \frac{1}{2b}\int_{x-b}^{x+b}(f(t) - f(x))dt \\ &= f'(x)\frac{1}{2b}\int_{x-b}^{x+b}(t-x)dt + \frac{1}{2b}\int_{x-b}^{x+b}\frac{1}{2}f''(\xi_t)(t-x)^2dt \\ &= \frac{1}{2b}\int_{x-b}^{x+b}\frac{1}{2}f''(\xi_t)(t-x)^2dt \end{aligned}$$

and

$$\begin{aligned} |S_{1,b}f(x) - f(x)| &\leq \frac{1}{2}\sup_{t\in[x-b,x+b]}|f''(t)|\frac{1}{2b}\int_{x-b}^{x+b}(t-x)^2dt \\ &= \frac{b^2}{3!}\sup_{t\in[x-b,x+b]}|f''(t)| \,. \end{aligned}$$

Letting $b := b_n(x) = a(x)/n$ and taking into account that f'' and a are bounded we can write

$$|S_{1,b_n} f(x) - f(x)| \le \frac{1}{n^2} \frac{a(x)^2}{3!} \sup_{t\in[x-b_n(x),x+b_n(x)]} |f''(t)| \,, \tag{3.1.11}$$

and since $f \in C(\overline{\mathbb{R}})$ and $a \in C(\overline{\mathbb{R}})$

$$|S_{1,b_n} f(x) - f(x)| \le \frac{\|a^2\| \|f''\|}{6} \frac{1}{n^2} \,.$$

Finally, from Proposition 3.1.2, we get

$$\|S_{k,b_n} f - f\| \le k \, \|S_{1,b_n} f - f\| \le \frac{\|a^2\| \|f''\|}{6} \frac{k}{n^2} \,.$$

□

Under the following further hypothesis on a

$$\exists\, \delta > 0, \; \exists\, M > 0 \quad : \quad \sup_{\substack{x,y\in\mathbb{R} \\ |x-y|\le\delta}} \frac{a^2(x)}{a^2(y)} \le M \,, \tag{3.1.12}$$

we can have (3.1.10) in terms of the operator A,

Proposition 3.1.4 *If condition* (3.1.12) *holds, then for every* $f \in D(A)$, *we have*

$$\|S_{k,b_n} f - f\| \le M \|Af\| \frac{k}{n^2} \,.$$

Proof. From (3.1.11) we have

$$|S_{1,b_n} f(x) - f(x)| \le \frac{1}{n^2} \frac{a(x)^2}{3!} \sup_{t\in[x-b_n(x),x+b_n(x)]} |f''(t)| \tag{3.1.13}$$

$$= \frac{1}{n^2} \frac{a(x)^2}{3!} |f''(\xi)| = \frac{1}{n^2} \frac{|a^2(\xi) f''(\xi)|}{6} \frac{a^2(x)}{a^2(\xi)} \le \frac{\|Af\|}{n^2} \left(\frac{a(x)}{a(\xi)}\right)^2 \,,$$

for some $\xi \in [x - a(x)/n, x + a(x)/n]$, that is $|\xi - x| \le \frac{\|a\|}{n}$. If we choose n such that $|\xi - x| \le \delta$ from condition (3.1.12) we have $\left(\frac{a(x)}{a(\xi)}\right)^2 \le M$ and from (3.1.13)

$$|S_{1,b_n} f(x) - f(x)| \le M \frac{\|Af\|}{n^2} \,.$$

Finally, from Proposition 3.1.2, we get

$$\|S_{k,b_n} f - f\| \le k \, \|S_{1,b_n} f - f\| \le M \|Af\| \, \frac{k}{n^2} \,.$$

□

Corollary 3.1.5 *For every $f \in C^2(\overline{\mathbb{R}})$,*

$$\|L_n(f) - f\| \leq \frac{\|a^2\|\|f''\|}{6}\frac{1}{n} \, .$$

Corollary 3.1.6 *If condition* (3.1.12) *holds, then for every $f \in D(A)$, we have*

$$\|L_n(f) - f\| \leq M\|Af\|\frac{1}{n} \, .$$

From the above results, we also obtain the following approximation properties of the sequence $(L_n)_{n\geq 1}$.

Theorem 3.1.7 *For every $f \in C(\overline{\mathbb{R}})$, we have*

$$\lim_{n\to+\infty} L_n(f) = f \qquad \textit{uniformly on } \mathbb{R} \, .$$

Proof. Indeed, it is clear that the uniform convergence of $(L_n(f))_{n\geq 1}$ to f holds true for every $f \in C^2(\overline{\mathbb{R}})$. Since $C^2(\overline{\mathbb{R}})$ is dense in $C(\overline{\mathbb{R}})$ and $(L_n)_{n\geq 1}$ is a sequence of positive contractions, the proof is complete. □

Our next aim is to obtain a quantitative Voronovskaja-type formula for Steklov operators. We begin with some properties of independent interest.

Proposition 3.1.8 *For every $f \in C^2(\overline{\mathbb{R}})$ we have*

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\| \leq \frac{\|a^2\|\|f''\|}{12}\frac{1}{n} \, .$$

Proof. Let $f \in C^2(\overline{\mathbb{R}})$ using Proposition 3.1.3, for every $x \in \mathbb{R}$ we can write

$$\begin{aligned}
\left| \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}f(x) - f(x) \right| &= \frac{1}{n}\left| \sum_{k=0}^{n-1} (S_{k,b_n}f(x) - f(x)) \right| \\
&\leq \frac{1}{n}\sum_{k=0}^{n-1} |S_{k,b_n}f(x) - f(x)| \\
&\leq \frac{1}{n}\sum_{k=0}^{n-1} \frac{\|a^2\|\|f''\|}{6}\frac{k}{n^2} = \frac{\|a^2\|\|f''\|}{6n^3}\sum_{k=0}^{n-1} k \\
&= \frac{\|a^2\|\|f''\|}{6n^3}\,\frac{n(n-1)}{2}
\end{aligned}$$

and this completes the proof. □

Proposition 3.1.9 *If condition* (3.1.12) *holds, then for every* $f \in D(A)$, *we have*

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\| \le M \frac{\|Af\|}{2} \frac{1}{n} \; .$$

Proposition 3.1.10 *For every* $f \in C(\overline{\mathbb{R}})$ *we have*

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} S_{k,b_n}(f) = f$$

uniformly on $\mathbb{R}$.

Proof. If follows from the density of $C^2(\overline{\mathbb{R}})$ into $C(\overline{\mathbb{R}})$ and from Proposition 3.1.8 □

Finally, we shall need the following lemma.

Lemma 3.1.11 *We have*

1. $S_{n,b_n}\mathbf{1}(x) - 1 = 0$,

2. $S_{n,b_n}(\mathrm{id} - x)(x) = 0$,

3. $S_{n,b_n}(\mathrm{id} - x)^2(x) = \dfrac{1}{3n} \left(\dfrac{1}{n} \displaystyle\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x) \right)$,

4. $|S_{n,b_n}((\mathrm{id} - x)^4)(x)| \le \dfrac{1}{n^2} \dfrac{\|a^2\|^2}{3}$.

Proof. We have already evaluated S_{n,b_n} at the functions $\mathbf{1}$, id and id^2, then a straightforward computations gives *1*, *2* and *3*.
Now, we also observe that

$$S_{n,b_n}(\mathrm{id}^3)(x) = x^3 + \sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id} \cdot b_n^2)(x) \; ,$$

$$S_{n,b_n}(\mathrm{id}^4)(x) = x^4 + 2 \sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}^2 \cdot b_n^2)(x) + \frac{1}{5} \sum_{k=0}^{n-1} S_{k,b_n}(b_n^4)(x) \; .$$

At this point we can easily evaluate $S_{n,b_n}\left((\mathrm{id}-x)^4\right)(x)$ and obtain

$$\begin{aligned}
S_{n,b_n}((\mathrm{id}-x)^4)(x) &= S_{n,b_n}(\mathrm{id}^4)(x) - 4xS_{n,b_n}(\mathrm{id}^3)(x) + 6x^2S_{n,b_n}(\mathrm{id}^2)(x) \\
&\quad -4x^3S_{n,b_n}(\mathrm{id})(x) + x^4 \\
&= x^4 + 2\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}^2\cdot b_n^2)(x) + \frac{1}{5}\sum_{k=0}^{n-1} S_{k,b_n}(b_n^4)(x) \\
&\quad -4x\left(x^3 + \sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}\cdot b_n^2)(x)\right) \\
&\quad +6x^2\left(x^2 + \frac{1}{3}\sum_{k=0}^{n-1} S_{k,b_n}(b_n^2)(x)\right) - 4x^3x + x^4 \\
&= 2\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}^2\cdot b_n^2)(x) - 4x\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}\cdot b_n^2)(x) \\
&\quad +2x^2\sum_{k=0}^{n-1} S_{k,b_n}(b_n^2)(x) + \frac{1}{5}\sum_{k=0}^{n-1} S_{k,b_n}(b_n^4)(x) \\
&= \frac{2}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}^2\cdot a^2)(x) - 4x\frac{1}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}(\mathrm{id}\cdot a^2)(x) \\
&\quad +2x^2\frac{1}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}(a^2)(x) + \frac{1}{5n^2}\frac{1}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}(a^4)(x)\ .
\end{aligned}$$

Consequently,

$$S_{n,b_n}((\mathrm{id}-x)^4)(x) = \frac{2}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}((\mathrm{id}-x)^2\cdot a^2)(x) + \frac{1}{5n^4}\sum_{k=0}^{n-1} S_{k,b_n}(a^4)(x)\ .$$

As regards the first term, for every $k = 1,\dots,n-1$, we have

$$\begin{aligned}
\left|S_{k,b_n}((\mathrm{id}-x)^2\cdot a^2)(x)\right| &\le \left\|a^2\right\| \left|S_{k,b_n}((\mathrm{id}-x)^2)(x)\right| \\
&= \left\|a^2\right\| \frac{1}{3n^2}\sum_{i=0}^{k-1} S_{i,b_n}(a^2)(x) \le \left\|a^2\right\|^2 \frac{k}{3n^2}
\end{aligned}$$

and consequently

$$\left|\frac{2}{n^2}\sum_{k=0}^{n-1} S_{k,b_n}((\mathrm{id}-x)^2\cdot a^2)(x)\right| \le \frac{2}{3n^4}\ \left\|a^2\right\|^2 \sum_{k=1}^{n-1} k = \frac{n(n-1)}{3n^4}\ \left\|a^2\right\|^2\ ;$$

since $a^4 \in C(\overline{\mathbb{R}})$ and S_{k,b_n} are positive contractions on $C(\overline{\mathbb{R}})$ with respect to the uniform norm, we can write

$$
\begin{aligned}
|S_{n,b_n}((\mathrm{id}-x)^4)(x)| &\le \frac{n(n-1)}{3n^4}\,\left\|a^2\right\|^2 + \frac{1}{5n^4}\sum_{k=0}^{n-1}\|a^4\| \\
&= \left(\frac{1}{n^2}-\frac{1}{n^3}\right)\frac{\left\|a^2\right\|^2}{3} + \frac{1}{n^3}\frac{\|a^4\|}{5} \\
&= \frac{1}{n^2}\frac{\left\|a^2\right\|^2}{3} + \frac{1}{n^3}\left(\frac{\|a^4\|}{5} - \frac{\left\|a^2\right\|_\infty^2}{3}\right) \le \frac{1}{n^2}\frac{\left\|a^2\right\|^2}{3}\,.
\end{aligned}
$$

□

Theorem 3.1.12 (Voronovskaja-type formula)
For every $f \in C^2(\overline{\mathbb{R}})$ we have

$$\lim_{n\to\infty} n\,(L_n f - f) = \frac{a^2}{6} f''\,.$$

PROOF. Let $f \in C^2(\overline{\mathbb{R}})$; we apply Theorem 2.2.2, taking $h_n = \frac{1}{n}$. The operators $\mathcal{A}_n f(x)$ became $n\frac{1}{2}L_n((\mathrm{id}-x)^2)(x)f''(x) = \frac{1}{6n}\sum_{k=0}^{n-1} S_{k,b_n}(a^2)(x)f''(x)$ which converges uniformly with respect $x \in \mathbb{R}$ to $\frac{1}{6}a^2(x)f''(x) = Af(x)$ from Proposition 3.1.10 and since $f \in C^2(\overline{\mathbb{R}})$. On the other hand $L_n\mathbf{1} - \mathbf{1} = 0$ and from Lemma 3.1.11 hypothesis (2.2.2,3) is also satisfied. Finally since $f \in C^2(\overline{\mathbb{R}})$, the second-order derivative is bounded and uniformly continuous. □

In order to consider a quantitative version of the above Voronovskaja-type formula we need to introduce the following space

$$C^{2,\alpha}(\overline{\mathbb{R}}) := \left\{f \in C^2(\overline{\mathbb{R}}) \mid f'' \in C^\alpha(\mathbb{R})\right\}\,.$$

Theorem 3.1.13 *If $a^2 \in C^2(\overline{\mathbb{R}})$ for every $f \in C^{2,\alpha}(\overline{\mathbb{R}})$ we have*

$$\|n(L_n f - f) - Af\| \le C_a \frac{M_f}{n^{\alpha/2}}\,, \tag{3.1.14}$$

where M_f is the seminorm defined by

$$M_f := L_{f''} + \|f''\| \tag{3.1.15}$$

and C_a is a constant depending on a defined by

$$C_a := \|a^2\| \max\left\{1, \frac{\|(a^2)''\|}{72\sqrt{n}}\right\}\,. \tag{3.1.16}$$

Proof. Let $f \in C^{2,\alpha}(\overline{\mathbb{R}})$ and let $\mathcal{A}_{S_{n,b_n}}$ be the operator (2.2.2) obtained by taking $L = S_{n,b_n}$ and which can be evaluated from Lemma 3.1.11,

$$\mathcal{A}_{S_{n,b_n}} f(x) = \frac{1}{6}\frac{1}{n}\left(\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right) f''(x) \ ,$$

we have

$$\begin{aligned}|n(L_n f(x) - f(x)) - Af| \quad &\leq \quad |n(L_n f(x) - f(x) - \mathcal{A}_{S_{n,b_n}} f(x)| \\ &\quad + |n\mathcal{A}_{S_{n,b_n}} f(x) - Af(x)| \ , \qquad (3.1.17)\end{aligned}$$

in regard to the first term of the the righthand side using Theorem 2.2.1 end Lemma 3.1.11 we have

$$\begin{aligned}&|n(L_n f(x) - f(x) - \mathcal{A}_{S_{n,b_n}} f(x)| \\ &\quad \leq \frac{L_{f''}}{2}\left(\frac{1}{3n^2}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right)^{\alpha/2}\left(\frac{1}{3}\left(\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right) + \frac{\|a^2\|}{\sqrt{3}}\right) \\ &\quad \leq \frac{L_{f''}}{2}\left(\frac{\|a^2\|}{3n}\right)^{\alpha/2}\frac{1+\sqrt{3}}{3}\|a^2\| \ .\end{aligned}$$

As regards the second term of (3.1.17) from Proposition 3.1.8, since $a^2 \in C^2(\overline{\mathbb{R}})$, we have

$$\begin{aligned}|n\mathcal{A}_{S_{n,b_n}} f(x) - Af(x)| &= \frac{1}{6}|f''(x)|\left|\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x) - a(x)\right| \\ &\leq \frac{1}{6}\|f''\|\frac{\|a^2\|\|(a^2)''\|}{12}\frac{1}{n} \\ &\leq \frac{1}{72n}\|f''\|\|a^2\|\|(a^2)''\| \ .\end{aligned}$$

Collecting the above inequalities we have

$$\begin{aligned}&|n(L_n f(x) - f(x)) - Af| \\ &\quad \leq \frac{L_{f''}}{2}\left(\frac{\|a^2\|}{3n}\right)^{\alpha/2}\frac{1+\sqrt{3}}{3}\|a^2\| + \frac{1}{72n}\|f''\|\|a^2\|\|(a^2)''\| \\ &\quad \leq \|a^2\|\left(\frac{L_{f''}}{n^{\alpha/2}} + \frac{\|f''\|\|(a^2)''\|}{72n}\right) \leq \frac{\|a^2\|}{n^{\alpha/2}}\left(L_{f''} + \|f''\|\frac{\|(a^2)''\|}{72\sqrt{n}}\right) \\ &\quad \leq C_a\frac{M_f}{n^{\alpha/2}}\end{aligned}$$

where M_f is the seminorm defined by (3.1.15) and C_a is a constant depending on a defined by (3.1.16). □

Observe that for n large enough C_a is equal to $\|a^2\|$.

At this point we deepen the connection with the differential operator A. We shall need the following core property of the operator $(A, D(A))$.

Proposition 3.1.14 *The space $C^2(\overline{\mathbb{R}})$ is a core for $(A, D(A))$.*

Proof. Let $u \in D(A)$ and $0 < \varepsilon < 1$. We show the existence of a function $v \in C^2(\overline{\mathbb{R}})$ such that

$$|u(x) - v(x)| \leq \varepsilon \,, \quad |Au(x) - Av(x)| \leq \varepsilon \,, \qquad x \in \mathbb{R} \,.$$

We argue only in the interval $[0, +\infty[$ since the same argument can be applied to $]-\infty, 0]$. Since $u \in C(\overline{\mathbb{R}})$ and

$$\lim_{x\to+\infty} Au(x) = 0 \,, \qquad \lim_{x\to+\infty} a(x) = 0 \,,$$

we can find $c > 0$ such that

$$|u(x) - \ell| < \varepsilon \,, \quad |Au(x)| < \varepsilon \,, \quad a(x)^2 < \varepsilon \,, \qquad x \geq c \,,$$

where $\ell := \lim_{x\to+\infty} u(x)$.

We observe that $\liminf_{x\to+\infty} u'(x) \leq 0 \leq \limsup_{x\to+\infty} u'(x)$, otherwise we could not have $u \in C(\overline{\mathbb{R}})$; consequently, we can choose $x_0 > c$ such that $|u'(x_0)| \leq \varepsilon$. Now, we consider $\delta_\varepsilon > 0$ such that

$$\frac{a(x)^2}{a(x_0)^2} \leq 2 \,, \qquad x \in [x_0, x_0 + \delta_\varepsilon]$$

and define

$$h_1 := \min\left\{\sqrt{\varepsilon}, \frac{\varepsilon}{|u''(x_0)| + 1}, \delta_\varepsilon\right\} \,, \quad h_2 := 3\sqrt{\varepsilon} \,.$$

We put, for simplicity,

$$x_1 := x_0 + h_1 \,, \quad x_3 := x_1 + h_2 \,;$$

finally, we consider $x_1 < x_2 < x_3$, and define

$$M := -\frac{2u'(x_0)}{h_2} - u''(x_0)\,\frac{h_1}{h_2} \,,$$

and the functions $w_2, w_1, w : [x_0, x_3] \to \mathbb{R}$ by setting, for every $x \in [x_0, x_3]$,

$$w_2(x) := \begin{cases} u''(x_0)\,\dfrac{x_1 - x}{x_1 - x_0} \,, & x \in [x_0, x_1] \,, \\[2ex] M\,\dfrac{x - x_1}{x_2 - x_1} \,, & x \in]x_1, x_2[\,, \\[2ex] M\,\dfrac{x_3 - x}{x_3 - x_2} \,, & x \in [x_2, x_3] \,, \end{cases}$$

$$w_1(x) := u'(x_0) + \int_{x_0}^{x} w_2(t)\,dt \,,$$

$$w(x) := u(x_0) + \int_{x_0}^{x} w_1(t)\,dt \,.$$

We easily get $w(x_0)=u(x_0)$, $w'(x_0)=w_1(x_0)=u'(x_0)$ and $w''(x_0)=w_2(x_0)=u''(x_0)$ and further

$$w''(x_3)=w_2(x_3)=0$$

and

$$\begin{aligned} w'(x_3) &= u'(x_0)+\int_{x_0}^{x_1} w_2(t)dt+\int_{x_1}^{x_2} w_2(t)dt+\int_{x_2}^{x_3} w_2(t)dt \\ &= u'(x_0)+\frac{1}{2}(x_1-x_0)u''(x_0)+\frac{1}{2}(x_2-x_1)M+\frac{1}{2}(x_3-x_2)M \\ &= u'(x_0)+\frac{1}{2}(x_1-x_0)u''(x_0)+\frac{1}{2}(x_3-x_1)M \end{aligned}$$

which yields $w'(x_3)=0$ from the definition of M.

Moreover, we observe that $h_1\le\sqrt{\varepsilon}$, $h_1|u''(x_0)|\le\varepsilon$, $h_2|M|\le 3\varepsilon$ and $|M|\le\sqrt{\varepsilon}$. Hence we obtain

$$\sup_{x\in[x_0,x_1]}|w'(x)|\le|w'(x_0)|+(x_1-x_0)\sup_{x\in[x_0,x_1]}|w''(x)|\le|u'(x_0)|+h_1|u''(x_0)|$$

and consequently

$$\begin{aligned} \sup_{x\in[x_1,x_3]}|w'(x)| &\le |w'(x_1)|+(x_3-x_1)\sup_{x\in[x_1,x_3]}|w''(x)| \\ &\le |u'(x_0)|+h_1|u''(x_0)|+h_2|M|\ ; \end{aligned}$$

therefore, in any case

$$\sup_{x\in[x_0,x_3]}|w'(x)|\le 5\varepsilon$$

and this implies

$$\sup_{x\in[x_0,x_3]}|w(x)-w(x_0)|\le(h_1+h_2)\sup_{x\in[x_0,x_3]}|w'(x)|\le 20\varepsilon\sqrt{\varepsilon}\ .$$

We conclude that

$$\begin{aligned} \sup_{x\in[x_0,x_3]}|w(x)-u(x)| &\le \sup_{x\in[x_0,x_3]}|w(x)-w(x_0)|+\sup_{x\in[x_0,x_3]}|u(x)-u(x_0)| \\ &\le 20\varepsilon\sqrt{\varepsilon}+2\varepsilon\ , \end{aligned}$$

and

$$\begin{aligned} \sup_{x\in[x_0,x_3]}|a(x)^2\,w''(x)| &\le \sup_{x\in[x_0,x_1]}|a(x)^2\,w''(x)|+\sup_{x\in[x_1,x_3]}|a(x)^2\,w''(x)| \\ &\le \sup_{x\in[x_0,x_1]}|a(x)^2\,u''(x_0)|+\sup_{x\in[x_1,x_3]}|a(x)^2\,M| \\ &\le \sup_{x\in[x_0,x_1]}\frac{a(x)^2}{a(x_0)^2}\,|a(x_0)^2\,u''(x_0)|+\varepsilon\sqrt{\varepsilon} \\ &\le 12\varepsilon+\varepsilon\sqrt{\varepsilon}\ . \end{aligned}$$

At this point it is clear that the function $v : [0, +\infty[\to \mathbb{R}$ defined by setting

$$v(x) := \begin{cases} u(x)\,, & 0 \le x < x_0\,, \\ w(x)\,, & x_0 \le x \le x_3\,, \\ w(x_3)\,, & x_3 < x\,, \end{cases}$$

is in $C^2([0, +\infty])$ and satisfies the required properties. □

We have the following main result.

Theorem 3.1.15 *The operator $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\ge0}$ of positive contractions in $C(\overline{\mathbb{R}})$ and, for every $t \ge 0$ and for every sequence $(k(n))_{n\ge1}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have*

$$T(t) = \lim_{n\to+\infty} L_n^{k(n)} \qquad \textit{strongly on } C(\overline{\mathbb{R}})\,, \tag{3.1.18}$$

moreover if $a^2 \in C^2(\overline{\mathbb{R}})$, for every $f \in C^{2,\alpha}(\overline{\mathbb{R}})$ we have

$$\begin{aligned} \left\| T(t)u - L_n^{k(n)}u \right\| \le & t\,\frac{C_a M_f}{n^{\alpha/2}} + \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n} \right) \\ & \times \left(\|Af\| + \frac{C_a M_f}{n^{\alpha/2}} \right) \end{aligned} \tag{3.1.19}$$

and choosing $k(n) = [nt]$

$$\left\| T(t)u - L_n^{[nt]}u \right\| \le t\,\frac{C_a M_f}{n^{\alpha/2}} + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}} \right) \left(\|Af\| + \frac{C_a M_f}{n^{\alpha/2}} \right)\,. \tag{3.1.20}$$

Proof. We already know that the operator $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\ge0}$ of contractions and hence, for every $\lambda > 0$, the range $(\lambda - A)(D(A))$ coincides with $C(\overline{\mathbb{R}})$. Moreover, by Proposition 3.1.14, $C^2(\overline{\mathbb{R}})$ is a core for $(A, D(A))$ and therefore $(\lambda - A)(C^2(\overline{\mathbb{R}}))$ is dense in $C(\overline{\mathbb{R}})$. Hence we can apply Trotter's approximation theorem [70] and obtain that the closure of the operator arising from the Voronovskaja-formula (Theorem 3.1.12) generates a C_0-semigroup represented by (3.1.18). Finally, this closure coincides with $(A, D(A))$ by the core property of $C^2(\overline{\mathbb{R}})$. The positivity of the semigroup is a consequence of the representation (3.1.18).

At this point we can apply Theorem 1.1.2. From Theorem 3.1.13 follows that the seminorms are given by $\psi_n(f) = C_a \frac{M_f}{n^{\alpha/2}}$ and $\varphi_n(f) = \|Af\| + C_a \frac{M_f}{n^{\alpha/2}}$, then taking into account that the growth bound of $(T(t))_{t\ge0}$ is equal to 0 and every $T(t)$ is a linear contraction, i.e. $\omega = 0$ and $M = 1$, the estimates (3.1.19) and (3.1.20) follow directly from (1.1.10) and (1.1.11), and this completes the proof. □

3.2 Steklov operators on bounded intervals

In this section we show how Steklov operators can be considered even in spaces of continuous functions on a bounded interval. For the sake of simplicity, we shall consider only the case of the interval $[0,1]$. Also in this case we can construct a suitable sequence of positive Steklov type operators which can be associated with the differential operator $A : D(A) \to C([0,1])$ defined by

$$Au(x) := \frac{1}{6}\, a(x)^2\, u''(x)\ , \qquad u \in D(A)\ , \qquad x \in [0,1]\ , \tag{3.2.1}$$

on the domain

$$D(A) := \{u \in C([0,1]) \cap C^2(]0,1[) \ \mid \ a^2 u'' \in C([0,1])\}\ , \tag{3.2.2}$$

where $a \in C([0,1])$ is strictly positive on $]0,1[$ and

$$a(x) = O(x(1-x))\ , \qquad \text{as } x \to 0,1 \tag{3.2.3}$$

(hence $a(0) = a(1) = 0$).

It is straightforward to check that 0 and 1 are natural endpoints and therefore the maximal domain $D(A)$ coincides with the following Ventcel's domain

$$D_V(A) := \{u \in C([0,1]) \cap C^2(]0,1[) \ \mid \ \lim_{x\to 0,1} a^2(x) u''(x) = 0\}\ .$$

Condition (3.2.3) also ensures that

$$b_n(x) := \frac{a(x)}{n} \le \frac{1}{2} - \left| x - \frac{1}{2} \right|$$

for n large enough and consequently for such integers the integral mean operators $M_{b_n} : L^1_{\mathrm{loc}}(0,1) \to C(]0,1[)$

$$M_{b_n} f(x) = \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} f(t)\, dt \tag{3.2.4}$$

are well-defined together with all their iterates. Observe that if $f \in C([0,1])$ then we can use the convention $M_{b_n} f(0) = f(0)$ and $M_{b_n} f(1) = f(1)$ and consider M_{b_n} as an operator acting on $C([0,1])$.

Now, consider the operators $L_n := S_{n,b_n} : C([0,1]) \to C([0,1])$.

We have the following properties:

i) $L_n \mathbf{1}(x) = 1$,

ii) $L_n \mathrm{id}(x) = x$,

iii) $L_n(\mathrm{id}^2)(x) = x^2 + \frac{1}{3}\sum_{i=0}^{n-1} S_{i,b_n}(b_n^2)(x) = x^2 + \frac{1}{3n}\left(\frac{1}{n}\sum_{i=0}^{n-1} S_{i,b_n}(a^2)(x)\right)$.

Since $|M_{b_n}(a^2)(x)| \le \|a^2\|$ we have that $|\frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(a^2)(x)| \le n\|a^2\|/n = \|a^2\|$, and therefore

$$\lim_{n\to\infty} L_n(\mathrm{id}^2) = \mathrm{id}^2$$

uniformly on $[0,1]$. From the classical first Korovkin's theorem [9, Theorem 4.2.7], we obtain the following result.

Theorem 3.2.1 *For every $f \in C([0,1])$,*

$$\lim_{n\to+\infty} L_n(f) = f \qquad \text{uniformly with respect to } x \in [0,1] \ .$$

As in the previous section, we can state the following results.

Proposition 3.2.2 *We have the following properties:*

1. *For every $f \in C^2([0,1])$, there exists a constant $C_f > 0$, depending on f, such that for every $n \ge 1$*

$$\|S_{k,b_n}(f) - f\| \le C_f \frac{k}{n^2} \ .$$

2. *For every $f \in C([0,1])$ we have*

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(f) = f$$

uniformly on $[0,1]$.

3. *We have*

$$|S_{n,b_n}((\mathrm{id}-x)^4)(x)| \le \frac{1}{n^2}\frac{\|a^2\|^2}{3} \qquad \text{for every } x \in [0,1] \ .$$

Proof. As regards the first property, we can reason as in Proposition 3.1.4 and obtain

$$C_f = \frac{1}{6}\,\|\alpha^2\|\|f''\| \ .$$

Property 2) follows from the density of $C^2([0,1])$ in $C([0,1])$ and the analogous result in the previous section.

Finally, property 3) can be shown in the same way as in Lemma 3.1.11. □

At this point we can state the following Voronovskaja-type formula and its quantitative version; the proof is similar to that of Theorem 3.1.12 and 3.1.13 and for the sake of brevity we shall omit it.

Theorem 3.2.3 (Voronovskaja-type formula)
For every $f \in C^2([0,1])$ *we have*

$$\lim_{n\to\infty} n\left(L_n(f) - f\right) = \frac{a^2}{6} f'' .$$

Theorem 3.2.4 (Quantitative Voronovskaja formula for Steklov operators)
If $a^2 \in C^2([0,1])$ *for every* $f \in C^{2,\alpha}([0,1])$ *we have*

$$\|n\left(L_n(f) - f\right) - Af\| \le C_a \frac{M_f}{n^{\alpha/2}} .$$

where M_f *is the seminorm defined by*

$$M_f := L_{f''} + \|f''\|$$

and C_a *is a constant depending on* a *defined by*

$$C_a := \|a^2\| \max\left\{1, \frac{\|(a^2)''\|}{72\sqrt{n}}\right\} .$$

We already know that the operator $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\ge0}$ of contractions (see, e.g., [44] or [48, Chapter VI, Section 4]) and hence, for every $\lambda > 0$, the range $(\lambda - A)(D(A))$ coincides with $C([0,1])$. If $C^2([0,1])$ is a core for $(A, D(A))$, then $(\lambda - A)(C^2([0,1]))$ is dense in $C([0,1])$ and we can apply Trotter's approximation theorem, which yields the representation of the semigroup $(T(t))_{t\ge0}$ in terms of iterates of the operators L_n with the same arguments of the preceding section. In some particular cases, it can be easily proved that $C^2([0,1])$ is a core for $(A, D(A))$.

Theorem 3.2.5 *Assume that*

$$a(x) = Cx(1-x) , \qquad 0 \le x \le 1 ,$$

for a suitable constant $C > 0$. *Then, the space* $C^2([0,1])$ *is a core for* $(A, D(A))$ *and, for every* $t \ge 0$ *and for every sequence* $(k(n))_{n\ge1}$ *of positive integers satisfying* $\lim_{n\to+\infty} k(n)/n = t$, *we have*

$$T(t) = \lim_{n\to+\infty} L_n^{k(n)} \qquad \textit{strongly on } C([0,1]) , \tag{3.2.5}$$

moreover if $f \in C^{2,\alpha}([0,1])$ *we have*

$$\left\|T(t)f - L_n^{k(n)} f\right\| \le t\,\frac{C_a M_f}{n^{\alpha/2}} + \left(\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\right)\left(\|Af\| + \frac{C_a M_f}{n^{\alpha/2}}\right) \tag{3.2.6}$$

and choosing $k(n) = [nt]$

$$\left\|T(t)f - L_n^{[nt]} f\right\| \le t\,\frac{C_a M_f}{n^{\alpha/2}} + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\right)\left(\|Af\| + \frac{C_a M_f}{n^{\alpha/2}}\right) . \tag{3.2.7}$$

Proof. The core property is well-known (see e.g. [26, Lemma 1.2]). Hence, we can apply Trotter's approximation theorem [70] and obtain the representation (3.2.5).

At this point estimates (3.2.6) and (3.2.7) follows from Theorem 1.1.2 taking into account of quantitative version of voronovskaja's formula obtained in Theorem 3.2.4 and this completes the proof. □

3.3 Steklov operators in weighted spaces

In this section we consider Steklov operators on the space

$$C_w(\overline{\mathbb{R}}) := \{f \in C(\mathbb{R}) \mid f \cdot w \in C(\overline{\mathbb{R}})\} ,$$

where $w : \mathbb{R} \to \mathbb{R}$ is a strictly positive continuous real function which tends to 0 at the points $\pm\infty$. The space $C_w(\overline{\mathbb{R}})$ is endowed with the norm

$$\|f\|_w := \sup_{x\in\mathbb{R}} |f(x) \cdot w(x)| \quad (= \|f \cdot w\|) , \qquad f \in C_w(\overline{\mathbb{R}}) .$$

In order to present a unified treatment we consider weight functions w having the form

$$w(x) := \frac{1}{1+|x|^p} , \qquad x \in \mathbb{R} ,$$

with $p \geq 2$ fixed, even if some partial results can be established in a more general setting.

Let $a \in C(\overline{\mathbb{R}})$ be a strictly positive function satisfying

$$\mathrm{id} \cdot a \in C(\overline{\mathbb{R}}) \tag{3.3.1}$$

and consider the differential operator $A_w : D(A_w) \to C_w(\overline{\mathbb{R}})$ defined by

$$A_w u(x) := \frac{1}{6}\, a(x)^2\, u''(x) , \qquad u \in D(A_w) , \quad x \in \mathbb{R} , \tag{3.3.2}$$

on the following maximal domain

$$D(A_w) := \{u \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid A_w u \in C_w(\overline{\mathbb{R}})\} .$$

We need to assume the following additional condition on a

$$\exists\, \delta > 0,\ \exists\, M > 0 \ : \qquad \sup_{\substack{x,y\in\mathbb{R} \\ |x-y|<\delta}} \frac{a(x)^2}{(1+|x|)\, a(y)^2} \leq M , \tag{3.3.3}$$

which is obviously satisfied if there exist $q \geq 1$ and $C_1, C_2 > 0$ such that, for every $x \in \mathbb{R}$,

$$\frac{C_1}{1+|x|^{q+1}} \leq a(x)^2 \leq \frac{C_2}{1+|x|^q} .$$

We can define $b_n := a/n$ and consider the operator S_{n,b_n}.

Different properties of Steklov operators are based on the behavior of the function $\omega_n : \mathbb{R} \to \mathbb{R}$ defined by setting, for every $x \in \mathbb{R}$,

$$\omega_n(x) := \frac{w(x)}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{w(t)}\, dt .$$

We have the following result.

Proposition 3.3.1 *There exists a constant $C_\omega > 0$ such that*

$$\|\omega_n - \mathbf{1}\| \le \frac{C_\omega}{n^2} \, .$$

PROOF. The property is obviously true on a neighborhood $[-\delta, \delta]$ of 0, hence by symmetry we can prove it only in the interval $[\delta, +\infty[$. Since $p \ge 2$, we have $1/w \in C^2(]0, +\infty[)$ and for every $x, t > 0$ we can write

$$\frac{1}{w(t)} - \frac{1}{w(x)} = \left(\frac{1}{w(x)}\right)' (t-x) + \frac{1}{2}\left(\frac{1}{w(\xi_t)}\right)'' (t-x)^2$$

for a suitable ξ_t in the interval with endpoints t and x.

Now, let $x \ge \delta$ and set for simplicity $b = b_n(x) = a(x)/n$; for n enough large we can assume that the interval of integration in the definition of the n-th Steklov operator is contained in $]0, +\infty[$ and consequently

$$\begin{aligned}
\left|S_{1,b}\left(\frac{1}{w}\right)(x) - \frac{1}{w(x)}\right| &= \left|\frac{1}{2b}\int_{x-b}^{x+b}\left(\frac{1}{w(t)} - \frac{1}{w(x)}\right)dt\right| \\
&= \Big|\left(\frac{1}{w(x)}\right)' \frac{1}{2b}\int_{x-b}^{x+b}(t-x)dt \\
&\quad + \frac{1}{2b}\int_{x-b}^{x+b}\frac{1}{2}\left(\frac{1}{w(\xi_t)}\right)''(t-x)^2 dt\Big| \\
&= \left|\frac{1}{2b}\int_{x-b}^{x+b}\frac{1}{2}\left(\frac{1}{w(\xi_t)}\right)''(t-x)^2 dt\right| \\
&\le \frac{1}{2b}\int_{x-b}^{x+b}\left|\frac{1}{2}p(p-1)\xi_t^{p-2}(t-x)^2\right| dt \\
&\le \frac{1}{2b}\frac{1}{2}p(p-1)(x+b)^{p-2}\int_{x-b}^{x+b}(t-x)^2 dt \\
&= \frac{b^2}{6}p(p-1)(x+b)^{p-2} \, .
\end{aligned}$$

Hence, we can conclude that

$$\begin{aligned}
|\omega_n(x) - 1| &= \left|w(x)S_{1,b_n}\left(\frac{1}{w}\right)(x) - w(x)\frac{1}{w(x)}\right| \\
&= w(x)\left|S_{1,b}\left(\frac{1}{w}\right)(x) - \frac{1}{w(x)}\right| \\
&\le \frac{1}{1+x^p}\frac{1}{n^2}\frac{a(x)^2}{6}p(p-1)\left(x + \frac{a(x)}{n}\right)^{p-2} \\
&\le \frac{1}{1+x^p}\frac{1}{6n^2}\|a\|^2 p(p-1)\left(x + \|a\|\right)^{p-2} \\
&\le \frac{C_\omega}{n^2}
\end{aligned}$$

and this completes the proof. □

From the preceding result, we have $\omega_n \le 1 + C_\omega/n^2$ and consequently we can find a constant $C_w > 0$ such that, for every $n \ge 1$,

$$\|\omega_n\|^n \le C_w \ . \tag{3.3.4}$$

Therefore, taking into account that, for every $f \in C_w(\overline{\mathbb{R}})$ and $x \in \mathbb{R}$, we have

$$|w(x)\, S_{1,b_n} f(x)| = \left| w(x) \, \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} f(t)\, w(t)\, \frac{1}{w(t)}\, dt \right| \le \|\omega_n\| \ \|f\|_w \ .$$

Arguing by induction on the integer $n \ge 1$ we get, for every $f \in C_w(\overline{\mathbb{R}})$,

$$\|S_{n,b_n}(f)\|_w \le \|\omega_n\|^n \ \|f\|_w \le C_w \, \|f\|_w \tag{3.3.5}$$

and we conclude that the operators S_{n,b_n} are equibounded.

Remark 3.3.2 The operators S_{n,b_n} map the space $C_w(\overline{\mathbb{R}})$ into itself.

Indeed, let $f \in C_w(\overline{\mathbb{R}})$; on a neighborhood of $+\infty$ we have

$$w(x)\, S_{1,b_n} f(x) = \frac{w(x)}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} f(t)\, dt = w(x)\, f(\xi_x) = \frac{w(x)}{w(\xi_x)}\, w(\xi_x)\, f(\xi_x) \ ,$$

for a suitable $\xi_x \in]x - b_n(x), x + b_n(x)[$; taking the limit as $x \to +\infty$ we have $\xi_x \to +\infty$ and consequently $w(\xi_x)\, f(\xi_x)$ tends to a finite limit; moreover $\lim_{x\to+\infty} w(x)/w(\xi_x) = 1$ since

$$\frac{1 + (x - b_n(x))^p}{1 + x^p} \le \frac{w(x)}{w(\xi_x)} \le \frac{1 + (x + b_n(x))^p}{1 + x^p} \ .$$

Now, a simple induction argument yields the existence of a finite limit of $S_{n,b_n} f(x)$ at the point $+\infty$.

A similar argument can be applied on a neighborhood of $-\infty$ and hence the property is completely established. □

As a consequence, we may now consider the operators $L_{w,n} : C_w(\overline{\mathbb{R}}) \to C_w(\overline{\mathbb{R}})$ defined by

$$L_{w,n} f(x) := S_{n,b_n} f(x) \ , \qquad f \in C_w(\overline{\mathbb{R}}) \ , \quad x \in \mathbb{R} \ . \tag{3.3.6}$$

Observe that the operators $L_{w,n}$ are equibounded and satisfy $\|L_{w,n}\| \le C_w$ for every $n \ge 1$.

Proposition 3.3.3 *For every $f \in C_w(\overline{\mathbb{R}})$, $k \ge 1$ and $x \in \mathbb{R}$,*

$$\|S_{k,b_n}(f) - f\|_w \le \|S_{1,b_n}(f) - f\|_w \sum_{i=0}^{k-1} \|\omega_n\|^i \ . \tag{3.3.7}$$

PROOF. We argue by induction on the integer $k \geq 1$. If $k = 1$ then (3.3.7) is obviously true. Now, assume that (3.3.7) holds for $k \geq 1$. We have

$$\begin{aligned}
&|w(x)(S_{k+1,b_n} f(x) - f(x))| \\
&\quad \leq |w(x) S_{1,b_n}(S_{k,b_n} f - f)(x)| + |w(x)(S_{1,b_n} f(x) - f(x))| \\
&\quad \leq \|\omega_n\| \, \|S_{k,b_n}(f) - f\|_w + \|S_{1,b_n}(f) - f\|_w \\
&\quad \leq \|\omega_n\| \sum_{i=0}^{k-1} \|\omega_n\|^i \|S_{1,b_n}(f) - f\|_w + \|S_{1,b_n}(f) - f\|_w \\
&\quad \leq \sum_{i=0}^{k} \|\omega_n\|^i \|S_{1,b_n}(f) - f\|_w
\end{aligned}$$

and this completes the induction argument. □

In the sequel we need to introduce the space

$$C_w^2(\overline{\mathbb{R}}) := \{ f \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid f'' \in C_w(\overline{\mathbb{R}}) \} .$$

Proposition 3.3.4 *For every $f \in C_w^2(\overline{\mathbb{R}})$ and for every $x \in \mathbb{R}$*

$$\|S_{1,b_n}(f) - f\|_w \leq C \frac{\|a^2\| \|f''\|_w}{n^2} , \tag{3.3.8}$$

where $C > 0$ is a suitable constant depending on a and the weight w.

PROOF. Let $f \in C_w^2(\overline{\mathbb{R}})$; for every $x, t \in \mathbb{R}$, we can write

$$f(t) - f(x) = f'(x)\,(t - x) + \frac{1}{2} f''(\xi_t)(t - x)^2$$

with ξ_t in the interval with endpoints t and x.

Then

$$\begin{aligned}
S_{1,b_n} f(x) - f(x) &= \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} (f(t) - f(x)) dt \\
&= f'(x) \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} (t - x) dt + \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{2} f''(\xi_t)(t - x)^2 dt \\
&= \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{2} f''(\xi_t)(t - x)^2 dt
\end{aligned}$$

and

$$\begin{aligned}
|S_{1,b_n(x)} f(x) - f(x)| &\leq \frac{1}{2} \sup_{t \in [x-b_n(x), x+b_n(x)]} |f''(t)| \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} (t - x)^2 dt \\
&= \frac{b_n(x)^2}{6} \sup_{t \in [x-b_n(x), x+b_n(x)]} |f''(t)| \ .
\end{aligned}$$

Observe that the functions $w\,f''$ and a are bounded and consequently, since

$$\sup_{t\in[x-b_n(x),x+b_n(x)]} \frac{|w(x)|}{|w(t)|} \leq \frac{1+(|x|+\|a\|)^p}{1+|x|^p}\ ,$$

the function $\sup_{t\in[x-b_n(x),x+b_n(x)]} \frac{|w(x)|}{|w(t)|}$ is bounded as well. Hence, we can write

$$\begin{aligned}
&|S_{1,b_n}f(x)-f(x)|_w \leq \frac{1}{n^2}\frac{a(x)^2}{6}w(x) \sup_{t\in[x-b_n(x),x+b_n(x)]} |f''(t)| \\
&\leq \frac{1}{n^2}\frac{a(x)^2}{6} \sup_{t\in[x-b_n(x),x+b_n(x)]} \frac{|w(x)|}{|w(t)|} \sup_{t\in[x-b_n(x),x+b_n(x)]} |w(t)f''(t)| \\
&\leq C\frac{\|a^2\|\|f''\|_w}{6n^2}\ ,
\end{aligned}$$

where $C := \sup_{x\in\mathbb{R}} \frac{1+(|x|+\|a\|)^p}{1+|x|^p}$. □

Proposition 3.3.5 *For every $f \in C_w^2(\overline{\mathbb{R}})$ we have*

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\|_w \leq C\frac{\|a^2\|\|f''\|_w}{n};,$$

where $C > 0$ is a suitable constant depending on a and the weight w.

Proof. We can apply Propositions 3.3.3, 3.3.4 and 3.3.1 and obtain

$$\begin{aligned}
\left\| \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\|_w &\le \frac{1}{n}\sum_{k=0}^{n-1} \|S_{k,b_n(x)}(f) - f\|_w \\
&\le \frac{1}{n}\|S_{1,b_n}(f) - f\|_w \sum_{k=0}^{n-1}\sum_{i=0}^{k-1} \|\omega_n\|^i \\
&\le C\frac{\|a^2\|\|f''\|_w}{6n^2}\frac{1}{n}\sum_{k=0}^{n-1}\sum_{i=0}^{k-1}(1 + C_\omega\frac{1}{n^2})^i \\
&= C\frac{\|a^2\|\|f''\|_w}{6C_\omega}\left(\frac{n}{C_\omega}\left(\left(1+\frac{C_\omega}{n^2}\right)^n - 1\right) - 1\right) \\
&= C\frac{\|a^2\|\|f''\|_w}{6C_\omega}\left(\frac{n}{C_\omega}\left(\sum_{k=0}^{n}\binom{n}{k}\left(\frac{C_\omega}{n^2}\right)^k - 1\right) - 1\right) \\
&= C\frac{\|a^2\|\|f''\|_w}{6C_\omega} \\
&\quad\times\left(\frac{n}{C_\omega}\left(1 + \frac{C_\omega}{n} + \frac{n-1}{2}\frac{C_\omega^2}{n^3} + \sum_{k=3}^{n}\binom{n}{k}\left(\frac{C_\omega}{n^2}\right)^k - 1\right) - 1\right) \\
&= C\frac{\|a^2\|\|f''\|_w}{6C_\omega}\left(\frac{n-1}{2}\frac{C_\omega}{n^2} + \frac{n}{C_\omega}\sum_{k=3}^{n}\binom{n}{k}\left(\frac{C_\omega}{n^2}\right)^k\right) \\
&\le C\frac{\|a^2\|\|f''\|_w}{6}\left(\frac{1}{2n} + \sum_{k=3}^{n}\binom{n}{k}\frac{C_\omega^{k-2}}{n^{2k-1}}\right) ,
\end{aligned}$$

we have $\sum_{k=3}^{n}\binom{n}{k}\frac{C_\omega^{k-2}}{n^{2k-1}} \le \frac{C_1}{2n}$, where C_1 depend on the weight w, and the proof is complete.
□

If condition (3.1.12) on a holds we can establish some quantitative results similar to the unweighted case.

Proposition 3.3.6 *If condition* (3.1.12) *holds, then we have the following properties:*

1. *For every $f \in D(A_w)$ we have*

$$\|S_{1,b_n}(f) - f\|_w \le C\frac{\|Af\|_w}{n^2} ,$$

where $C > 0$ is a suitable constant depending on a and the weight w.

2. *For every $f \in D(A_w)$ we have*

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} S_{k,b_n}(f) - f \right\|_w \le C\frac{\|Af\|_w}{n} ,$$

where $C > 0$ is a suitable constant depending on a and the weight w.

At this point, we can state the following main results.

Theorem 3.3.7 *For every $f \in C_w(\overline{\mathbb{R}})$, we have*

$$\lim_{n\to+\infty} L_{w,n}(f) = f$$

with respect to the norm $\|\cdot\|_w$.

Proof. Since $C_w^2(\overline{\mathbb{R}})$ is dense in $C_w(\overline{\mathbb{R}})$ we can apply Propositions 3.3.3, 3.3.4 and 3.3.1 and obtain

$$\begin{aligned}
\|L_{w,n}(f) - f\|_w &\leq \|S_{1,b_n}(f) - f\|_w \sum_{i=0}^{n} \|\omega_n\|^i \\
&\leq C\frac{\|a^2\|\|f''\|_w}{6n^2} \sum_{i=0}^{n-1} \|\omega_n\|^i \\
&\leq C\frac{\|a^2\|\|f''\|_w}{6n^2} \sum_{i=0}^{n-1} \left(1 + \frac{C_\omega}{n^2}\right)^i \\
&= C\frac{\|a^2\|\|f''\|_w}{6} \frac{1}{C_\omega} \left(\left(1 + \frac{C_\omega}{n^2}\right)^n - 1\right) .
\end{aligned}$$

Thus, it is enough to take the limit as $n \to +\infty$ and the proof is complete.
□

Theorem 3.3.8 (Weighted Voronovskaja-type formula)
For every $f \in C_w^2(\overline{\mathbb{R}})$ we have

$$\lim_{n\to\infty} \left\| n\left(L_{w,n}(f) - f\right) - \frac{a^2}{6} f'' \right\|_w = 0 .$$

Proof. Let $f \in C_w^2(\overline{\mathbb{R}})$; we apply Theorem 2.2.2, taking $h_n = \frac{1}{n}$. The operators $\mathcal{A}_n f(x)$ become $\frac{n}{2} L_n((\mathrm{id} - x)^2)(x) f''(x) = \frac{1}{6n}\sum_{k=0}^{n-1} S_{k,b_n}(a^2)(x) f''(x)$ which converge with respect the weighted norm to $\frac{1}{6}a^2(x)f''(x) = Af(x)$ taking into account Proposition 3.1.10 and since $f \in C_w^2(\overline{\mathbb{R}})$. On the other hand $L_n\mathbf{1} - \mathbf{1} = 0$ and from Lemma 3.1.11 $nL_n((\mathrm{id} - x)^4)$ converges to zero even with respect the weighted norm. Finally since $f \in C_w^2(\overline{\mathbb{R}})$, the seconorder derivative is bounded uniformly continuous with respect the weighted norm. □

In order to obtain a quantitative estimate of the Voronovskaja'a formula we need to consider the class of functions

$$C_w^{2,\alpha}(\overline{\mathbb{R}}) := \{f \in C_w^2(\overline{\mathbb{R}}) \mid wf'' \in C^\alpha(\mathbb{R})\} .$$

Theorem 3.3.9 (Weighted quantitative Voronovskaja-type formula)

If $a^2 \in C^2(\overline{\mathbb{R}})$ *for every* $f \in C_w^{2,\alpha}(\overline{\mathbb{R}})$ *we have*

$$\left\| n\left(L_{w,n}(f) - f\right) - \frac{a^2}{6} f'' \right\|_w \le C_a \frac{M_f}{n^{\alpha/2}} ,$$

where M_f *is the seminorm defined by*

$$M_f = L_{wf''} + \|f''\|_w \tag{3.3.9}$$

and

$$C_a := \|a^2\| \max\left\{ \frac{4}{3}, \frac{\|(a^2)''\|}{72\sqrt{n}} \right\} . \tag{3.3.10}$$

Proof. Let $f \in C_w^{2,\alpha}(\overline{\mathbb{R}})$; for every $x, t \in \mathbb{R}$, we can write

$$f(t) - f(x) = f'(x)\,(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \eta(t,x)(t-x)^2 \tag{3.3.11}$$

where $\eta : \mathbb{R}^2 \to \mathbb{R}$ satisfies $\eta(t,x) = \frac{1}{2}\left(f''(\xi(t)) - f''(x)\right)$ for some $\xi(t) \in [x,t]$.
Then

$$\begin{aligned}
n\left(L_{w,n}f(x) - f(x)\right) &= n\, S_{n,b_n}(f - f(x)\cdot \mathbf{1})(x) \\
&= n\, S_{n,b_n}\left(f'(x)\,(\mathrm{id}-x) + \frac{1}{2} f''(x)(\mathrm{id}-x)^2 + \eta(\mathrm{id},x)(\mathrm{id}-x)^2 \right)(x) \\
&= n\, f'(x)\, S_{n,b_n}(\mathrm{id}-x)(x) + n\frac{1}{2} f''(x)\, S_{n,b_n}\left((\mathrm{id}-x)^2\right)(x) \\
&\quad + n\, S_{n,b_n}\left(\eta(\mathrm{id},x)(\mathrm{id}-x)^2\right)(x) \\
&= n\frac{1}{2} f''(x)\, S_{n,b_n}\left((\mathrm{id}-x)^2\right)(x) + n\, S_{n,b_n}\left(\eta(\mathrm{id},x)(\mathrm{id}-x)^2\right)(x) ,
\end{aligned}$$

and moreover

$$\begin{aligned}
&|w(x)(n\left(L_{w,n}f(x) - f(x)\right) - Af(x))| \\
&\quad \le \left| w(x)\left(n\frac{1}{2} f''(x)\, S_{n,b_n}\left((\mathrm{id}-x)^2\right)(x) - Af(x) \right) \right| \\
&\qquad + \left| w(x) n\, S_{n,b_n}\left(\eta(\mathrm{id},x)(\mathrm{id}-x)^2\right)(x) \right| .
\end{aligned}$$

We can write

$$\begin{aligned}
w(x)\eta(t,x) &= w(x)(f''(\xi) - f''(x)) \\
&= w(\xi)f''(\xi) - w(x)f''(x) + w(x)f''(\xi) - w(\xi)f''(\xi) \\
&= \left(w(\xi)f''(\xi) - w(x)f''(x)\right) + f''(\xi)w(\xi)\left(\frac{w(x)}{w(\xi)} - 1\right) ,
\end{aligned}$$

and then

$$
\begin{aligned}
&|w(x)(n\,(L_{w,n}f(x)-f(x))-Af(x))| && (3.3.12)\\
&\quad\le \left|w(x)\left(n\frac{1}{2}f''(x)\,S_{n,b_n}\left((\mathrm{id}-x)^2\right)(x)-Af(x)\right)\right| && (3.3.13)\\
&\qquad+\left|n\,S_{n,b_n}\left(\left((w\circ\xi)(f''\circ\xi)-w(x)f''(x)\right)(\mathrm{id}-x)^2\right)(x)\right|\\
&\qquad+\left|n\,S_{n,b_n}\left((f''\circ\xi)(w\circ\xi)\left(\frac{w(x)}{w\circ\xi}-1\right)(\mathrm{id}-x)^2\right)(x)\right|\ .
\end{aligned}
$$

As regards the first addend, since $a^2\in C^2(\overline{\mathbb{R}})$, from Proposition 3.1.8 we have

$$
\begin{aligned}
&\left|w(x)f''(x)\left(\frac{1}{2}n\,S_{n,b_n}((\mathrm{id}-x)^2)(x)-\frac{1}{6}a^2(x)\right)\right|\\
&\quad=\left|\frac{w(x)f''(x)}{6}\right|\left|\frac{1}{n}\sum_{k=0}^{n-1}S_{k,b_n}(a^2)(x)-a^2(x)\right|\\
&\quad\le\frac{\|f''\|_w}{6}\frac{\|a^2\|\|(a^2)''\|}{12n}\\
&\quad=\frac{\|f''\|_w\|a^2\|\|(a^2)''\|}{72n}\ .
\end{aligned}
$$

As regards the second term in (3.3.12) since $f\in C^{2,\alpha}(\overline{\mathbb{R}})$ and $|w(\xi)f''(\xi)-w(x)f''(x)|\le L_{wf''}|\xi-x|^\alpha\le L_{wf''}|t-x|^\alpha$, we have

$$
\begin{aligned}
&\left|n\,S_{n,b_n}\left(\left((w\circ\xi)(f''\circ\xi)-w(x)f''(x)\right)(\mathrm{id}-x)^2\right)(x)\right|\\
&\quad\le L_{wf''}nS_{n,b_n}\left(|\mathrm{id}-x|^{2+\alpha}\right)(x)\ .
\end{aligned}
$$

For every $\delta>0$ we have

$$
|t-x|^{2+\alpha}\le\delta^\alpha\left(\delta^2+\frac{(t-x)^4}{\delta^2}\right)\ ;
$$

choosing $n\delta^2=\|a^2\|$ and taking into account Lemma 3.1.11 we obtain

$$
\begin{aligned}
&\left|w(x)n\,S_{n,b_n}\left(\eta(\mathrm{id},x)(\mathrm{id}-x)^2\right)(x)\right|\le\delta^\alpha\left(n\delta^2+\frac{n}{\delta^2}S_{n,b_n}\left((\mathrm{id}-x)^4\right)(x)\right)\\
&\le\frac{L_{wf''}}{n^{\alpha/2}}(\|a^2\|+\frac{n^2}{\|a^2\|}S_{n,b_n}\left((\mathrm{id}-x)^4\right)(x))\\
&\le\frac{4}{3}\frac{L_{wf''}}{n^{\alpha/2}}\|a^2\|\ .
\end{aligned}
$$

As regards the last term in (3.3.12) using the Cauchy-Schwartz inequality

$$
\begin{aligned}
&\left|n\,S_{n,b_n}\left((f''\circ\xi)(w\circ\xi)\left(\frac{w(x)}{w\circ\xi}-1\right)(\mathrm{id}-x)^2\right)(x)\right|\\
&\quad\le n\|f''\|_w\left(S_{n,b_n}\left(\left(\frac{w(x)}{w\circ\xi}-1\right)^2\right)(x)\right)^{1/2}\left(S_{n,b_n}(\mathrm{id}-x)^4)(x)\right)^{1/2}\ .
\end{aligned}
$$

In order to estimate the term $S_{n,b_n}\left(\left(\frac{w(x)}{w\circ\xi}-1\right)^2\right)(x)$ we observe that $S_{1,b_n}\left(\left(\frac{w(x)}{w\circ\xi}-1\right)^2\right)(x)=\left(\frac{w(x)}{w(\xi(t_0))}-1\right)^2$ for some $t_0\in[x-b_n(x),x+b_n(x)]$; since $\xi_0:=\xi(t_0)\in[x,t_0]$, it follows $\xi_0\in[x-b_n(x),x+b_n(x)]$. Consequently we have

$$\left(\frac{w(x)}{w(\xi_0)}-1\right)^2=\left(\frac{1+|\xi_0|^p}{1+|x|^p}-1\right)^2\le\left(\frac{1+|x+\|\alpha\|/n|^p}{1+|x|^p}-1\right)^2$$
$$=\left(\frac{|x+\|\alpha\|/n|^p-|x|^p}{1+|x|^p}\right)^2\le\frac{C}{n^2}$$

in the case $x\ge 0$; if $x\le 0$ we can argue similarly. From Proposition 3.1.2, since $\frac{w(x)}{w\circ\xi(x)}-1=0$ we have

$$\left\|S_{n,b_n}\left(\frac{w(x)}{w\circ\xi}-1\right)^2\right\|\le\frac{C}{n}\,,$$

and from Lemma 3.1.11 the last term in (3.3.12) can be estimated as follows

$$\left|n\,S_{n,b_n}\left((f''\circ\xi)(w\circ\xi)\left(\frac{w(x)}{w\circ\xi}-1\right)(\mathrm{id}-x)^2\right)(x)\right|\le\|f''\|_w\frac{C\|a^2\|}{\sqrt{3}}\frac{1}{\sqrt{n}}\,.$$

Finally we have

$$\begin{aligned}&|n(L_{w,n}(f)(x)-f(x)-A_wf''(x)|\\&\quad\le\frac{L_{wf''}}{n^{\alpha/2}}\frac{4\|a^2\|}{3}+\frac{\|f''\|_w\|a^2\|\|(a^2)''\|}{72n}+\|f''\|_w\frac{C\|a^2\|}{\sqrt{3}}\frac{1}{\sqrt{n}}\\&\quad\le C_a\frac{M_f}{n^{\alpha/2}}\,,\end{aligned}$$

where M_f is the seminorm defined by (3.3.9) and C_a is a constant defined by (3.3.10). □

The following lemma will allow us to state the core property of $C^2_w(\overline{\mathbb{R}})$ for $(A_w,D(A_w))$. Under additional assumptions and in different settings, the core property has been considered also in [59, 13, 14, 8].

Lemma 3.3.10 *For every $u\in C^2(\mathbb{R})$ and $h\ge 0$, the following statements are equivalent:*

a) $(wu)''(x)=O(x^h)$ *(respectively,* $(wu)''(x)=o(x^h)$*) as* $x\to\pm\infty$*;*

b) $wu''(x)=O(x^h)$ *(respectively,* $wu''(x)=o(x^h)$*) as* $x\to\pm\infty$*;*

c) $(wu')'(x)=O(x^h)$ *(respectively,* $(wu')'(x)=o(x^h)$*) as* $x\to\pm\infty$*.*

PROOF. Assume that $(wu)''(x) = O(x^h)$ as $x \to \pm\infty$. Then, for every $x \in \mathbb{R}$, $x \neq 0$,

$$\begin{aligned}
(wu'')(x) &= (wu)''(x) + 2(wu)'(x)w(x)\,(1/w)'(x) + (wu)(x)\,w(x)\,(1/w)''(x)\\
&= (wu)''(x) + 2(wu)'(x)\,\frac{px}{1/|x|^{p-2}+x^2}\\
&\quad +(wu)(x)\,p(p-1)\frac{1}{1/|x|^{p-2}+x^2}\\
&\approx (wu)''(x) + (wu)'(x)\,\frac{x}{1+x^2} + (wu)(x)\,\frac{1}{1+x^2}\,.
\end{aligned}$$

Condition $(wu)''(x) = O(x^h)$ implies $(wu)'(x) = O(x^{h+1})$ which in turn yields $(wu)(x) = O(x^{h+2})$; hence $(wu'')(x) = O(x^h)$ as $x \to \pm\infty$.

Now, let $(wu'')(x) = O(x^h)$ as $x \to \pm\infty$. We have

$$\begin{aligned}
(wu)''(x) &= (wu'')(x) + 2(wu')(x)\frac{w'(x)}{w(x)} + (wu)(x)\frac{w''(x)}{w(x)}\\
&= (wu'')(x) + 2(wu')(x)\frac{w'(x)}{w(x)}\\
&\quad +(wu)(x)\left(2\left(\frac{w'(x)}{w(x)}\right)^2 - w(x)\left(\frac{1}{w(x)}\right)''\right)\\
&= (wu'')(x) - 2(wu')(x)\,\frac{px}{1/|x|^{p-2}+x^2}\\
&\quad +(wu)(x)\left(2\left(\frac{px}{1/|x|^{p-2}+x^2}\right)^2 - p(p-1)\frac{1}{1/|x|^{p-2}+x^2}\right)\\
&\approx (wu'')(x) + (wu')(x)\frac{x}{1+x^2} + (wu)(x)\,\frac{1}{1+x^2}\,;
\end{aligned}$$

as before, from $(wu'')(x) = O(x^h)$, we obtain $u''(x) = O(x^{h+p})$ and in turn $u'(x) = O(x^{h+p+1})$ and $u(x) = O(x^{h+p+2})$; hence $(wu)''(x) = O(x^h)$.

The equivalence between a) and c) can be proved similarly. □

Remark 3.3.11 We observe that the Ventcel domain of A_w

$$D_V(A_w) := \{u \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid \lim_{x\to\pm\infty} w(x)A_w u(x) = 0\}$$

coincides with the maximal domain already defined.

Indeed let $u \in D(A_w)$ and by contradiction assume that $u \notin D_V(A_w)$, for example $\lim\limits_{x\to+\infty} w(x)A_w u(x) = \ell \neq 0$. Then

$$\lim_{x\to+\infty} \frac{1}{1+x^p}\,\frac{1}{6x^2}\,a^2(x)x^2u''(x) = \ell$$

and therefore $\lim_{x\to+\infty} \frac{1}{x^{p+2}} u''(x) \neq 0$; consequently $\lim_{x\to+\infty} \frac{1}{x^{p+4}} u(x) \neq 0$, contradicting the condition $w\,u \in C(\overline{\mathbb{R}})$. The same reasoning holds at the point $-\infty$. □

Proposition 3.3.12 *The space $C^2_w(\overline{\mathbb{R}})$ is a core for $(A_w, D(A_w))$.*

Proof. We consider the canonical isometry $\Gamma : C_w(\overline{\mathbb{R}}) \to C(\overline{\mathbb{R}})$ introduced in [8] and defined by setting, for every $f \in C_w(\overline{\mathbb{R}})$,

$$\Gamma(f) = f \cdot w\ .$$

Denote by $(A, D(A))$ the differential operator obtained by (3.3.2) with $w = 1$ on the domain

$$D(A) := \left\{ u \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid a^2\, u'' \in C(\overline{\mathbb{R}}) \right\}\ .$$

From [33, Proposition 2.10] we know that $C^2(\overline{\mathbb{R}})$ is a core for $(A, D(A))$.

Now, we show that $\Gamma(D(A_w)) = D(A)$. Let $v = \Gamma u \in D(A)$; we have $u \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R})$ and further

$$\begin{aligned}
|\Gamma(Au)(x)| &= \left| \left(w \frac{a^2}{6} u'' \right)(x) \right| \\
&\leq \frac{a(x)^2}{6} |v''(x)| + \frac{C_1}{1+|x|} \frac{a(x)^2}{6} |v'(x)| + \frac{a(x)^2}{6} \frac{C_2}{1+x^2} |v(x)| \\
&\leq |Av(x)| + \frac{C_1}{1+|x|} \frac{a(x)^2}{6} |v'(x)| + \frac{C_3}{1+x^2} |v(x)|\ .
\end{aligned}$$

Using Taylor's formula we can write $v'(x) = \frac{v(x+\delta)-v(x)}{\delta} - \frac{v''(\xi)}{2}\delta$ for some $\xi \in]x, x+\delta[$, where δ is given by (3.3.3), and consequently

$$\begin{aligned}
\left| \frac{1}{1+|x|} \frac{a(x)^2}{6} v'(x) \right| &= \left| \frac{1}{1+|x|} \frac{a(x)^2}{6} \frac{v(x+\delta)-v(x)}{\delta} - \frac{1}{1+|x|} \frac{a(x)^2}{6} v''(\xi) \frac{\delta}{2} \right| \\
&\leq \frac{1}{1+|x|} \frac{1}{3\delta} \|a^2\| \|v\| + \left| \frac{1}{1+|x|} \frac{a(x)^2}{a(\xi)^2} \frac{a(\xi)^2}{6} v''(\xi) \frac{\delta}{2} \right| \\
&\leq \frac{C_4}{1+|x|} \|v\| + \frac{\delta}{2} M \left| \frac{a(\xi)^2}{6} v''(\xi) \right| \\
&= \frac{C_4}{1+|x|} \|v\| + \frac{\delta}{2} M |Av(\xi)|
\end{aligned}$$

which implies

$$|\Gamma(Au)(x)| \leq |Av(x)| + \frac{C_4}{1+|x|} \|v\| + \frac{\delta}{2} M |Av(\xi)| + \frac{C_3}{1+x^2} \|v\|. \tag{3.3.14}$$

Since $\|v\| < \infty$ and $\lim_{x\to\pm\infty} Av(x) = 0$ (see Remark 3.3.11), we have $\lim_{x\to\pm\infty} \Gamma(Au)(x) = 0$ and $u \in D(A_w)$.

Conversely let $u \in D(A_w)$, we have $\Gamma(u) \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R})$ and further

$$\begin{aligned}
|A(\Gamma u)(x)| &= \left|\frac{a(x)^2}{6}(wu)''(x)\right| \\
&\leq \left|\frac{a(x)^2}{6}(wu'')(x)\right| + \frac{C_1}{1+|x|}\frac{a(x)^2}{6}|(wu')(x)| \\
&\quad + C_2\left|\frac{a(x)^2}{6}(wu)(x)\,\frac{1}{1+x^2}\right| \\
&\leq |\Gamma(Au)(x)| + \frac{C_1}{1+|x|}\frac{a(x)^2}{6}|(wu')(x)| + \frac{C_3}{1+x^2}|\Gamma u(x)|\ .
\end{aligned}$$

Again from Taylor's formula we have $u'(x) = \frac{u(x+\delta)-u(x)}{\delta} - \frac{u''(\xi)}{2}\,\delta$ where $\xi \in]x, x+\delta[$ and δ is given by (3.3.3), and consequently

$$\begin{aligned}
\left|\frac{1}{1+|x|}\frac{a(x)^2}{6}(wu')(x)\right| &\leq \left|\frac{1}{1+|x|}\frac{a(x)^2}{6}w(x)\frac{u(x+\delta)-u(x)}{\delta}\right| \\
&\quad + \left|\frac{1}{1+|x|}\frac{a(x)^2}{6}w(x)u''(\xi)\frac{\delta}{2}\right| \\
&\leq \frac{\|a^2\|}{6\delta}\frac{1}{1+|x|}\frac{w(x)}{w(x+\delta)}|\Gamma u(x+\delta)| \\
&\quad + \frac{\|a^2\|}{6\delta}\frac{1}{1+|x|}|\Gamma u(x)| \\
&\quad + \left|\frac{1}{1+|x|}\frac{a(x)^2}{a(\xi)^2}\,\frac{w(x)}{w(\xi)}\,w(\xi)\,\frac{a(\xi)^2}{6}\,u''(\xi)\frac{\delta}{2}\right| \\
&\leq \frac{C_4}{1+|x|}\|\Gamma(u)\| + C_5|\Gamma(Au)(\xi)|\ .
\end{aligned}$$

Then we have

$$\begin{aligned}
|A(\Gamma u)(x)| &\leq |\Gamma(Au)(x)| + \frac{C_4}{1+|x|}\|\Gamma(u)\| \\
&\quad + C_5|\Gamma(Au)(\xi)| + \frac{C_3}{1+x^2}\|\Gamma u\|\ ;
\end{aligned} \tag{3.3.15}$$

and taking the limit as $x \to \pm\infty$ we obtain $\lim_{x\to\pm\infty} A(\Gamma u)(x) = 0$ (see Remark 3.3.11) and consequently $\lim_{x\to\pm\infty} \Gamma u(x) = 0$.

Now, we observe that $\Gamma(C^2_w(\overline{\mathbb{R}})) = \Gamma(\{u \in C_w(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid w\,u'' \in C(\overline{\mathbb{R}})\})$ and $C^2(\overline{\mathbb{R}}) = \{v \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid v'' \in C(\overline{\mathbb{R}})\}$ and taking into account that $wu'' \in C(\overline{\mathbb{R}})$ if and only if $(wu)'' = (\Gamma u)'' \in C(\overline{\mathbb{R}})$, we deduce the equality

$$\Gamma(C^2_w(\overline{\mathbb{R}})) = C^2(\overline{\mathbb{R}})\ .$$

Finally from (3.3.14), it immediately follows that if $\Gamma(u) \in D(A)$ then

$$\|\Gamma(Au)\| \leq C\left(\|A(\Gamma u)\| + \|\Gamma(u)\|\right) \tag{3.3.16}$$

for some constant $C > 0$.

Taking into account that $C^2(\overline{\mathbb{R}})$ is a core for $(A, D(A))$, the proof is complete. □

Finally, we discuss the representation of the semigroup generated by A by means of iterates of weighted Steklov operators.

Proposition 3.3.13 *The operator $(A_w, D(A_w))$ generates a strongly continuous semigroup on $C_w(\overline{\mathbb{R}})$.*

Proof. Let $\lambda > 0$ and consider the problem

$$\lambda u - A(u) = f \qquad f \in C_w(\overline{\mathbb{R}})\ , \tag{3.3.17}$$

which is equivalent to

$$\lambda v - B(v) = g\ , \qquad g \in C(\overline{\mathbb{R}})\ , \tag{3.3.18}$$

where $v = uw$, $g = wf$, and

$$B(v) = wA\left(\frac{v}{w}\right)\ .$$

From $\left(\frac{v}{w}\right)' = \frac{v'}{w} - \frac{w'}{w^2}v$ and $\left(\frac{v}{w}\right)'' = \frac{v''}{w} - 2\frac{w'}{w^2}v' - \left(\frac{w'}{w^2}\right)' v$, we obtain

$$B(v) = \frac{a^2}{6}v'' - 2\frac{a^2}{6}\frac{w'}{w}v' - \frac{a^2}{6}w\left(\frac{w'}{w^2}\right)' v =: \alpha_B v'' + \beta_B v' + \gamma_B v\ .$$

Using Feller's classification of the endpoints (see [48]), we prove that B generates a C_0-semigroup on its maximal domain $D(B) := \{u \in C(\overline{\mathbb{R}}) \mid Bu \in C(\overline{\mathbb{R}})\}$. In our case the functions W, R and Q in [48, p. 391–393]) are given by

$$W_B(x) = \exp\left(-\int_{x_0}^{x} \frac{\beta_B}{\alpha_B}\right) = \exp\left(\int_{x_0}^{x} 2\frac{w'}{w}\right) = \frac{w(x)^2}{w(x_0)^2}\ ,$$

$$R_B(x) = W_B(x)\int_{x_0}^{x} \frac{1}{\alpha_B W_B} = w(x)^2 \int_{x_0}^{x} \frac{6}{a^2 w^2}$$

and

$$Q_B(x) = \frac{1}{\alpha_B(x) W_B(x)} \int_{x_0}^{x} W_B = \frac{6}{a(x)^2 w(x)^2} \int_{x_0}^{x} w^2\ .$$

Since $w(x) = \frac{1}{1+|x|^p}$, it can be readily seen that

$$Q_B(x) = \frac{6(1+|x|^p)^2}{a(x)^2} \int_0^x \frac{1}{(1+|t|^p)^2}\, dt$$

and hence Q_B is not integrable on $[0,+\infty[$ and on $]-\infty,0]$; moreover,

$$R_B(x) = \frac{6}{(1+|x|^p)^2} \int_0^x \frac{(1+|t|^p)^2}{a(t)^2}\,dt$$

and from the estimate

$$R_B(x) \geq \frac{C}{1+x^{2p}} \int_0^x t^{2p+2}dt = C_1\,\frac{x^{2p+3}}{1+x^{2p}}\ ,$$

it follows that R_B is not integrable on $[0,+\infty[$ and on $]-\infty,0]$ as well.

So the endpoint $-\infty$ and ∞ are both natural for the operator B, and consequently the operator $Bv-\gamma_B v$ generates a strongly continuous semigroup on its maximal domain on $C(\overline{\mathbb{R}})$. Observe also that the maximal domain coincides with the Ventcel domain in the case of natural endpoints, as an immediate consequence of the classical generation results by Clément and Timmermans [44] and Timmermans [69] (see Chapter II).

Taking into account that

$$\gamma_B(x) = \frac{a(x)^2}{6}\,\frac{p(p-1)|x|^{p-2}}{1+|x|^p}$$

we conclude that γ_B is bounded and hence $(B, D(B))$ generates a strongly continuous semigroup on $C(\overline{\mathbb{R}})$.

Finally, we observe that if $v \in C(\overline{\mathbb{R}})$ is a solution of (3.3.18), then $u := v/w \in C_w(\overline{\mathbb{R}})$ is a solution of (3.3.17). Moreover $u \in D(A)$ if and only if $u \cdot w \in D(B)$ since Γ is an isometry between $C_w(\overline{\mathbb{R}})$ and $C(\overline{\mathbb{R}})$, and from the generation property of $(B, D(B))$ we deduce that $(A_w, D(A_w)$ generates a C_0-semigroup on $C_w(\overline{\mathbb{R}})$. □

Finally we can state the following representation theorem.

Theorem 3.3.14 *The operator $(A_w, D(A_w))$ generates a positive C_0-semigroup $(T(t))_{t\geq 0}$ in $C_w(\overline{\mathbb{R}})$ satisfying $\|T(t)\| \leq e^{Ct}$ and, for every $t \geq 0$ and for every sequence $(k(n)_{n\geq 1})$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have*

$$T(t) = \lim_{n\to+\infty} L_n^{k(n)} \qquad \textit{strongly on } C(\overline{\mathbb{R}})\ ,$$

moreover if $a \in C^2(\overline{\mathbb{R}})$, for every $f \in C_w^{2,\alpha}(\mathbb{R})$ we have

$$\begin{aligned}\left\|T(t)u - L_n^{k(n)}u\right\|_w \leq & t\,\frac{C_\alpha M_f}{n^{\alpha/2}} + \left(\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\right) \\ & \times \left(\|Af\|_w + \frac{C_\alpha M_f}{n^{\alpha/2}}\right)\end{aligned} \qquad (3.3.19)$$

and choosing $k(n) = [nt]$

$$\left\|T(t)u - L_n^{[nt]}u\right\|_w \leq t\,\frac{C_\alpha M_f}{n^{\alpha/2}} + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\right)\left(\|Af\|_w + \frac{C_\alpha M_f}{n^{\alpha/2}}\right)\ . \qquad (3.3.20)$$

PROOF. From Proposition 3.3.13 we know that the operator $(A_w, D(A_w))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ and hence, for every $\lambda > 0$, the range $(\lambda - A_w)(D(A_w))$ coincides with $C_w(\overline{\mathbb{R}})$. Moreover, by Proposition 3.3.12, $C_w^2(\overline{\mathbb{R}})$ is a core for $(A_w, D(A_w))$ and therefore $(\lambda - A_w)(C_w^2(\overline{\mathbb{R}}))$ is dense in $C_w(\overline{\mathbb{R}})$. We observe that we have

$$\|L_{w,n}\| \leq \|\omega_n\|_\infty^n \leq (1 + \|\omega_n - 1\|)^n \leq \left(1 + \frac{C}{n^2}\right)^n$$

as a consequence of (3.3.5) and Proposition 3.3.1, and consequently

$$\|L_{w,n}^k\| \leq \left(\left(1 + \frac{C}{n^2}\right)^{n^2}\right)^{k/n} \leq e^{Ck/n} \,.$$

Hence we can apply Trotter's approximation theorem [70] and obtain that the closure of the operator arising from the Voronovskaja's formula (Theorem 3.3.8) generates a C_0-semigroup represented by (3.3.14). Finally, this closure coincides with $(A_w, D(A_w))$ since $C_w^2(\overline{\mathbb{R}})$ is a core by Proposition 3.3.12. The positivity of the semigroup is a consequence of the representation (3.3.14).

At this point we can apply Theorem 1.1.2. From Theorem 3.3.9 follows that the seminorms are given by $\psi_n(f) = C_a \frac{M_f}{n^{\alpha/2}}$ and $\varphi_n(f) = \|Af\|_w + C_a \frac{M_f}{n^{\alpha/2}}$, then taking into account that the growth bound of $(T(t))_{t\geq 0}$ is equal to 0 and every $T(t)$ is a linear contraction, i.e. $\omega = 0$ and $M = 1$, the estimates (3.3.19) and (3.3.20) follow directly from (1.1.10) and (1.1.11), and this completes the proof. $\square$

3.4 Steklov operators in weighted spaces on $[0,1]$

In this section we consider the weighted space of continuous functions on the interval $[0,1]$. We fix $a \in C([0,1])$ satisfying $a(0)=a(1)=0$ and require that it is differentiable at 0 and 1, i.e.

$$\lim_{x\to 0,1} \frac{a(x)}{x(1-x)} \in \mathbb{R}\ . \tag{3.4.1}$$

We consider the weight function

$$w(x) := x^p(1-x)^q\ , \qquad x\in[0,1]\ , \qquad p,q\geq 2\ . \tag{3.4.2}$$

In the sequel, we shall set $b_n(x) := a(x)/n$. From (3.4.1), it follows $[x-b_n(x), x+b_n(x)] \subset [0,1]$ for every $x\in[0,1]$ and large enough n; hence, we can define the functions $\omega_n : [0,1]\to\mathbb{R}$ by setting

$$\omega_n(x) := \frac{w(x)}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{w(t)}\,dt\ , \qquad x\in]0,1[\ , \tag{3.4.3}$$

and

$$\omega_n(0) := \lim_{x\to 0^+} \omega_n(x)\ , \qquad \omega_n(1) := \lim_{x\to 1^-}\omega_n(x)\ . \tag{3.4.4}$$

Remark 3.4.1 Let $n\in\mathbb{N}$ be large enough; then the limits in (3.4.4) exist and are finite at the points 0 and 1.

Since the discussion is at all similar on neighborhoods of the endpoints 0 and 1, in the sequel we shall limit ourselves to consider only a neighborhood of 0, where the weight function can be taken of the form

$$w(x) := x^p\ .$$

We observe that, for every $x\in]0,1[$,

$$\begin{aligned}
\omega_n(x) &= x^p \frac{1}{2b_n(x)} \int_{x-b_n(x)}^{x+b_n(x)} \frac{1}{t^p}\,dt \qquad (3.4.5)\\
&= \frac{1}{1-p}\frac{1}{2b_n(x)}\left[\frac{1}{(x+b_n(x))^{p-1}} - \frac{1}{(x-b_n(x))^{p-1}}\right] x^p\\
&= \frac{1}{p-1}\frac{1}{2b_n(x)} \frac{(x+b_n(x))^{p-1}-(x-b_n(x))^{p-1}}{(x^2-b_n(x)^2)^{p-1}}\, x^p\\
&= \frac{1}{p-1}\frac{x^{2p-1}}{2b_n(x)\,x^{2p-2}} \frac{(1+b_n(x)/x)^{p-1}-(1-b_n(x)/x)^{p-1}}{(1-b_n(x)^2/x^2)^{p-1}}\\
&= \frac{x}{2(p-1)b_n(x)} \frac{(1+b_n(x)/x)^{p-1}-(1-b_n(x)/x)^{p-1}}{(1-b_n(x)^2/x^2)^{p-1}}\ .
\end{aligned}$$

Since a is differentiable at 0, we have

$$\lim_{x\to 0^+}\omega_n(x) \;=\; \begin{cases} 1\,, & a'(0)=0\,,\\ \dfrac{1}{2(p-1)}\,\dfrac{\left(1+\frac{\gamma}{n}\right)^{p-1}-\left(1-\frac{\gamma}{n}\right)^{p-1}}{\frac{\gamma}{n}\left(1-\frac{\gamma^2}{n^2}\right)^{p-1}}\,, & a'(0)\neq 0\,,\end{cases}$$

where $\gamma := a'(0)$. □

In the next Proposition 3.4.2 we estimate ω_n.

Proposition 3.4.2 *There exists a constant* $C>0$ *such that*

$$\|\omega_n-\mathbf{1}\|\le \frac{C}{n^2}\,.$$

PROOF. For n large enough, we have $0\le b_n(x)/x\le 1/2$ for every $x\in]0,1[$ and hence, using the Taylor's expansions of $(1\pm y)^{p-1}$ at 0, we get the existence of $\xi_x,\theta_x,\eta_x\in]0,1/2[$ such that

$$\begin{aligned}\left(1+\frac{b_n(x)}{x}\right)^{p-1} &= 1+(p-1)\frac{b_n(x)}{x}+\frac{1}{2}(p-1)(p-2)\,\frac{b_n(x)^2}{x^2}\\ &\quad+\frac{1}{6}(p-1)(p-2)(p-3)\,(1+\xi_x)^{p-4}\,\frac{b_n(x)^3}{x^3}\,,\\ \left(1-\frac{b_n(x)}{x}\right)^{p-1} &= 1-(p-1)\frac{b_n(x)}{x}+\frac{1}{2}(p-1)(p-2)\,\frac{b_n(x)^2}{x^2}\\ &\quad-\frac{1}{6}(p-1)(p-2)(p-3)\,(1-\theta_x)^{p-4}\,\frac{b_n(x)^3}{x^3}\,,\\ \left(1-\frac{b_n(x)^2}{x^2}\right)^{p-1} &= 1-(p-1)\,(1-\eta_x)^{p-2}\,\frac{b_n(x)^2}{x^2}\,.\end{aligned}$$

Consequently, using (3.4.5),

$$\begin{aligned}
\omega_n(x) - 1 &= \frac{1}{2(p-1)b_n(x)/x} \, \frac{(1+b_n(x)/x)^{p-1} - (1-b_n(x)/x)^{p-1}}{(1-b_n(x)^2/x^2)^{p-1}} \\
&\quad - \frac{(1-b_n(x)^2/x^2)^{p-1}}{(1-b_n(x)^2/x^2)^{p-1}} \\
&= \frac{1}{(1-b_n(x)^2/x^2)^{p-1}} \times \\
&\quad \times \Big(1 + \frac{1}{12}(p-2)(p-3)\,\big((1+\xi_x)^{p-4} + (1-\theta_x)^{p-4}\big)\,\frac{b_n(x)^2}{x^2} \\
&\quad -1 + (p-1)\,(1-\eta_x)^{p-2}\,\frac{b_n(x)^2}{x^2}\Big) \\
&= \frac{b_n(x)^2/x^2}{(1-b_n(x)^2/x^2)^{p-1}} \,\Big(\frac{1}{12}(p-2)(p-3) \times \\
&\quad \times \ \big((1+\xi_x)^{p-4} + (1-\theta_x)^{p-4}\big) + (p-1)\,(1-\eta_x)^{p-2}\Big) \\
&= \frac{1}{n^2}\,\frac{a(x)^2/x^2}{(1-a(x)^2/(n^2\,x^2))^{p-1}}\,\varphi(x)\,, \qquad (3.4.6)
\end{aligned}$$

where

$$\varphi(x) := \frac{1}{12}(p-2)(p-3)\,\big((1+\xi_x)^{p-4} + (1-\theta_x)^{p-4}\big) + (p-1)\,(1-\eta_x)^{p-2}\,.$$

Since φ is bounded as well as $a(x)/x$, we get the desired result. □

Now, we consider the weighted space

$$C_w([0,1]) := \{f \in C(]0,1[) \mid \exists \lim_{x\to 0,1} w(x)\,f(x) \in \mathbb{R}\}$$

endowed with the norm

$$\|f\|_w = \sup_{x\in[0,1]} |f(x)\,w(x)|\,, \qquad f \in C_w([0,1])\,.$$

As in the preceding section we define the operators $L_{w,n} : C_w([0,1]) \to C_w([0,1])$ by setting, for every $f \in C_w([0,1])$, $L_{w,n}(f) := S_{n,b_n}(f)$.

We also need to define the subspace

$$C_w^2([0,1]) := \{f \in C_w([0,1]) \cap C^2(]0,1[) \mid f'' \in C_w([0,1])\}\,.$$

If n is large enough, then for every $f \in C_w([0,1])$, the function $L_{w,n}(f)$ is well-defined by (3.4.1). Moreover we have,

$$\|L_{w,n}(f)\|_w = \|S_{n,b_n}(f)\|_w \le \|\omega_n\|^n \,\|f\|_w \le C_w\,\|f\|_w\,, \qquad (3.4.7)$$

and hence $L_{w,n}(f) \in C_\omega([0,1])$.

The proofs of the following properties are at all similar to the unbounded weighted case and therefore we shall omit them.

Proposition 3.4.3 *We have the following properties:*

1. *For every* $f \in C(0,1)$, $k \geq 1$ *and* $x \in \mathbb{R}$,

$$\|S_{k,b_n}(f) - f\|_w \leq \|S_{1,b_n}(f) - f\|_w \sum_{i=0}^{k} \|\omega_n\|^i \ . \tag{3.4.8}$$

2. *For every* $f \in C_w^2([0,1])$, *there exists a constant* $C_f > 0$, *depending on* f, *such that for every* $x \in (0,1)$

$$\|S_{1,b_n}(f) - f\|_w \leq \frac{C_f}{n^2} \ . \tag{3.4.9}$$

3. *For every* $f \in C_w([0,1])$ *we have*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} S_{k,b_n}(f) = f$$

with respect to the norm $\|\cdot\|_w$.

As a consequence, also in the present setting we can state the following results. The proof is omitted since it is at all similar to the unweighted case.

Theorem 3.4.4 *For every* $f \in C_w([0,1])$, *we have*

$$\lim_{n\to+\infty} L_{w,n}(f) = f$$

uniformly with respect to the weighted norm $\|\cdot\|_w$.

Theorem 3.4.5 (Weighted Voronovskaja-type formula)
For every $f \in C_w^2([0,1])$ *we have*

$$\lim_{n\to\infty} \left\| n\left(L_{w,n}(f) - f\right) - \frac{a^2}{6} f'' \right\|_w = 0 \ .$$

In order to obtain a quantitative estimate of the Voronovskaja'a formula we need to consider the class of functions

$$C_w^{2,\alpha}([0,1]) := \{ f \in C_w^2([0,1]) \mid wf'' \in C^\alpha([0,1]) \} \ .$$

Theorem 3.4.6 (Weighted quantitative Voronovskaja-type formula)

If $a^2 \in C^2([0,1])$ *for every* $f \in C_w^{2,\alpha}([0,1])$ *we have*

$$\left\| n\left(L_{w,n}(f) - f\right) - \frac{a^2}{6} f'' \right\|_w \leq C_a \frac{M_f}{n^{\alpha/2}} \ ,$$

where M_f is the seminorm defined by

$$M_f = L_{wf''} + \|f''\|_w \tag{3.4.10}$$

and

$$C_a := \|a^2\| \max\left\{\frac{4}{3}, \frac{\|(a^2)''\|}{72\sqrt{n}}\right\} . \tag{3.4.11}$$

Finally, we shall be concerned with the core property and the representation of the semigroup generated by the differential operator arising from the Voronovskaja-type formula.

The differential operator is $A_w u(x) := a(x)^2\, u''(x)/6$ on the following Ventcel's domain

$$D_V(A_w) := \{u \in C_w([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0,1} w(x)\, A_w u(x) = 0\} .$$

As in the preceding section, we consider only first-order degeneracy of the function a at the endpoints. Different generation results in the space of continuous functions vanishing at the endpoints are available in [8] (see also [6] and the references given there).

Theorem 3.4.7 *Assume that*

$$a(x) = Cx(1-x) , \qquad 0 \le x \le 1 ,$$

for a suitable constant $C > 0$. Then, the space $C^2([0,1])$ (and hence $C^2_w([0,1])$) is a core for $(A_w, D_V(A_w))$; moreover, the operator $(A_w, D_V(A_w))$ generates a C_0-semigroup $(T(t))_{t\ge 0}$ in $C_w([0,1])$ and, for every $t \ge 0$ and for every sequence $(k(n))_{n\ge 1}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have

$$T(t) = \lim_{n\to+\infty} L_{w,n}^{k(n)} \qquad \text{strongly on } C_w([0,1]) , \tag{3.4.12}$$

moreover for every $f \in C^{2,\alpha}_w([0,1])$ we have

$$\left\|T(t)u - L_n^{k(n)}u\right\|_w \le t\,\frac{C_\alpha M_f}{n^{\alpha/2}} + \left(\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{k(n)}}{n}\right) \times \left(\|Af\|_w + \frac{C_\alpha M_f}{n^{\alpha/2}}\right) \tag{3.4.13}$$

and choosing $k(n) = [nt]$

$$\left\|T(t)u - L_n^{[nt]}u\right\|_w \le t\,\frac{C_\alpha M_f}{n^{\alpha/2}} + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\right)\left(\|Af\|_w + \frac{C_\alpha M_f}{n^{\alpha/2}}\right) . \tag{3.4.14}$$

PROOF. We reason only in the interval $[0,1/2]$ and assume $a(x)=Cx$. Let $u\in D_V(A_w)$ and fix $\varepsilon>0$. We have $u''(x)=o(1/x^{p+2})$ as $x\to 0$ and hence $u'(x)=o(1/x^{p+1})$ and $u(x)=o(1/x^p)$, that is $\lim_{x\to 0} w(x)\,x\,u'(x)=0$ and $\lim_{x\to 0} w(x)\,u(x)=0$. Hence, we can choose $\delta>0$ such that $\delta<1/2$ and

$$\begin{aligned} |w(x)\,u(x)| &< \varepsilon\,, \qquad x\in[0,\delta]\,,\\ |w(x)\,x\,u'(x)| &< \varepsilon\,, \qquad x\in]0,\delta]\,,\\ |w(x)\,x^2\,u''(x)| &< \varepsilon\,, \qquad x\in]0,\delta]\,. \end{aligned}$$

Let $v:[0,1/2]\to\mathbb{R}$ as follows

$$v(x):=\begin{cases} u(\delta)+u'(\delta)(x-\delta)+u''(\delta)\,\dfrac{(x-\delta)^2}{2}\,, & x\in[0,\delta[\,,\\ u(x)\,, & x\in\left[\delta,\dfrac{1}{2}\right]\,. \end{cases}$$

Then, $v\in C^2([0,1/2])$ and, for every $x\in]0,\delta]$, we have

$$\begin{aligned} |w(x)\,Av(x)| &= \left|w(x)\,\frac{C^2\,x^2}{6}\,v''(x)\right| = \left|\frac{C^2\,x^{p+2}}{6}\,u''(\delta)\right|\\ &\le \left|\frac{C^2\,\delta^{p+2}}{6}\,u''(\delta)\right| < \frac{C^2}{6}\,\varepsilon \end{aligned}$$

and consequently

$$|w(x)(Au(x)-Av(x))|\le \frac{C^2}{6}\,\varepsilon+\frac{C^2}{6}\,\varepsilon=\frac{C^2}{3}\,\varepsilon\,, \qquad x\in\left[0,\frac{1}{2}\right]\,.$$

Finally, for every $x\in[0,\delta]$,

$$\begin{aligned} |w(x)(u(x)-v(x))| &\le |w(x)\,u(x)|+|w(x)\,u(\delta)|+|w(x)\,u'(\delta)\,(x-\delta)|\\ &\quad +\left|w(x)\,u''(\delta)\,\frac{(x-\delta)^2}{2}\right|\\ &\le \varepsilon+|w(\delta)\,u(\delta)|+|w(\delta)\,\delta\,u'(\delta)|+\left|w(\delta)\,\frac{\delta^2}{2}\,u''(\delta)\right|\\ &\le \varepsilon+\varepsilon+\varepsilon+\frac{\varepsilon}{2}=\frac{7}{2}\,\varepsilon\,. \end{aligned}$$

The same inequality obviously extends to the interval $[0,1/2]$ and since ε is arbitrary, this completes the proof of the core property.

In order to apply Trotter's approximation theorem, we need to establish the stability estimate $\|L_{w,n}^k\|\le Me^{ck/n}$ for every $n,k\ge 1$ for a suitable constant $c\ge 0$.

Indeed, from (3.4.7) and Proposition 3.4.2, we get

$$\|L_{w,n}\|\le\|\omega_n\|^n\le(1+\|\omega_n-\mathbf{1}\|)^n\le\left(1+\frac{C}{n^2}\right)^n\,,$$

where C is the constant in Proposition 3.4.2, and consequently

$$\|L_{w,n}^k\| \leq \left(\left(1+\frac{C}{n^2}\right)^{n^2}\right)^{k/n} \leq e^{Ck/n} \ .$$

Hence, we can apply Trotter's approximation theorem [70] and obtain that the closure $(\bar{A}, D(\bar{A}))$ of the restriction of A_w to $C^2([0,1])$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $C_w([0,1])$ which can be represented by (3.4.12). Moreover, the stability estimate also gives $\|T(t)\| \leq e^{Ct}$ for every $t \geq 0$.

Now, we show that $(A_w, D_V(A_w))$ is closed and this, together with the core property, will imply that $(\bar{A}, D(\bar{A})) = (A_w, D_V(A_w))$ and complete the proof.

Let $(u_n)_{n\geq 1}$ be a sequence in $D_V(A_w)$ and $u, v \in C_w([0,1])$ such that $(w \cdot u_n)_{n\geq 1}$ converges uniformly to $w \cdot u$ and $(w \cdot A(u_n))_{n\geq 1}$ converges uniformly to $w \cdot v$. Since α and w are continuous and strictly positive in $]0,1[$, they have a positive minimum in every interval $[a,b] \subset]0,1[$ and consequently $(u_n)_{n\geq 1}$ converges uniformly to u and $(A(u_n))_{n\geq 1}$ converges uniformly to v in $[a,b]$; from the classical theory, we have that $u \in C^2([a,b])$ and $Au = v$ in $[a,b]$. Since the interval $[a,b]$ is arbitrary, we get $u \in C^2(]0,1[)$ and $Au = v$ in $]0,1[$. Finally, from the uniform convergence of $(w \cdot A(u_n))_{n\geq 1}$ to $w \cdot v$ and the condition $\lim_{x\to 0,1} w(x)\, Au_n(x) = 0$, we also have $\lim_{x\to 0,1} w(x)\, Au(x) = 0$ and hence $u \in D_V(A_w)$.

At this point we can apply Theorem 1.1.2. From Theorem 3.4.6 follows that the seminorms are given by $\psi_n(f) = C_a \frac{M_f}{n^{\alpha/2}}$ and $\varphi_n(f) = \|Af\|_w + C_a \frac{M_f}{n^{\alpha/2}}$, then taking into account that the growth bound of $(T(t))_{t\geq 0}$ is equal to 0 and every $T(t)$ is a linear contraction, i.e. $\omega = 0$ and $M = 1$, the estimates (3.4.13) and (3.4.14) follow directly from (1.1.10) and (1.1.11), and this completes the proof. □

3.5 An extension to the multivariate case

In this final section we briefly consider a possible extension to the multivariate case; we establish some approximation results and a Voronovskaja's formula.

The following results have been published in [42].

For the sake of simplicity, we limit ourselves to the two-variables case since a similar construction can be extended in a straightforward way in more variables.

The case considered in this section is of particular interest since it involves a second-order partial differential operator on the whole space $\mathbb{R}^2$ whose coefficients of the second-order partial derivatives may be even unbounded or degenerate.

First, we define the mean integral operator over a rotated rectangle. Let $a, b : \mathbb{R}^2 \to \mathbb{R}$ be strictly positive continuous functions satisfying the following condition

$$a(x,y), b(x,y) \leq c_1 + c_2(|x| + |y|) \ , \qquad (x,y) \in \mathbb{R}^2 \ , \tag{3.5.1}$$

for some suitable constants $c_1, c_2 > 0$ and let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function.

Moreover, denote by $L^1_{\text{loc}}(\mathbb{R}^2)$ the space of locally integrable functions on $\mathbb{R}^2$.

The mean integral operator $M_{a,b,\theta} : L^1_{\text{loc}}(\mathbb{R}^2) \to C(\mathbb{R}^2)$ is defined by setting, for every $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $(x,y) \in \mathbb{R}^2$,

$$M_{a,b,\theta} f(x,y) := \frac{1}{|R[x,y]|} \iint_{R[x,y]} f(\xi,\eta)\, d\xi\, d\eta \ , \tag{3.5.2}$$

where

$$\begin{aligned} R[x,y] \ := \ & \{(\xi,\eta) \in \mathbb{R}^2 \mid \\ & |(\xi - x)\cos\theta(x,y) + (\eta - y)\sin\theta(x,y)| \leq a(x,y) \ , \\ & |-(\xi - x)\sin\theta(x,y) + (\eta - y)\cos\theta(x,y)| \leq b(x,y)\} \end{aligned} \tag{3.5.3}$$

is the rectangle with center (x,y), sides $2\,a(x,y)$ and $2\,b(x,y)$ and rotated anticlockwise of an angle $\theta(x,y)$; moreover $|R[x,y]| := 4\,a(x,y)\,b(x,y)$ denotes the Lebesgue area of $R[x,y]$.

Using the parallel variables to the sides of the rectangle $R[x,y]$, from (3.5.2) we easily get, for every $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $(x,y) \in \mathbb{R}^2$,

$$M_{a,b,\theta} f(x,y) := \frac{1}{4\,a(x,y)\,b(x,y)} \int_{-a(x,y)}^{a(x,y)} \int_{-b(x,y)}^{b(x,y)} f \circ \varphi_{x,y}\,(\xi,\eta)\ d\xi\, d\eta \ , \tag{3.5.4}$$

where

$$\varphi_{x,y}(\xi,\eta) := (x+\xi\cos\theta(x,y)-\eta\sin\theta(x,y), y+\xi\sin\theta(x,y)+\eta\cos\theta(x,y))$$

defines the change of variables.

Denote by $C^{(b)}(\mathbb{R}^2)$ the space of all continuous bounded real functions on $\mathbb{R}^2$ and by $C_0(\mathbb{R}^2)$ the subspace consisting of all continuous functions vanishing at the point at infinity of $\mathbb{R}^2$. These spaces are endowed with the usual uniform norm

$$\|f\| := \sup_{(x,y)\in\mathbb{R}^2} |f(x,y)| , \qquad f \in C^{(b)}(\mathbb{R}^2) .$$

Moreover, we shall consider the function $w : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$w(x,y) := \frac{1}{1+x^2+y^2} , \qquad (x,y)\in\mathbb{R}^2 , \tag{3.5.5}$$

and the space

$$C_w^{(b)}(\mathbb{R}^2) := \left\{ f \in C(\mathbb{R}^2) \mid w\, f \in C^{(b)}(\mathbb{R}^2) \right\}$$

endowed with the norm

$$\|f\|_w := \sup_{(x,y)\in\mathbb{R}^2} |w(x,y)\, f(x,y)| .$$

Observe that $M_{a,b,\theta}$ maps $C^{(b)}(\mathbb{R}^2)$ into itself and is a positive contraction when considered as an operator on this space. Moreover, $M_{a,b,\theta}$ maps the space of compactly supported functions into $C_0(\mathbb{R}^2)$ and by continuity also $C_0(\mathbb{R}^2)$ into itself.

Now, we show that $M_{a,b,\theta}$ maps $C_w^{(b)}(\mathbb{R}^2)$ into itself. First, we put

$$r(x,y) := \sqrt{a(x,y)^2+b(x,y)^2} , \tag{3.5.6}$$

for every $(x,y)\in\mathbb{R}^2$ and we observe that (see (3.5.3))

$$R[x,y] \subset [x-r(x,y), x+r(x,y)] \times [y-r(x,y), y+r(x,y)] . \tag{3.5.7}$$

Moreover, from (3.5.1), we have

$$r(x,y) \le \sqrt{2}\ (c_1 + c_2(|x|+|y|))$$

and hence, using (3.5.6) and (3.5.7),

$$
\begin{aligned}
\frac{|M_{a,b,\theta} f(x,y)|}{1+x^2+y^2} &\le \frac{1}{(1+x^2+y^2)\,|R[x,y]|} \times \qquad (3.5.8)\\
&\quad \times \iint_{R[x,y]} \frac{f(\xi,\eta)}{1+\xi^2+\eta^2}\,(1+\xi^2+\eta^2)d\xi\,d\eta \\
&\le \frac{1}{1+x^2+y^2} \sup_{(\xi,\eta)\in R[x,y]} (1+\xi^2+\eta^2)\,\|f\|_w \\
&\le \frac{1+x^2+y^2+2r(x,y)^2+2(|x|+|y|)r(x,y)}{1+x^2+y^2}\,\|f\|_w \\
&\le \left(1+\frac{c_3+c_4(x^2+y^2)}{1+x^2+y^2}\right)\|f\|_w \le (1+c_5)\,\|f\|_w
\end{aligned}
$$

for some suitable constant $c_3, c_4 > 0$.

Then $M_{a,b,\theta}$ is a bounded operator when considered on the space $C_w^{(b)}(\mathbb{R}^2)$ endowed with the norm $\|\cdot\|_w$.

We are now in a position to define our Steklov operators. For every $n \ge 1$, set $a_n := a/n$, $b_n := b/n$ and consider the n-th Steklov operator $S_n : C_w^{(b)}(\mathbb{R}^2) \to C_w^{(b)}(\mathbb{R}^2)$ defined by setting, for every $f \in C_w^{(b)}(\mathbb{R}^2)$ and $(x,y) \in \mathbb{R}^2$,

$$S_n f(x,y) := M^n_{a_n,b_n,\theta} f(x,y)\ , \qquad (3.5.9)$$

where, as usual, $M^n_{a_n,b_n,\theta}$ denotes n-th iterate of the operator $M_{a_n,b_n,\theta}$.

We shall write M_n in place of $M_{a_n,b_n,\theta}$ if no confusion arises.

As a consequence of the properties of $M_{a,b,\theta}$, from (3.5.9) we have that S_n is well-defined as an operator on $C_w^{(b)}(\mathbb{R}^2)$ and maps the spaces $C^{(b)}(\mathbb{R}^2)$ and $C_0(\mathbb{R}^2)$ into themselves. When necessary, we shall consider the Steklov operators acting on these spaces too; we also observe that S_n is a positive contraction when acting on $C^{(b)}(\mathbb{R}^2)$ and $C_0(\mathbb{R}^2)$ endowed with the uniform norm.

In order to estimate the norm of S_n with respect to the weighted uniform norm in $C_w^{(b)}(\mathbb{R}^2)$, we use (3.5.8) taking into account that in this case we have to consider $r_n := \sqrt{a^2+b^2}/n$ in place of r and obtain

$$
\begin{aligned}
\frac{|M_n f(x,y)|}{1+x^2+y^2} &\le \frac{1+x^2+y^2+2r_n(x,y)^2+2(|x|+|y|)r_n(x,y)}{1+x^2+y^2}\,\|f\|_w \\
&\le \left(1+\frac{C}{n}\right)\|f\|_w\ , \qquad (x,y)\in\mathbb{R}^2\ , \qquad (3.5.10)
\end{aligned}
$$

for a suitable constant $C > 0$; this yields

$$\|S_n(f)\|_w = \|M_n^n(f)\|_w \le \left(1+\frac{C}{n}\right)^n \|f\|_w \le e^C\,\|f\|_w\ ,$$

and hence the sequence $(S_n)_{n\geq 1}$ is equibounded with respect to the weighted uniform norm in $C_w^{(b)}(\mathbb{R}^2)$; moreover, it is also a sequence of positive contractions on $C^{(b)}(\mathbb{R}^2)$ and $C_0(\mathbb{R}^2)$.

We have the following preliminary properties.

Lemma 3.5.1 *The following equalities hold, for every $n \in \mathbb{N}$,*

1) $S_n(\mathbf{1}) = \mathbf{1}$.

2) $S_n(\mathrm{pr}_i) = \mathrm{pr}_i$, $i = 1, 2$.

3) *For every $(x, y) \in \mathbb{R}$, we have*

$$\begin{aligned} S_n(\mathrm{pr}_1^2)(x,y) &= x^2 + \frac{1}{3n^2}\sum_{k=0}^{n-1} M_n^k(a^2\cos^2\theta + b^2\sin^2\theta)(x,y) , \\ S_n(\mathrm{pr}_2^2)(x,y) &= y^2 + \frac{1}{3n^2}\sum_{k=0}^{n-1} M_n^k(a^2\sin^2\theta + b^2\cos^2\theta)(x,y) , \\ S_n(\mathrm{pr}_1\,\mathrm{pr}_2)(x,y) &= x\,y + \frac{1}{3n^2}\sum_{k=0}^{n-1} M_n^k\left((a^2-b^2)\cos\theta\,\sin\theta\right)(x,y) . \end{aligned}$$

Proof. It is obvious that $M_n\mathbf{1} = \mathbf{1}$ and hence property 1) is true. Moreover, from (3.5.4) we easily obtain $M_n\mathrm{pr}_i = \mathrm{pr}_i$, $i = 1, 2$, and this yields property 2). Finally a straightforward calculation based on (3.5.4) gives

$$\begin{aligned} M_n(\mathrm{pr}_1^2)(x,y) &= x^2 + \frac{1}{3n^2}\left(a(x,y)^2\cos^2\theta(x,y)\right. \\ &\quad \left.+ b(x,y)^2\sin^2\theta(x,y)\right) . \\ M_n(\mathrm{pr}_2^2)(x,y) &= y^2 + \frac{1}{3n^2}\left(a(x,y)^2\sin^2\theta(x,y)\right. \\ &\quad \left.+ b(x,y)^2\cos^2\theta(x,y)\right) . \\ M_n(\mathrm{pr}_1\,\mathrm{pr}_2)(x,y) &= x\,y + \frac{1}{3n^2}\left((a(x,y)^2 - b(x,y)^2)\times\right. \\ &\quad \left.\times\ \cos\theta(x,y)\,\sin\theta(x,y)\right) \end{aligned} \tag{3.5.11}$$

and an induction argument on the integer $n \geq 1$ yields property 3). □

The convergence of the sequence $(S_n)_{n\geq 1}$ of Steklov-type operators will be obtained studying the behavior of these operators on the subspace

$$C_w^{2,(b)}(\mathbb{R}^2) := \left\{ f \in C_w^{(b)}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2) \mid \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x\,\partial y} \in C^{(b)}(\mathbb{R}^2) \right\} \tag{3.5.12}$$

of all functions in $C_w^{(b)}(\mathbb{R}^2)$ with bounded second-order partial derivatives.

Proposition 3.5.2 *For every $f \in C_w^{2,(b)}(\mathbb{R}^2)$ there exists a constant $C_f \geq 0$, depending on f, such that*

$$\|M_n f - f\|_w \leq C_f \frac{1}{n^2} \,. \tag{3.5.13}$$

PROOF. For every $x, y, s, t \in \mathbb{R}$, there exist $\xi, \eta \in \mathbb{R}$ such that

$$\begin{aligned} f(s,t) - f(x,y) &= \frac{\partial f}{\partial x}(x,y)(s-x) + \frac{\partial f}{\partial y}(t-y) + \frac{\partial^2 f}{\partial x^2}(\xi,\eta)\frac{(s-x)^2}{2} \\ &\quad + \frac{\partial^2 f}{\partial y^2}(\xi,\eta)\frac{(t-y)^2}{2} + \frac{\partial^2 f}{\partial x \partial y}(\xi,\eta)(s-x)(t-y) \end{aligned}$$

and consequently

$$\begin{aligned} M_n f(x,y) - f(x,y) &= \frac{\partial f}{\partial x}(x,y) M_n(\mathrm{pr}_1 - x)(x,y) \\ &+ \frac{\partial f}{\partial y}(x,y) M_n(\mathrm{pr}_2 - y)(x,y) + M_n\left(\frac{\partial^2 f}{\partial x^2}(\xi,\eta)\frac{(\mathrm{pr}_1 - x)^2}{2}\right)(x,y) \\ &+ M_n\left(\frac{\partial^2 f}{\partial y^2}(\xi,\eta)\frac{(\mathrm{pr}_2 - y)^2}{2}\right)(x,y) \\ &+ M_n\left(\frac{\partial^2 f}{\partial x \partial y}(\xi,\eta)(\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y)\right)(x,y) \,. \end{aligned}$$

Let $C := \max\{\|\partial^2 f/\partial x^2\|, \|\partial^2 f/y^2\|, 2\|\partial^2 f/\partial x \partial y\|\}$; then

$$\begin{aligned} |M_n f(x,y) - f(x,y)| &\leq \frac{C}{2}\big(M_n(\mathrm{pr}_1 - x)^2 + M_n(\mathrm{pr}_2 - y)^2 \\ &\quad + M_n(\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y)\big) \\ &= \frac{C}{6n^2}\left(a^2 + b^2 + (a^2 - b^2)\sin(2\theta)\right)(x,y) \,. \end{aligned}$$

Multiplying by $w(x,y)$ and taking the maximum, from (3.5.1) we obtain the desired estimate. □

At this point, we introduce the function $w_n : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\omega_n(x,y) := w(x,y)\, M_n\left(\frac{1}{w}\right)(x,y)$$

(see (3.5.5)). Observe that w_n is bounded since $1/w \in C_w^{(b)}(\mathbb{R}^2)$ and consequently $M_n(1/w) \in C_w^{(b)}(\mathbb{R}^2)$ too.

Proposition 3.5.3 *For every $f \in C_w^{(b)}(\mathbb{R}^2)$ and $k \geq 1$,*

$$\|M_n^k(f) - f\|_w \leq \|M_n(f) - f\|_w \sum_{i=0}^{k-1} \|\omega_n\|^i \,. \tag{3.5.14}$$

Proof. We argue by induction on the integer $k \geq 1$. If $k = 1$ then (3.5.14) is obviously true. Now, assume that (3.5.14) holds for $k \geq 1$. We have

$$\begin{aligned}
&|w(x,y)(M_n^{k+1} f(x,y) - f(x,y))| \\
&\quad \leq |w(x,y) M_n (M_n^k f - f)(x,y)| + |w(x,y)(M_n f(x,y) - f(x,y))| \\
&\quad \leq \omega_n(x,y) \|M_n^k(f) - f\|_w + \|M_n(f) - f\|_w \\
&\quad \leq \omega_n(x,y) \sum_{i=0}^{k-1} \omega_n(x,y)^i \, \|M_n(f) - f\|_w + \|M_n(f) - f\|_w \\
&\quad \leq \sum_{i=1}^{k} \omega_n(x,y)^i \, \|M_n(f) - f\|_w + \|M_n(f) - f\|_w \\
&\quad \leq \sum_{i=0}^{k} \omega_n(x,y)^i \, \|S_{1,b_n}(f) - f\|_w
\end{aligned}$$

and this completes the induction argument. □

In order to deduce the convergence of Steklov operators from the above proposition, we need to estimate the convergence of the sequence $(w_n)_{n\geq 1}$.

Proposition 3.5.4 *There exists a constant $C > 0$ such that*

$$\|\omega_n - \mathbf{1}\| \leq \frac{C}{n^2}\,.$$

Proof. First, we observe that

$$\begin{aligned}
|\omega_n(x) - 1| &= \left| w(x) S_{1,b_n}\left(\frac{1}{w}\right) x - w(x) \frac{1}{w(x)} \right| \\
&= \left| w(x) \left[S_{1,b}\left(\frac{1}{w}\right) x - \frac{1}{w(x)} \right] \right| \\
&\leq \left\| M_n\left(\frac{1}{w}\right) - \frac{1}{w} \right\|_w \,.
\end{aligned}$$

Since $1/w \in C_w^{(b)}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ and its second-order partial derivatives are bounded, we can apply Proposition 3.5.2 and obtain a constant $C > 0$ such that

$$\left\| M_n\left(\frac{1}{w}\right) - \frac{1}{w} \right\|_w \leq \frac{C}{n^2}$$

and this completes the proof. □

Now, we are in a position to state the convergence property of the sequence $(S_n)_{n\geq 1}$.

Theorem 3.5.5 *For every $f \in C_w^{2,(b)}(\mathbb{R}^2)$, we have*

$$\lim_{n\to+\infty} \|S_n(f) - f\|_w = 0\ .$$

PROOF. Let $f \in C_w^{2,(b)}(\mathbb{R}^2)$; from Propositions 3.5.2, 3.5.3 and 3.5.4 we obtain

$$\begin{aligned}\|S_n(f) - f\|_w &\leq \|M_n(f) - f\|_w \sum_{i=0}^{n-1} \|\omega_n\|^i \leq \frac{C_f}{n^2} \sum_{i=0}^{n-1} \|\omega_n\|^i \\ &\leq \frac{C_f}{n^2} \sum_{i=0}^{n-1} \left(1 + C\frac{1}{n^2}\right)^i = \frac{C_f}{C} \left(\left(1 + \frac{C}{n^2}\right)^n - 1\right)\end{aligned}$$

and consequently $\lim_{n\to+\infty} \|S_n(f) - f\|_w = 0$. □

As a consequence of Theorem 3.5.5, we have that the sequence $(S_n(f))_{n\geq 1}$ converges to f uniformly on every compact subset of $\mathbb{R}^2$ whenever $f \in C_w^{(b)}(\mathbb{R}^2)$.

In the following result, we study the uniform convergence of the sequence of Steklov operators in the space $C_0(\mathbb{R}^2)$.

We observe that from Proposition 3.5.2, it follows that if $f \in C_w^{(b)}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ has bounded second-order partial derivatives, we have

$$|M_n f(x,y) - f(x,y)| \leq \frac{C_f}{6n^2} \left(a^2 + b^2 + (a^2 - b^2)\sin(2\theta)\right)(x,y)\ ,\ \ (x,y) \in \mathbb{R}^2$$

and hence if a, b are bounded, we also obtain

$$\|M_n(f) - f\| \leq \frac{C_f}{n^2} \left(\|a^2\| + \|b^2\|\right)\ . \tag{3.5.15}$$

Theorem 3.5.6 *Assume that $a, b \in C^{(b)}(\mathbb{R}^2)$. Then, for every $f \in C_0(\mathbb{R}^2)$, we have*

$$\lim_{n\to+\infty} \|S_n f - f\| = 0\ . \tag{3.5.16}$$

PROOF. Let $f \in C^2(\mathbb{R}^2)$ with bounded second-order partial derivatives and observe that the second member of (3.5.15) tends uniformly to 0 as $n \to +\infty$. Moreover, in this case the operators M_n are positive contractions with respect to the uniform norm, and consequently

$$\|S_n(f) - f\| = \|M_n^n(f) - f\| \leq n\|M_n(f) - f\|$$

which yields $\lim_{n\to+\infty} \|S_n(f) - f\| = 0$. The general case where $f \in C_0(\mathbb{R}^2)$ follows from a density argument. □

Now we establish a Voronovskaja-type formula for the operators S_n. We need some preliminary properties of independent interest which establishes the convergence of the mean of iterates of the operators M_n.

Proposition 3.5.7 *For every $f \in C_w^{2,(b)}(\mathbb{R}^2)$, we have*

$$\lim_{n\to+\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} M_n^k(f) - f \right\|_w = 0 \,. \tag{3.5.17}$$

Proof. Let $f \in C_w^{2,(b)}(\mathbb{R}^2)$; from Propositions 3.5.3 and 3.5.4 it follows

$$\begin{aligned}\left\| \frac{1}{n} \sum_{k=0}^{n-1} M_n^k(f) - f \right\|_w &\le \frac{1}{n} \sum_{k=0}^{n-1} \left\| M_n^k(f) - f \right\|_w \\ &\le \frac{1}{n} \|M_n(f) - f\|_w \sum_{k=0}^{n-1}\sum_{i=0}^{k-1} \|\omega_n\|^i \le C_f \frac{1}{n}\frac{1}{n^2} \sum_{k=0}^{n-1}\sum_{i=0}^{k-1} (1 + C\frac{1}{n^2})^i \\ &= \frac{C_f}{C} \left(\frac{n}{C} \left(\left(1 + \frac{C}{n^2}\right)^n - 1\right) - 1 \right) ,\end{aligned}$$

and taking the limit as $n \to +\infty$, we have the validity of (3.5.17). □

Remark 3.5.8 If we consider the subspace

$$C_{0,w}(\mathbb{R}^2) := \left\{ f \in C_w^{(b)}(\mathbb{R}^2) \mid w\, f \in C_0(\mathbb{R}^2) \right\} , \tag{3.5.18}$$

we can observe that the preceding proposition is still true for every $f \in C_{0,w}(\mathbb{R}^2)$.

Indeed, from (3.5.10), we get

$$\|M_n^k\| \le \left(1 + \frac{C}{n}\right)^k \le e^C$$

with respect to the norm in $C_w^{(b)}(\mathbb{R}^2)$ and hence the mean operators $\sum_{k=0}^{n-1} M_n^k/n$ are equibounded when acting on the space $C_w^{(b)}(\mathbb{R}^2)$. Since $C_w^{2,(b)}(\mathbb{R}^2)$ is dense in $C_{0,w}(\mathbb{R}^2)$, Proposition 3.5.7 can be applied to every $f \in C_{0,w}(\mathbb{R}^2)$. □

Finally, we can establish a Voronovskaja-type formula with respect to the weighted uniform norm.

Theorem 3.5.9 (Voronovskaja-type formula) *Assume that $a, b \in C_0(\mathbb{R}^2)$ and consider the second-order partial differential operator $A : C^2(\mathbb{R}^2) \to C(\mathbb{R}^2)$ defined by*

$$A(f) := \frac{1}{6} \left(\alpha \frac{\partial^2 f}{\partial x^2} + \beta \frac{\partial^2 f}{\partial y^2} + \gamma \frac{\partial^2 f}{\partial x \partial y} \right) , \qquad f \in C^2(\mathbb{R}^2) , \tag{3.5.19}$$

where the coefficients $\alpha, \beta, \gamma : \mathbb{R}^2 \to \mathbb{R}$ *are given by*

$$\alpha := a^2 \cos^2\theta + b^2 \sin^2\theta \ , \ \beta := a^2 \sin^2\theta + b^2 \cos^2\theta \ , \ \gamma := (a^2 - b^2)\sin(2\theta) \ .$$

Then, for every $f \in C_w^{2,(b)}(\mathbb{R}^2)$ *with uniformly continuous second-order partial derivatives, we have*

$$\lim_{n\to\infty} \|n\,(S_n f - f) - A(f)\|_w = 0 \ .$$

Proof. Let $f \in C_w^{2,(b)}(\mathbb{R}^2)$ have uniformly continuous second-order partial derivatives. For every $x, y, s, t \in \mathbb{R}$, we can write

$$\begin{aligned} f(s,t) - f(x,y) &= \frac{\partial f}{\partial x}(x,y)(s-x) + \frac{\partial f}{\partial y}(t-y) \\ &+ \frac{\partial^2 f}{\partial x^2}(x,y)\frac{(s-x)^2}{2} + \frac{\partial^2 f}{\partial y^2}(x,y)\frac{(t-y)^2}{2} + \frac{\partial^2 f}{\partial x \partial y}(s-x)(t-y) \\ &+ \eta(s,t,x,y)\left((s-x)^2 + (t-y)^2\right) \end{aligned}$$

where $\eta : \mathbb{R}^4 \to \mathbb{R}$ satisfies $\lim_{(s,t)\to(x,y)} \eta(s,t,x,y) = 0$ uniformly with respect to $(x,y) \in \mathbb{R}^2$. Then

$$\begin{aligned} n\,(S_n f(x,y) - f(x,y)) &= n\,S_n(f - f(x,y))(x,y) \qquad (3.5.20) \\ &= n\,\frac{\partial f}{\partial x}(x,y) S_n(\mathrm{pr}_1 - x)(x,y) + n\,\frac{\partial f}{\partial y}(x,y) S_n(\mathrm{pr}_1 - y)(x,y) \\ &+ \frac{n}{2}\,\frac{\partial^2 f}{\partial x^2}(x,y) S_n((\mathrm{pr}_1 - x)^2)(x,y) \\ &+ \frac{n}{2}\,\frac{\partial^2 f}{\partial y^2}(x,y) S_n((\mathrm{pr}_2 - y)^2)(x,y) \\ &+ n\,\frac{\partial^2 f}{\partial x \partial y} S_n((\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y))(x,y) \\ &+ n\,S_n\left(\eta(\mathrm{pr}_1, \mathrm{pr}_2, x, y)\left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2\right)\right)(x,y) \ . \end{aligned}$$

We observe that the first two addends in (3.5.20) vanishes.
Moreover, we have

$$\begin{aligned} \frac{n}{2}\,S_n\left((\mathrm{pr}_1 - x)^2\right)(x,y) &= n\left(S_n(\mathrm{pr}_1^2)(x,y) - 2x S_n(\mathrm{pr}_1)(x,y) + x^2\right) \\ &= \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k(a^2 \cos^2\theta + b^2 \sin^2\theta)(x,y) \\ &= \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k(\alpha)(x,y) \ , \end{aligned}$$

$$\begin{aligned}\frac{n}{2}\, S_n\left((\mathrm{pr}_2 - y)^2\right)(x,y) &= n\left(S_n(\mathrm{pr}_2^2)(x,y) - 2yS_n(\mathrm{pr}_2)(x,y) + y^2\right)\\ &= \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k(a^2\,\sin^2\theta + b^2\,\cos^2\theta)(x,y)\\ &= \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k(\beta)(x,y)\ ,\end{aligned}$$

$$\begin{aligned}&n\,S_n((\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y))(x,y)\\ &\quad = n(S_n(\mathrm{pr}_1\mathrm{pr}_2)(x,y) - yS_n(\mathrm{pr}_1) - xS_n(\mathrm{pr}_2)(x,y) + xy)\\ &\quad = \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k\left((a^2 - b^2)\,\sin(2\theta)\right)(x,y)\\ &\quad = \frac{1}{6n}\sum_{k=0}^{n-1} M_n^k\,(\gamma)\,(x,y)\ .\end{aligned}$$

Our assumptions on the functions a and b ensure that all the functions α, β and γ are in $C_{0,w}(\mathbb{R}^2)$ and hence from Proposition 3.5.7 (see also Remark 3.5.8), it follows

$$\begin{aligned}\lim_{n\to+\infty}\frac{n}{2}\, S_n\left((\mathrm{pr}_1 - x)^2\right)(x,y) &= \frac{1}{6}\,\alpha(x,y)\\ \lim_{n\to+\infty}\frac{n}{2}\, S_n\left((\mathrm{pr}_2 - y)^2\right)(x,y) &= \frac{1}{6}\,\beta(x,y)\\ \lim_{n\to+\infty} nS_n((\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y))(x,y) &= \frac{1}{6}\,\gamma(x,y)\end{aligned}$$

uniformly with respect to the weighted uniform norm in $C_w^{(b)}(\mathbb{R}^2)$.

Finally, we have only to show that the last addend in (3.5.20) converges to 0 with respect to the weighted uniform norm. To this end, let $\varepsilon > 0$ and consider $\delta > 0$ such that $|\eta(s,t,x,y)| \le \varepsilon$ whenever $(s-x)^2 + (t-y)^2 \le \delta^2$. Moreover, take $M > 0$ such that $|\eta(s,t,x,y)| \le M$ for every $(x,y), (s,t) \in \mathbb{R}^2$. We have

$$|\eta(s,t,x,y)|\,((s-x)^2 + (t-y)^2) \le \varepsilon((s-x)^2 + (t-y)^2)$$

if $(s-x)^2 + (t-y)^2 \le \delta^2$ and

$$|\eta(s,t,x,y)|\,((s-x)^2 + (t-y)^2) \le \frac{M}{\delta^2}\,((s-x)^2 + (t-y)^2)^2\ ,$$

whenever $(s-x)^2 + (t-y)^2 > \delta^2$. In any case

$$|\eta(s,t,x,y)|\,((s-x)^2 + (t-y)^2) \le \varepsilon((s-x)^2 + (t-y)^2) + \frac{M}{\delta^2}((s-x)^2 + (t-y)^2)^2$$

and hence

$$\begin{aligned}
&\left| n\, S_n \left(\eta(\mathrm{pr}_1, \mathrm{pr}_2, x, y) \left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right) \right) (x, y) \right| \\
&\quad \le n\, S_n \left(|\eta(s, t, x, y)| \left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right) \right) (x, y) \\
&\quad \le n\, S_n \left(\varepsilon ((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2) \right) (x, y) \\
&\qquad\quad + n\, S_n \left(\frac{M}{\delta^2} ((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2)^2 \right) (x, y) \\
&\quad \le \varepsilon n\, S_n \left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right) (x, y) \\
&\qquad\quad + \frac{M}{\delta^2} n\, S_n \left(\left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right)^2 \right) (x, y) \ .
\end{aligned}$$

Observe that

$$\lim_{n \to +\infty} n\, S_n ((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2)(x, y) = \frac{1}{3} (a^2(x, y) + b^2(x, y))$$

uniformly with respect to the weighted norm and hence, from the arbitrarily of ε, it remains only to show that

$$\lim_{n \to +\infty} \left\| n\, S_n \left(\left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right)^2 \right) \right\|_w = 0 \ . \tag{3.5.21}$$

Indeed, a straightforward calculation yields

$$\begin{aligned}
&M_n \left(\left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right)^2 \right) = ((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2)^2 \\
&\quad + \frac{1}{n^4} \left(\frac{1}{5} a^4 + \frac{2}{9} a^2 b^2 + \frac{1}{5} b^4 \right) + \frac{2}{n^2} \alpha \left((\mathrm{pr}_1 - x)^2 + \frac{1}{3} (\mathrm{pr}_2 - y)^2 \right) \\
&\quad + \frac{2}{n^2} \beta \left(\frac{1}{3} (\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right) + \frac{4}{3n^2} \gamma\, (\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y) \ ,
\end{aligned}$$

and taking n iterations of the above formula, we have

$$\begin{aligned}
&n\, S_n \left(\left((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right)^2 \right) = ((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2)^2 \\
&\quad + \frac{1}{n^2} \left(\frac{1}{n} \sum_{k=0}^{n-1} M_n^k \left(\frac{1}{5} a^4 + \frac{2}{9} a^2 b^2 + \frac{1}{5} b^4 \right) \right) \\
&\quad + \frac{1}{n} \sum_{k=0}^{n-1} M_n^k \left(2\alpha \left((\mathrm{pr}_1 - x)^2 + \frac{1}{3} (\mathrm{pr}_2 - y)^2 \right) \right) \\
&\quad + \frac{1}{n} \sum_{k=0}^{n-1} M_n^k \left(2\beta \left(\frac{1}{3} (\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right) \right) \\
&\quad + \frac{1}{n} \sum_{k=0}^{n-1} M_n^k \left(\frac{4}{3} \gamma (\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y) \right) \ .
\end{aligned}$$

Now, we discuss the convergence in $C_w^{(b)}(\mathbb{R}^2)$ of the preceding addends evaluated at (x,y). The first addend $((\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2)^2$ vanishes identically at (x,y). As regards to the second addend, we observe that the assumptions on a and b ensure that $a^4, b^4, a^2b^2 \in C_{0,w}(\mathbb{R}^2)$ and hence, from Proposition 3.5.7 and Remark 3.5.8, we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} M_n^k \left(\frac{1}{5}\, a^4 + \frac{2}{9}\, a^2\, b^2 + \frac{1}{5}\, b^4 \right) = \frac{1}{5}\, a^4 + \frac{2}{9}\, a^2\, b^2 + \frac{1}{5}\, b^4$$

for the weighted uniform norm. Therefore the second addend converges to 0 uniformly with respect to the weighted norm due to the factor $1/n^2$. Finally, the same argument can be applied to the last three addends and we find that they converge in $C_w^{(b)}(\mathbb{R}^2)$ respectively to the functions

$$2\alpha \left((\mathrm{pr}_1 - x)^2 + \frac{1}{3}(\mathrm{pr}_2 - y)^2 \right) , \qquad 2\beta \left(\frac{1}{3}(\mathrm{pr}_1 - x)^2 + (\mathrm{pr}_2 - y)^2 \right) ,$$

and

$$\frac{4}{3}\, \gamma\, (\mathrm{pr}_1 - x)(\mathrm{pr}_2 - y)$$

which vanish identically at (x,y).

Hence (3.5.21) has been established and the proof is complete. □

Under additional assumptions on the functions a and b, we can state the Voronovskaja-type formula in the space $C_0(\mathbb{R}^2)$ with respect to the uniform norm.

We begin with the analogous of Proposition 3.5.7 in the space $C_0(\mathbb{R}^2)$.

Proposition 3.5.10 *For every $f \in C^2(\mathbb{R}^2)$ having bounded second-order partial derivatives, we have*

$$\lim_{n\to+\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} M_n^k(f) - f \right\| = 0 \, . \tag{3.5.22}$$

Proof. Indeed, since the operators M_n are positive contractions, from (3.5.15) we have

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=0}^{n-1} M_n^k(f) - f \right\| &\le \frac{1}{n} \sum_{k=0}^{n-1} \left\| M_n^k(f) - f \right\| \\ &\le \frac{1}{n}\, \|M_n(f) - f\| \sum_{k=0}^{n-1} k \le \frac{1}{n}\, \frac{C_f(\|a^2\| + \|b^2\|)}{n^2}\, \frac{n(n-1)}{2} \, . \end{aligned}$$

Taking the limit as $n \to +\infty$, we have the validity of (3.5.22). □

Theorem 3.5.11 (Voronovskaja-type formula in $C_0(\mathbb{R}^2)$) *Assume that*

$$a(\mathbf{1} + \mathrm{pr}_1 + \mathrm{pr}_2) \in C_0(\mathbb{R}^2) , \qquad b(\mathbf{1} + \mathrm{pr}_1 + \mathrm{pr}_2) \in C_0(\mathbb{R}^2)$$

and consider the differential operator A defined by (3.5.19)*.*

Then, for every $f \in C_0^2(\mathbb{R}^2)$, we have

$$\lim_{n\to\infty} \| n\,(S_n f - f) - A(f) \| = 0 \ .$$

PROOF. We observe that Proposition 3.5.10 continues to hold for every function in $C_0(\mathbb{R}^2)$ by a density argument, and that our assumptions ensure that α, β and γ and all the functions involved in the remainder estimates are in $C_0(\mathbb{R}^2)$. Using this remark in place of Proposition 3.5.7, we can proceed exactly as in the proof of Theorem 3.5.9. □

Chapter 4

Best decomposition

The representation of the solutions of parabolic problems by means of iterates of approximating operators may be more effective if we choose appropriately the sequence of operators. Even the quantitative estimates between the semigroup and the iterates may be affected by this choice. In this chapter we introduce a method which can be useful in order to consider a combinations of different sequences of operators in the approximation of the same problem.

Using a general procedure we consider some combination of different approximation processes by means of projections on orthogonal subspaces. We concentrate our attention on some particular positive approximation processes in spaces of L^2-real functions in order to satisfy a prescribed Voronovskaja-type formula. Some similar questions have also been considered in [31] and in [32].

The results in this chapter are contained in [40]

4.1 Direct sums of approximation processes

We are mainly interested in the application of a general and simple method which consists in constructing a new approximation process starting with a decomposition of a Hilbert space into the direct sum of orthogonal subspaces and associating to each subspace an assigned approximation process.

In this way we obtain some noteworthy results regarding the possibility of obtaining new Voronovskaja-type formulas from assigned ones and extending the class of differential problems under consideration.

The general method can actually be applied in different settings. Indeed, we may have the necessity of using different approximation processes on orthogonal subspaces as done in Section 4.2 in connection with Bernstein-Kantorovich and Bernstein-Durrmeyer operators; this may happen for example in studying diffusion models in population genetics where different factors may depend on the subspace containing the initial condition. Indeed, it is well-known that the differential operator arising from the Voronovskaja's formula for both Bernstein-Kantorovich and Bernstein-Durrmeyer operators describes the evolution process associated with some diffusion models in population genetics through the representation given in (4.2.13) which depends only on the initial condition u_0 in (4.2.14). Hence the method used in Section 4.2 allows us to arrange better the choice of the subspace V and the approximating operators to the initial condition. A different motivation can be the preservation of some functions by a modified classical approximation process; this was already realized in [31] for some sequences of algebraic polynomials and now we have also considered an example concerned with convolution operators in Section 4.3. Different applications to projections onto splines can also be considered; here we have not dealt with this case due to the large literature already existing in this field (see [46, Section 13.4]) and also because we are only interested in the possibility of approximating the solution of wider classes of differential problems and consequently only to more general Voronovskaja-type formulas.

The method is based on some simple properties of Hilbert spaces. Consider an Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and a decomposition

$$\mathcal{H} = \bigoplus_{i\in I} V_i$$

of $\mathcal{H}$ into the direct sum of orthogonal closed subspaces V_i, $i \in I$ and for every $i \in I$ denote by P_i the canonical orthogonal projection onto the subspace V_i.

Now, let $(L_i)_{i\in I}$ be a family of linear operators from $\mathcal{H}$ into itself and consider the linear operator $L : \mathcal{H} \to \mathcal{H}$ defined by setting, for every $u \in \mathcal{H}$,

$$L(u) = \sum_{i\in I} P_i(L_i(u)) . \tag{4.1.1}$$

In this way we associate the operator L to the families $(V_i)_{i\in I}$ and $(L_i)_{i\in I}$.

Observe that if $u, v \in \mathcal{H}$ and $L_i(u) = v$ for every $i \in I$ then we have $L(u) = v$ too. In particular if all the operators L_i, $i \in I$ coincide with an operator T we also have $L = T$.

Moreover, it is also interesting to observe that we can also study perturbations of an operator L having the form (4.1.1) by modifying some of its components L_i; this will be performed in Section 4.3 in connection with Jackson convolution operators.

At this point, we apply the preceding procedure to a sequence of families $(L_{i,n})_{i\in I}$ of linear operators and using (4.1.1) we define the new sequence $(L_n)_{n\in\mathbb{N}}$ of linear operators given by

$$L_n(u) = \sum_{i\in I} P_i(L_{i,n}(u)) . \tag{4.1.2}$$

It is immediate to check that if every sequence $(L_{i,n})_{n\in\mathbb{N}}$, $i \in I$, is an approximation process on $\mathcal{H}$, then $(L_n)_{n\in\mathbb{N}}$ satisfies the same property. Moreover, if every sequence $(L_{i,n})_{n\geq 1}$ satisfies an abstract Voronovskaja-type formula

$$\lim_{n\to+\infty} n(L_{i,n}u - u) = A_i(u) \ , \qquad u \in D \ , \tag{4.1.3}$$

where $A_i : D \to \mathcal{H}$ is a linear operator and D is a subspace of $\mathcal{H}$, then the sequence $(L_n)_{n\geq 1}$ satisfies the Voronovskaja's formula

$$\lim_{n\to+\infty} n(L_n u - u) = \sum_{i\in I} P_i(A_i(u)) \ , \qquad u \in D \ . \tag{4.1.4}$$

Using this general scheme, we pass to consider some cases of particular interest in different settings where we can add more details on the convergence of the constructed operators and their Voronovskaja-type formulas.

It will be useful to observe that if a finite-dimensional subspace V of $\mathcal{H}$ is generated by the independent system $\{\alpha_1, \ldots, \alpha_m\}$, then the projection P_V of $\mathcal{H}$ onto V can be easily obtained by considering the square matrix $A := (\langle \alpha_i, \alpha_j \rangle)_{i,j=1,\ldots,m}$ and taking into account that for every $f, g \in \mathcal{H}$ we have $P_V(f) = g$ if and only if $AG = F$ where F is the column vector with components $(\langle f, \alpha_i \rangle)_{i=1,\ldots,m}$ and $G = (g_i)_{i=1,\ldots,m}$ is the vector of the components of $P_V(f)$ in the subspace V, i.e. $P_V(f) = \sum_{i=1}^m g_i \alpha_i$; imposing $\langle P_V(f) - f, \alpha_i \rangle = 0$ for every $i = 1, \ldots, m$ we find

$$G = A^{-1} \cdot F \tag{4.1.5}$$

and in particular, if $\{\alpha_1, \ldots, \alpha_m\}$ is an orthogonal system

$$g_i = \frac{\langle f, \alpha_i \rangle}{\|\alpha_i\|^2} \ . \tag{4.1.6}$$

4.2 Bernstein-Kantorovich-Durrmeyer operators

In this section we split the space $L^2(0,1)$ into two components and consider a combination of the classical Bernstein-Kantorovich and Bernstein-Durrmeyer operators. Obviously the same construction may be carried on by considering different orthogonal subspaces of $L^2(0,1)$ or different sequences of operators.

First, we recall that for every $n \geq 1$ the n-th Bernstein-Kantorovich $K_n : L^2(0,1) \to L^2(0,1)$ and respectively the n-th Bernstein-Durrmeyer operator $M_n : L^2(0,1) \to L^2(0,1)$ are defined by setting, for every $f \in L^2(0,1)$ and $x \in [0,1]$,

$$K_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \ , \tag{4.2.1}$$

and respectively

$$M_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)\, dt \ , \tag{4.2.2}$$

where, as usual, $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$.

We also recall that (see, e.g., [58, p. 31] and [9, Section 5.3.7, 5.3.8])

$$\begin{aligned} K_n(\mathbf{1}) &= \mathbf{1}\ , \quad K_n(\mathrm{id})(x) = \frac{2nx+1}{2(n+1)}\ , && (4.2.3)\\ K_n(\mathrm{id}^2)(x) &= \frac{3n(n-1)x^2+6nx+1}{3(n+1)^2}\ , \\ M_n(\mathbf{1}) &= \mathbf{1}\ , \quad M_n(\mathrm{id})(x) = \frac{nx+1}{n+2}\ , && (4.2.4)\\ M_n(\mathrm{id}^2)(x) &= \frac{n(n-1)x^2+4nx+2}{(n+2)(n+3)}\ , \end{aligned}$$

for every $x \in [0,1]$ and these formulas ensure the convergence of the sequences $(K_n)_{n\geq 1}$ and $(M_n)_{n\geq 1}$ to the identity operator by the classical Korovkin's theorem (see e.g. [9, Theorem 4.2.7]).

Moreover, estimates of the convergence can be found with respect to the classical modulus of continuity $\omega(f,\delta)$ in spaces of continuous functions (see [9, (5.3.38)–(5.3.42) and (5.3.51)–(5.3.53)])

$$|K_n f(x) - f(x)| \leq 2\omega\left(f, \frac{\sqrt{(n-1)\,x(1-x)}}{n+1}\right)\ ,$$

$$|M_n f(x) - f(x)| \leq 2\omega\left(f, \sqrt{\frac{2(n-3)\,x(1-x)+2}{(n+2)(n+3)}}\right)$$

which give

$$\|K_n f(x) - f(x)\| \le 2\omega\left(f, \frac{1}{\sqrt{n}}\right) \ , \quad \|M_n f(x) - f(x)\| \le 2\omega\left(f, \frac{1}{\sqrt{n}}\right)$$

for every $f \in C([0,1])$ and with respect to the averaged modulus of smoothness $\tau(f,\delta)_2 := \left(\int_0^1 \omega(f,\delta,x)^2\,dx\right)^{1/2}$

$$\|K_n f - f\|_2 \ \le \ 748\,\tau\left(f, \frac{1}{\sqrt{n+1}}\right)_2 \ , \tag{4.2.5}$$

$$\|M_n f - f\|_2 \ \le \ 748\,\tau\left(f, \frac{1}{\sqrt{n+1}}\right)_2 \tag{4.2.6}$$

for every $f \in L^2(0,1)$.

Finally, we also recall the following Voronovskaja-type formulas

$$\lim_{n\to+\infty} n(K_n(f) - f) \ = \ \frac{1}{2}A(f) \ , \tag{4.2.7}$$

$$\lim_{n\to+\infty} n(M_n(f) - f) \ = \ A(f) \ , \tag{4.2.8}$$

which are satisfied for every $f \in C^2([0,1])$, where $A : C^2([0,1]) \to C([0,1])$ denotes the differential operator defined by

$$Au(x) := \frac{d}{dx}\left(x(1-x)\,u'(x)\right) \ , \qquad u \in C^2([0,1]) \ , \quad x \in [0,1] \ .$$

Now, let V be the subspace of $L^2(0,1)$ consisting of all linear functions on $[0,1]$ and its orthogonal subspace given by

$$W := \left\{ v \in L^2(0,1) \mid \int_0^1 (a+bt)\,v(t)\,dt = 0 \text{ for every } a,b \in \mathbb{R} \right\} \ ;$$

it is easy to recognize that

$$\begin{aligned} W \ &= \ \left\{ v \in L^2(0,1) \mid \int_0^1 v(t)\,dt = 0 \, , \int_0^1 t\,v(t)\,dt = 0 \right\} \\ &= \ \left\{ v \in L^2(0,1) \mid \int_0^1 t\,v(t)\,dt = 0 \, , \int_0^1 (1-t)\,v(t)\,dt = 0 \right\} \ ; \end{aligned}$$

moreover, P_V and P_W denote the orthogonal projections onto the subspaces V and respectively W.

According to the general procedure, we can define the new sequence $(L_n)_{n\ge1}$ of linear operators on $L^2(0,1)$ by setting

$$L_n f(x) := P_V(K_n(f))(x) + P_W(M_n(f))(x) \ , \quad f \in L^2(0,1) \ , \ x \in [0,1] \ . \tag{4.2.9}$$

In order to write a more explicit expression of the operators L_n, we consider the orthogonal basis of V consisting of the two functions $\mathbf{1}$ and $\mathbf{1} - 2\mathrm{id}$.

Using (4.1.5), for every $f \in L^2(0,1)$ and $x \in [0,1]$, we get

$$
\begin{aligned}
P_V(K_n(f))(x) &= \int_0^1 K_n f(t)\,dt + \frac{\int_0^1 (1-2t) K_n f(t)\,dt}{\int_0^1 (1-2t)^2\,dt}\,(1-2x) \\
&= (n+1)\sum_{k=0}^{n} \binom{n}{k} \frac{k!(n-k)!}{(n+1)!} \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds \\
&\quad +3(n+1)\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k!(n-k)!}{(n+1)!} - 2\frac{(k+1)!(n-k)!}{(n+2)!} \right) \\
&\quad \times \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\;(1-2x) \\
&= \int_0^1 f(s)\,ds + 3\sum_{k=0}^{n} \left(1 - 2\,\frac{k+1}{n+2}\right) \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\;(1-2x) \\
&= (4-6x)\int_0^1 f(s)\,ds - \frac{6}{n+2}\sum_{k=0}^{n}(k+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\;(1-2x)
\end{aligned}
$$

and consequently

$$
\begin{aligned}
P_W(M_n(f))(x) &= M_n f(x) - P_V(M_n(f))(x) \\
&= M_n f(x) - \int_0^1 M_n f(t)\,dt - \frac{\int_0^1 (1-2t) M_n f(t)\,dt}{\int_0^1 (1-2t)^2\,dt}\,(1-2x) \\
&= M_n f(x) - \sum_{k=0}^{n} \int_0^1 p_{n,k}(s) f(s)\,ds \\
&\quad -3\sum_{k=0}^{n} \left(1 - 2\,\frac{k+1}{n+2}\right) \int_0^1 p_{n,k}(s)\, f(s)\,ds\;(1-2x) \\
&= M_n f(x) - \int_0^1 f(s)\,ds - 3\sum_{k=0}^{n} \int_0^1 p_{n,k}(s)\, f(s)\,ds\;(1-2x) \\
&\quad +\frac{6}{n+2}\sum_{k=0}^{n}(k+1)\int_0^1 p_{n,k}(s) f(s)\,ds\;(1-2x) \\
&= M_n f(x) + (-4+6x)\int_0^1 f(s)\,ds + \frac{6}{n+2}\int_0^1 (ns+1) f(s)\,ds\;(1-2x)\,.
\end{aligned}
$$

Hence, from (4.2.9) we obtain

$$\begin{aligned}
L_n f(x) &= M_n f(x) \qquad\qquad (4.2.10)\\
&+\frac{6(1-2x)}{n+2}\left(\int_0^1 (ns+1)f(s)\,ds-\sum_{k=0}^{n}(k+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\right)\\
&= M_n f(x)+\frac{6n(1-2x)}{n+2}\sum_{k=0}^{n}\int_{k/(n+1)}^{(k+1)/(n+1)}\left(s-\frac{k}{n}\right)f(s)\,ds
\end{aligned}$$

for every $f \in L^2(0,1)$ and $x \in [0,1]$.

The convergence of $(L_n)_{n\geq 1}$ to the identity operator on $L^2(0,1)$ is ensured by (4.2.9) and the analogous properties of the sequences $(K_n)_{n\geq 1}$ and $(M_n)_{n\geq 1}$.

As regards to a quantitative estimate of the convergence, again from (4.2.9) and (4.2.5)–(4.2.6) we get, for every $f \in L^2(0,1)$,

$$\|L_n f - f\|_2 \leq 1496\,\tau\left(f, \frac{1}{\sqrt{n+1}}\right)_2 .$$

We explicitly observe that

$$\begin{aligned}
L_n\mathbf{1} &= \mathbf{1}\,,\\
L_n\mathrm{id}(x) &= \frac{nx+1}{n+2}+\frac{n(2x-1)}{2(n+1)(n+2)} = \frac{n}{n+1}\,x+\frac{1}{2(n+1)}\,,\\
L_n\mathrm{id}^2(x) &= \frac{n(n-1)x^2+4nx+2}{(n+2)(n+3)}+\frac{n(2x-1)}{2(n+1)(n+2)}\\
&= \frac{n(n-1)}{(n+2)(n+3)}\,x^2+\frac{n(5n+7)}{(n+1)(n+2)(n+3)}\,x\\
&\quad -\frac{n^2-n-4}{2(n+1)(n+2)(n+3)}\,.
\end{aligned}$$

Moreover, the following result establishes a Voronovskaja's formula for the sequence $(L_n)_{n\geq 1}$.

Theorem 4.2.1 *For every $f \in C^2([0,1])$, we have*

$$\lim_{n\to+\infty} n(L_n f(x) - f(x)) = Af(x) + 3(1-2x)\int_0^1 (1-2t)\,f(t)\,dt\,. \qquad (4.2.11)$$

uniformly with respect to $x \in [0,1]$.

Proof. Indeed, from (4.1.4) and (4.2.7)–(4.2.8) and using twice the integration by parts, for every $f \in C^2([0,1])$ and $x \in [0,1]$ we have

$$\begin{aligned}
\lim_{n\to+\infty} n(L_n f(x) - f(x)) &= P_V\left(\frac{1}{2}Af\right)(x) + P_W\,(Af)\,(x) \\
&= \frac{1}{2}\int_0^1 \big(t(1-t)\,f'(t)\big)'\,dt + \frac{3}{2}\,(1-2x)\int_0^1 (1-2t)\,\big(t(1-t)\,f'(t)\big)'\,dt \\
&\quad + Af(x) - \int_0^1 \big(t(1-t)\,f'(t)\big)'\,dt \\
&\quad - 3(1-2x)\int_0^1 (1-2t)\,\big(t(1-t)\,f'(t)\big)'\,dt \\
&= Af(x) - 3(1-2x)\int_0^1 t(1-t)\,f'(t)\,dt \\
&= Af(x) + 3(1-2x)\int_0^1 (1-2t)\,f(t)\,dt
\end{aligned}$$

and this completes the proof. □

Finally, we observe that the differential operator $B : C^2([0,1]) \to C([0,1])$ defined by

$$Bu(x) := Au(x) + 3(1-2x)\int_0^1 (1-2t)\,u(t)\,dt\;, \qquad u \in C^2([0,1])\;, \quad x \in [0,1]\;,$$

may be considered as a bounded perturbation of the operator A since

$$\int_0^1 \left(3(1-2x)\int_0^1 (1-2t)\,u(t)\,dt\right)^2 dx \le 9\left(\int_0^1 u(t)\,dt\right)^2 \le 9\|u\|_2^2\;;$$

hence $A - B$ is bounded and $\|A - B\| \le 3$.

It is well-known that the closure $(A, D(A))$ of $(A, C^2([0,1]))$ is defined on the domain

$$\begin{aligned}
D(A) := \{f \in L^2(0,1) \quad | \quad & f \text{ is locally absolutely continuous in }]0,1[\\
& \text{and } x(1-x)\,f'(x) \in W_0^{1,2}(0,1)\}\;,
\end{aligned}$$

and generates a C_0-semigroup $(T(t))_{t\ge0}$ of contraction on $L^2(0,1)$ which is analytic (with angle $\pi/2$) and immediately compact (see e.g. [1, Theorem 2.3]). From the classical perturbation theory of C_0-semigroup (see e.g. [48, Section III.1] or also [64, Section 3.1]) we conclude that also $(B, D(A))$ generates an analytic C_0-semigroup $(S(t))_{t\ge0}$ on $L^2(0,1)$ with angle $\pi/2$ on the same domain $D(A)$. From this it also follows that $C^2([0,1])$ is a core for $(B, D(A))$ and further

$$\|S(t)\| \le e^{\|A-B\|t}\|T(t)\| \le e^{3t}\;. \tag{4.2.12}$$

Moreover, in connection with the operators L_n we have the following representation of the semigroup $(S(t))_{t\ge0}$.

Theorem 4.2.2 *For every $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have*

$$\lim_{n\to+\infty} L_n^{k(n)} = S(t) \qquad \textit{strongly on } L^2(0,1) \ . \tag{4.2.13}$$

PROOF. Since $(B, D(A))$ generates a C_0-semigroup in $L^2(0,1)$ with growth bound ≤ 3, the range of $\lambda - B$ coincides with $L^2(0,1)$ for every $\lambda > 3$. Moreover, for every $n \geq 1$ we have

$$\begin{aligned}\|L_n(f) - M_n(f)\|_2^2 &\leq 36 \left(\sum_{k=0}^{n} \int_{k/(n+1)}^{(k+1)/(n+1)} \left(s - \frac{k}{n} \right) f(s)\, ds \right)^2 \\ &\leq \frac{36}{(n+1)^2} \left(\int_0^1 f(s)\, ds \right)^2 \leq \frac{36}{(n+1)^2} \|f\|_2^2\end{aligned}$$

and consequently $\|L_n\| \leq \|M_n\| + 6/(n+1) \leq 1 + 6/(n+1)$ which yields, for every $k \geq 1$,

$$\|L_n^k\| \leq \left(1 + \frac{6}{n+1}\right)^k = \left(\left(1 + \frac{6}{n+1}\right)^n\right)^{k/n} \leq e^{6k/n} \ .$$

Hence, the stability condition in Trotter's Theorem II.1.1 is satisfied and its application yields completely the proof. □

The preceding result ensures the possibility of approximating the solutions of the evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial u}{\partial x}\left(x(1-x)\, \dfrac{\partial u}{\partial x}(t,x) \right) + 3(1-2x) \displaystyle\int_0^1 (1-2s)\, u(t,s)\, ds \ , \\ \hfill t \geq 0 \ , \ x \in [0,1] \ , \\ u(0,x) = u_0(x) \ , \hfill u_0 \in L^2(0,1) \ , \end{cases} \tag{4.2.14}$$

using iterates of the operators L_n applied to the initial condition; namely, for every $t \geq 0$ and $x \in [0,1]$ we have

$$u(x,t) = S(t)u_0(x) = \lim_{n\to+\infty} L_n^{[nt]} u_0(x) \ ,$$

in the norm L^2 with respect to $x \in [0,1]$ and uniformly in compact intervals with respect to $t \geq 0$.

Quantitative estimates of the above convergence formulas can be obtained on suitable subspaces using the results in Chapter 1, provided that we have a quantitative versions of (4.2.7) and (4.2.8); for the sake of brevity we state it only for Bernstein-Durrmeyer operators, since the same methods can be applied to obtain a similar estimate for Bernstein-Kantorovich operators.

Lemma 4.2.3 *We have*

1. $M_n\mathbf{1}(x) - 1 = 0$,

2. $M_n(\mathrm{id} - x)(x) = \left(\dfrac{nx+1}{n+2} - x\right)$,

3. $M_n((\mathrm{id} - x)^2)(x) = \left(\dfrac{n(n-1)x^2 + 4nx + 2}{(n+2)(n+3)} - 2x\dfrac{nx+1}{n+2} + x^2\right)$,

4. $|M_n((\mathrm{id} - x)^4)(x)| \leq \dfrac{C}{n^2}$.

Proof. The statements *1*, *2* and *3* follow easily from (4.2.4).
Now a straightforward calculus gives, for every $m \geq 1$,

$$\begin{aligned} M_n(\mathrm{id}^m)(x) &= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 \binom{n}{k} t^k(1-t)^{n-k}t^m\,dt \\ &= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\binom{n}{k}\beta(k+m+1, n-k+1) \\ &= \sum_{k=0}^{n} p_{n,k}(x)\frac{(k+1)\cdots(k+m)}{(n+2)\cdots(n+m+1)} , \end{aligned}$$

where $\beta(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ is the Euler's beta function. So we have

$$\begin{aligned} M_n(\mathrm{id}^3)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\frac{(k+1)(k+2)(k+3)}{(n+2)(n+3)(n+4)} \\ &= \frac{n^3B_n(\mathrm{id}^3)(x) + 6n^2B_n(\mathrm{id}^2)(x) + 11nB_n(\mathrm{id})(x) + 6}{(n+2)(n+3)(n+4)} \\ &= \frac{nx(1+3x(n-1)+x^2(n-1)(n-2)) + 6nx(1+x(n-1)) + 11nx + 6}{(n+2)(n+3)(n+4)} \end{aligned}$$

and

$$\begin{aligned} M_n(\mathrm{id}^4)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\frac{(k+1)(k+2)(k+3)(k+4)}{(n+2)(n+3)(n+4)(n+5)} \\ &= \frac{n^4B_n(\mathrm{id}^4)(x) + 10n^3B_n(\mathrm{id}^3)(x) + 35n^2B_n(\mathrm{id}^2)(x) + 50nB_n(\mathrm{id})(x) + 24}{(n+2)(n+3)(n+4)(n+5)} \\ &= \frac{1}{(n+2)(n+3)(n+4)(n+5)}\times \\ &\quad \times \big[nx(1+7x(n-1)+6x^2(n-1)(n-2)+x^3(n-1)(n-2)(n-3)) + \\ &\quad 10nx(1+3x(n-1)+x^2(n-1)(n-2)) + 35nx(1+x(n-1)) + 50nx + 24\big] \end{aligned}$$

and consequently, using (4.2.4) we obtain

$$\begin{aligned}
&M_n((\mathrm{id}-x)^4)(x)\\
&= M_n(\mathrm{id}^4)(x) - 4xM_n(\mathrm{id}^3)(x) + 6x^2M_n(\mathrm{id}^2)(x) - 4x^3M_n(\mathrm{id})(x) + x^4\\
&= x^4\Big(1 - \frac{4\,n}{n+2} + \frac{6\,n(n-1)}{(n+2)(n+3)} - \frac{4\,n(n-1)(n-2)}{(n+2)(n+3)(n+4)}\\
&\quad + \frac{n(n-1)(n-2)(n-3)}{(n+2)(n+3)(n+4)(n+5)}\Big) + x^3\Big(-\frac{4}{n+2} + \frac{24\,n}{(n+2)(n+3)}\\
&\quad - \frac{36\,n(n-1)}{(n+2)(n+3)(n+4)} + \frac{16\,n(n-1)(n-2)}{(n+2)(n+3)(n+4)(n+5)}\Big)\\
&\quad + x^2\Big(\frac{12}{(n+2)(n+3)} - \frac{72\,n}{(n+2)(n+3)(n+4)}\\
&\quad + \frac{72n(n-1)}{(n+2)(n+3)(n+4)(n+5)}\Big) + x\Big(\frac{24}{(n+2)(n+3)(n+4)}\\
&\quad + \frac{96}{(n+2)(n+3)(n+4)(n+5)}\Big) + \frac{24}{(n+2)(n+3)(n+4)(n+5)}\\
&= \frac{12\,n^2}{(n+2)(n+3)(n+4)(n+5)}\,x^2(1-x)^2 + o\left(\frac{1}{n^2}\right)\\
&\le \frac{3}{4n^2} + \frac{C_1}{n^3} \le \frac{C}{n^2}\,.
\end{aligned}$$

□

Proposition 4.2.4 *Let $0 < \alpha \le 1$; then there exist $C_1, C_2 > 0$ such that, for every $f \in C^{2,\alpha}([0,1])$*

$$\|n(M_n(f) - f) - Af\| \quad \le \quad C\frac{M_f}{n^{\alpha/2}}\,,$$

where M_f is the seminorm defined by

$$M_f := \|f'\| + \|f''\| + L_{f''} \tag{4.2.15}$$

Proof. Let $f \in C^{2,\alpha}(\overline{\mathbb{R}})$ and let $\mathcal{A}_{M_n}$ be the operator defined by (2.2.2) taking $L = M_n$. From Lemma 4.2.3 we get

$$\begin{aligned}
\mathcal{A}_{M_n}f(x) &= f'(x)M_n(\mathrm{id}-x)(x) + \frac{1}{2}f''(x)M_n\left((\mathrm{id}-x)^2\right)(x)\\
&= f'(x)\left(\frac{nx+1}{n+2} - x\right)\\
&\quad + f''(x)\frac{1}{2}\left(\frac{n(n-1)x^2 + 4nx + 2}{(n+2)(n+3)} - 2x\frac{nx+1}{n+2} + x^2\right)\,.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&|n(M_nf(x) - f(x)) - Af(x)| \qquad (4.2.16)\\
&\quad \le |n(L_nf(x) - f(x) - \mathcal{A}_{M_n}f(x)| + |n\mathcal{A}_{M_n}f(x) - Af(x)|\,.
\end{aligned}$$

In regard to the first term in (4.2.16) we use Theorem 1.1.2 and taking into account Lemma 4.2.3, we obtain the existence of $C_1, C_2 > 0$ such that

$$M_n\left((\mathrm{id}-x)^2\right)(x) \le \frac{C_1}{n}\,,\quad M_n\left((\mathrm{id}-x)^4\right)(x) \le \frac{C_2}{n^2}\,.$$

Thus

$$|n\left(L_n f(x) - f(x) - \mathcal{A}_{M_n} f(x)\right)| \le n\frac{L_{f''}}{2}\left(\frac{C_1}{n}\right)^{\alpha/2}\left(\frac{C_1^2}{n^2}+\frac{C_2}{n^2}\right)^{1/2}\,.$$

As regards the second term in (4.2.16) we have

$$\begin{aligned}
&n\mathcal{A}_{M_n} f(x) - Af(x)\\
&\quad= f'(x)n\left(\frac{nx+1}{n+2}-x\right) + f''(x)\frac{n}{2}\left(\frac{n(n-1)x^2+4nx+2}{(n+2)(n+3)} - 2x\frac{nx+1}{n+2}+x^2\right)\\
&\qquad-(1-2x)f'(x) - x(1-x)f''(x)\\
&\quad= f'(x)2\frac{2x-1}{n+1} + f''(x)\frac{n(8x^2-8x+1)-6x(1-x)}{(n+2)(n+3)}\,,
\end{aligned}$$

and this yields

$$\|n\mathcal{A}_{M_n} f - Af\| \le \frac{16}{n}\|f''\| + \frac{6}{n}\|f'\|\,.$$

Finally collecting the above inequalities we obtain

$$\begin{aligned}
&|n(M_n f(x) - f(x)) - Af(x)|\\
&\quad\le n\frac{L_{f''}}{2}\left(\frac{C_1}{n}\right)^{\alpha/2}\left(\frac{C_1^2}{n^2}+\frac{C_2}{n^2}\right)^{1/2} + \frac{16}{n}\|f''\| + \frac{6}{n}\|f'\|\\
&\quad\le C\frac{M_f}{n^{\alpha/2}}\,,
\end{aligned}$$

where M_f is the seminorm defined by (4.2.15). □

A similar estimate holds for Bernstein-Kantorovich operators (with different constants) and the same estimates continue to hold for both operators with respect to the L^2-norm.

Hence, for every $f \in C^{2,\alpha}([0,1])$ and $n \ge 1$, we have

$$\|n(L_n(f)-f) - Bf\|_w \le \psi_n(f)\,,\qquad \|n(L_n(f)-f)\|_w \le \varphi_n(f)\,, \quad (4.2.17)$$

where

$$\psi_n(f) := C\frac{M_f}{n^{\alpha/2}}\,,\qquad \varphi_n(f) := \|B(f)\|_w + C\frac{M_f}{n^{\alpha/2}}\,.$$

From (4.2.12) the growth bound of the semigroup $(S(t))_{t\ge 0}$ is less or equal than 3 (and constant $M = 1$) and therefore, applying Theorem 1.1.2, we obtain the following result.

Proposition 4.2.5 *For every $t \geq 0$, $(k(n))_{n\geq 1}$ sequence of positive integers and $f \in C^{2,\alpha}([0,1])$, we have*

$$\begin{aligned}\left\|L_n^{k(n)}u - S(t)u\right\|_w &\leq t\,\exp(3\,e^{3/n}\,t)\,\psi_n(u) \qquad (4.2.18)\\ &+\left(\exp(3\,e^{3/n}\,t_n)\,\left|\frac{k(n)}{n}-t\right| + \sqrt{\frac{2}{\pi}}\,e^{3\,k(n)/n}\,\frac{\sqrt{k(n)}}{n}\right.\\ &\left.+\frac{3}{n}\,\frac{k(n)}{n}\,\exp\left(3\,e^{3/n}\,\frac{k(n)}{n}\right)\right)\,\varphi_n(u)\,,\end{aligned}$$

where $t_n := \sup\{t, k(n)/n\}$.

In particular, if we take $k(n) = [nt]$, we obviously have $t_n = t$ and $\left|\frac{[nt]}{n} - t\right| = \frac{nt}{n} - \frac{[nt]}{n} \leq \frac{1}{n}$. Hence (4.2.18) yields

$$\begin{aligned}\left\|L_n^{k(n)}u - S(t)u\right\|_2 &\leq t\,\exp(3\,e^{3/n}\,t)\,\psi_n(u) \qquad (4.2.19)\\ +\frac{1}{\sqrt{n}}&\left(\frac{\exp(3\,e^{3/n}\,t)}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\,e^{3\,t} + \frac{3\,t}{\sqrt{n}}\,\exp\left(3\,e^{3/n}\,t\right)\right)\,\varphi_n(u)\,.\end{aligned}$$

Of course the definition of $(L_n)_{n\geq 1}$ depends also on the decomposition of the space $L^2(0,1)$. Using different decompositions, we can describe the solution of different evolution problems in terms of iterates of suitable operators.

A different interesting example can be performed using the one-dimensional subspace X generated by the function $\mathrm{id}(\mathbf{1}-\mathrm{id})$ and its orthogonal subspace Y given by

$$Y := \left\{v \in L^2(0,1) \mid \int_0^1 t(1-t)\,v(t)\,dt = 0\right\}\,.$$

Taking the same sequences as before in this case we obtain the operator $(Q_n)_{n\geq 1}$ of linear operators on $L^2(0,1)$ by setting

$$Q_n f(x) := P_X(K_n(f))(x) + P_Y(M_n(f))(x)\,, \quad f \in L^2(0,1)\,,\; x \in [0,1]\,. \qquad (4.2.20)$$

Similarly to the preceding case, from (4.1.6) we obtain, for every $f \in L^2(0,1)$ and $x \in [0,1]$,

$$\begin{aligned}P_X(K_n(f))(x) &= \frac{\int_0^1 t(1-t)K_n f(t)\,dt}{\int_0^1 t^2(1-t)^2\,dt}\;x(1-x)\\ &= 30(n+1)\sum_{k=0}^n \binom{n}{k}\frac{(k+1)!(n-k+1)!}{(n+3)!}\int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\;x(1-x)\\ &= \frac{30}{(n+2)(n+3)}\sum_{k=0}^n (k+1)(n-k+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\;x(1-x)\end{aligned}$$

and consequently

$$\begin{aligned}
P_Y(M_n(f))(x) &= M_n f(x) - P_X(M_n(f))(x)\\
&= M_n f(x) - 30\int_0^1 t(1-t)M_n f(t)\,dt\ x(1-x)\\
&= M_n f(x) - \frac{30}{(n+2)(n+3)}\\
&\quad\times \sum_{k=0}^{n}(k+1)(n-k+1)\int_0^1 \binom{n}{k} s^k(1-s)^{n-k} f(s)\,ds\ x(1-x)\ .
\end{aligned}$$

Hence, from (4.2.20),

$$\begin{aligned}
Q_n f(x) = M_n f(x) - \frac{30}{(n+2)(n+3)}\sum_{k=0}^{n}(k+1)(n-k+1) \qquad (4.2.21)\\
\times\left(\int_0^1 \binom{n}{k} s^k(1-s)^{n-k} f(s)\,ds - \int_{k/(n+1)}^{(k+1)/(n+1)} f(s)\,ds\right) x(1-x)
\end{aligned}$$

for every $f \in L^2(0,1)$ and $x \in [0,1]$.

Clearly, the sequence $(Q_n)_{n\geq 1}$ converges to the identity operator on $L^2(0,1)$ and a quantitative estimate of the convergence can be obtained as before from (4.2.20) and (4.2.5)–(4.2.6).

A Voronovskaja's formula for the sequence $(Q_n)_{n\geq 1}$ can be also established using the same arguments of Theorem 4.2.1 and yields, for every $f \in C^2([0,1])$

$$\lim_{n\to+\infty} n(L_n f(x) - f(x)) = Af(x) - 15\,x(1-x)\int_0^1 (6t^2-6t+1)\,f(t)\,dt\ . \qquad (4.2.22)$$

uniformly with respect to $x \in [0,1]$.

Finally, the differential operator arising from the (4.2.22) is again a bounded perturbation of the operator A and consequently its closure generates an analytic C_0-semigroup $(Q(t))_{t\geq 0}$ in $L^2(0,1)$ with angle $\pi/2$ on the same domain $D(A)$. The semigroup $(Q(t))_{t\geq 0}$ can be represented as

$$\lim_{n\to+\infty} Q_n^{k(n)} = Q(t) \qquad \text{strongly on } L^2(0,1)\ .$$

whenever $t \geq 0$ and $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$.

Hence, even in this case we have the possibility of approximating the solutions of the associated evolution problem using iterates of the operators Q_n evaluated at the initial point.

Remark 4.2.6 It is worthwhile mentioning that if we consider the identity operator in place of one of the preceding sequences we obtain the operators considered in [31] in connection with a best approximation property with respect to a linear operator.

Hence, the problem considered in [31] in the one-dimensional setting can be completely framed in the more general setting considered here.

In the following section we give an example of such situation by considering the case of convolution operators. □

4.3 Best perturbation of Jackson convolution operators

In this section we consider a perturbation of the classical Jackson convolution operators obtained by imposing a best approximation property on the subspace of all trigonometric polynomials having degree less or equal to 2. Since the treatment of this case is very similar to the preceding one, we shall omit several details and we shall only describe the main steps.

We consider the space $L^2_{2\pi}$ of all real 2π-periodic functions which are square summable on the interval $[-\pi,\pi]$ endowed with the usual scalar product

$$\langle f,g\rangle_{2\pi} := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx\ , \qquad f,g\in L^2_{2\pi}\ .$$

For every $n\geq 1$, we recall that the n-th Jackson operator $J_n : L^2_{2\pi}\to L^2_{2\pi}$ is defined by setting, for every $f\in L^2_{2\pi}$ and $x\in\mathbb{R}$,

$$J_n f(x) := \frac{3}{2\pi n(2n^2+1)}\int_{-\pi}^{\pi} f(x-t)\,\frac{\sin^4 n\,t/2}{\sin^4 t/2}\,dt\ . \tag{4.3.1}$$

It is well-known that $J_n(f)$ is a trigonometric polynomial of degree $2n-2$ and the following estimate is satisfied for every $f\in L^2_{2\pi}$ (see [63, pp. 79–84], [21, p. 60] and also [9, (5.4.45)])

$$\|J_n(f)-f\|_{2\pi} \leq (1+\pi)\ \omega^{(2)}\left(f,\frac{1}{n+1}\right)\ ,$$

where $\omega^{(2)}(f,\delta) := \sup_{|h|\leq\delta}\|f(\cdot+h)-f\|_{2\pi}$.

We consider the subspace V of $L^2_{2\pi}$ generated by the trigonometric polynomials with degree less or equal to 2 and its orthogonal subspace W. If P_V and P_W denote the orthogonal projection onto the subspaces V and respectively W, we can define the sequence $(H_n)_{n\geq 1}$ of linear operators on $L^2_{2\pi}$ by setting

$$H_n f := P_V(f) + P_W(J_n(f)) = J_n(f) + P_V(f - J_n(f))\ , \quad f\in L^2_{2\pi}\ . \tag{4.3.2}$$

Taking into account that Jackson convolution operators preserve the trigonometric polynomials having degree less or equal to 1, from (4.1.5) we get, for every $f\in L^2_{2\pi}$ and $x\in\mathbb{R}$,

$$\begin{aligned} H_n f(x) &= J_n f(x) - \frac{\cos 2x}{\pi}\int_{-\pi}^{\pi}(J_n f(t)-f(t))\cos 2t\,dt \qquad (4.3.3)\\ &\quad -\frac{\sin 2x}{\pi}\int_{-\pi}^{\pi}(J_n f(t)-f(t))\sin 2t\,dt \end{aligned}$$

for every $f\in L^2_{2\pi}$ and $x\in\mathbb{R}$.

It is clear from (4.3.3) that $(H_n)_{n\in\mathbb{N}}$ converges to the identity operator on $L^2_{2\pi}$; moreover, for every $f \in L^2_{2\pi}$,

$$\|H_n f - f\|_{2\pi} \le (1+\pi)\,\omega^{(2)}\left(f, \frac{1}{n+1}\right)$$

since

$$H_n f - f = P_V(f) + P_W(J_n(f)) - (P_V(f) + P_W(f)) = P_W(J_n(f) - f)\,.$$

Since the Jackson convolution operators preserves the trigonometric polynomials of degree less or equal to 1, the same happens for the operators H_n; moreover, by definition H_n also preserves all trigonometric polynomials having degree less or equal to 2.

Finally, we recall that Jackson convolution operators satisfy the following Voronovskaja-type formula, for every $f \in C^1_{2\pi}$

$$\lim_{n\to+\infty} n(J_n(f) - f) = \frac{\sqrt{3}}{2}\,\pi\, f'\,, \tag{4.3.4}$$

(see also [21] and [9, 365–369 and 357]).

Consequently, the operators H_n satisfy the following Voronovskaja-type formula, for every $f \in C^1_{2\pi}$,

$$\begin{aligned}
&\lim_{n\to+\infty} n(H_n(f) - f)\\
&\quad= \frac{\sqrt{3}}{2}\,\pi\left(f' - \frac{\cos 2\mathrm{id}}{\pi}\int_{-\pi}^{\pi} f'(t)\,\cos 2t\,dt - \frac{\sin 2\mathrm{id}}{\pi}\int_{-\pi}^{\pi} f'(t)\,\sin 2t\,dt\right)\\
&\quad= \frac{\sqrt{3}}{2}\,\pi\, f' - \sqrt{3}\cos 2\mathrm{id}\int_{-\pi}^{\pi} f(t)\sin 2t\,dt + \sqrt{3}\sin 2\mathrm{id}\int_{-\pi}^{\pi} f(t)\cos 2t\,dt
\end{aligned}$$

In this case, if we denote by C the differential operator arising from the preceding Voronovskaja's formula, we can also point out that the closure of $(C^2, C^1(\mathbb{R}))$ generates a cosine function $(C(t))_{t\in\mathbb{R}}$ on $L^2_{2\pi}$ and every $C(t)$ is the strong limit of iterates of the operators H_n (see [35, Theorem 1.2] or Chapter 5 for more details).

Chapter 5

Quantitative approximation of cosine functions

In the preceding chapters we have studied the possibility of approximating the solutions of a suitable parabolic problems using semigroup's theory. A similar approach can be used for the representation of the solutions of suitable hyperbolic problems using the generation of a cosine function (see [50] and [68] for more details on this approach).

Here we are interested to a cosine version of Trotter's theorem on the approximation of C_0-semigroups and we give a general quantitative estimate of the convergence of the iterates of a sequence of trigonometric polynomials. Moreover we introduce some suitable sequences of linear operators approximating the resolvent operators associated with the generator of the cosine functions. Some applications to particular sequences of classical trigonometric polynomials are also furnished.

The results in this chapter are collected in [43].

5.1 Approximation processes for cosine functions

First, we establish a cosine version of Trotter's approximation theorem [70, Theorem 5.3] and provide a quantitative estimate of the convergence. A partial result on the generation of cosine functions is also stated in [35, Theorem 1.2] without quantitative estimates.

Theorem 5.1.1 *Let E be a Banach space, let $(L_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ be two sequences of linear operators from E in itself and assume that there exists $M \geq 1$ and $\omega \geq 0$ such that*

$$\|L_n^k\| \leq M\, e^{\omega\, k/n}\ , \quad \|M_n^k\| \leq M\, e^{\omega\, k/n}\ , \qquad n, k \geq 1\ . \tag{5.1.1}$$

Moreover, assume that D is a dense subspace of E such that, for every

$u \in D$ and $n \geq 1$, we have

$$\|n(L_n u - u)\| \leq \varphi_n(u) \ , \qquad \|n(M_n u - u)\| \leq \varphi_n(u) \ , \tag{5.1.2}$$

and the following estimates of the Voronovskaja-type formula hold

$$\|n(L_n u - u) - Au\| \leq \psi_n(u) \ , \qquad \|n(M_n u - u) + Au\| \leq \psi_n(u) \ , \tag{5.1.3}$$

where $A : D \to E$ is a linear operator on E and $\varphi_n, \psi_n : D \to [0, +\infty[$ are seminorms on the subspace D such that $\lim_{n\to\infty} \psi_n(u) = 0$ for every $u \in D$.

If $(\lambda - A)(D)$ is dense in E for some $\lambda > \omega$, then the square A^2 of the closure of (A, D) generates a cosine function $(C(t))_{t\in\mathbb{R}}$ in E and, for every $t \geq 0$,

$$C(t) = \frac{1}{2} \lim_{n\to\infty} \left(L_n^{k(n)} + M_n^{k(n)} \right) \ , \tag{5.1.4}$$

where $(k(n)_n)_{n\in\mathbb{N}}$ is a sequence of positive integers such that $\lim_{n\to+\infty} k(n)/n = t$ (in particular, we can take $k(n) = [n\,t]$). Consequently, for every $t \in \mathbb{R}$, we have $\|C(t)\| \leq M\, e^{\omega\,|t|}$.

Moreover, for every $t \geq 0$ and for every increasing sequence $(k(n))_{n\geq 1}$ of positive integers and $u \in D$, we have

$$\begin{aligned} \left\| C(t)u - \frac{1}{2}\left(L_n^{k(n)} u + M_n^{k(n)} u \right) \right\| &\leq M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u) \qquad (5.1.5) \\ &+ M \left(\exp(\omega\, e^{\omega/n}\, t_n) \left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\, e^{\omega\, k(n)/n}\, \frac{\sqrt{k(n)}}{n} \right. \\ &\left. + \frac{\omega}{n} \frac{k(n)}{n} \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n} \right) \right) \varphi_n(u) \end{aligned}$$

where $t_n := \sup\{t, k(n)/n\}$.

Proof. From the classical Trotter's Theorem II.1.1 it follows that the closure of the operators A and $-A$ generate a C_0-semigroup $(T_+(t))_{t\geq 0}$ and respectively $(T_-(t))_{t\geq 0}$ in E. Consequently, the closure of A generates a C_0-group $(G(t))_{t\in\mathbb{R}}$ in E and, for every $t \geq 0$,

$$G(t) = T_+(t) \ , \qquad G(-t) = T_-(t) \ .$$

Moreover, again from Trotter's Theorem, we obtain the representation of the group $(G(t))_{t\in\mathbb{R}}$ in terms of iterates of the operators L_n and M_n; indeed, for every $t \geq 0$ and for every sequence $(k(n)_n)_{n\in\mathbb{N}}$ of positive integers such that $\lim_{n\to+\infty} k(n)/n = t$, we have

$$G(t) = \lim_{n\to+\infty} L_n^{k(n)} \ , \qquad G(-t) = \lim_{n\to+\infty} M_n^{k(n)} \ .$$

Consequently, it follows that the square of the closure of (A, D) generates a cosine function $(C(t))_{t\in\mathbb{R}}$ in E (see [17, Example 3.14.15, p. 217]) and, for every $t \in \mathbb{R}$, $C(t) = (G(t) + G(-t))/2$. Hence the representation of the cosine function is a consequence of the representation of $(G(t))_{t\in\mathbb{R}}$ and the estimate $\|C(|t|)\| \le M\, e^{\omega\, t}$ follows from (5.1.1) and (5.1.4).

Finally, we show the validity of (5.1.5).

Let $t \ge 0$, $(k(n))_{n\ge 1}$ an increasing sequence of positive integers and $u \in D$. From Theorem 1.1.2 we get

$$\begin{aligned}\left\|T_+(t)u - L_n^{k(n)}u\right\| &\le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u)\\ &+M\left(\exp(\omega\, e^{\omega/n}\, t_n)\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\, e^{\omega\, k(n)/n}\, \frac{\sqrt{k(n)}}{n}\right.\\ &\left.+\frac{\omega}{n}\,\frac{k(n)}{n}\, \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n}\right)\right)\, \varphi_n(u)\end{aligned}$$

and

$$\begin{aligned}\left\|T_-(t)u - M_n^{k(n)}u\right\| &\le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u)\\ &+M\left(\exp(\omega\, e^{\omega/n}\, t_n)\left|\frac{k(n)}{n} - t\right| + \sqrt{\frac{2}{\pi}}\, e^{\omega\, k(n)/n}\, \frac{\sqrt{k(n)}}{n}\right.\\ &\left.+\frac{\omega}{n}\,\frac{k(n)}{n}\, \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n}\right)\right)\, \varphi_n(u)\ .\end{aligned}$$

Taking into account that

$$\begin{aligned}&\left\|C(t)u - \frac{1}{2}\left(L_n^{k(n)}u + M_n^{k(n)}u\right)\right\|\\ &\qquad = \frac{1}{2}\left\|T_+(t)u + T_-(t)u - L_n^{k(n)}u - M_n^{k(n)}u\right\|\\ &\qquad \le \frac{1}{2}\left(\left\|T_+(t)u - L_n^{k(n)}u\right\| + \left\|T_-(t)u - M_n^{k(n)}u\right\|\right)\end{aligned}$$

from the preceding inequalities the proof is completed. □

Remark 5.1.2 In many applications it is natural to consider the sequence $k(n) = [nt]$ for which $t_n = t$ and $|[nt]/n - t| = nt/n - [nt]/n \le 1/n$. Hence estimate (5.1.5) yields

$$\begin{aligned}\left\|C(t)u - \frac{1}{2}\left(L_n^{[nt]}u + M_n^{[nt]}u\right)\right\| &\le M^2\, t\, \exp(\omega\, e^{\omega/n}\, t)\, \psi_n(u) \qquad (5.1.6)\\ +\frac{M}{\sqrt{n}}\left(\frac{\exp(\omega\, e^{\omega/n}\, t)}{\sqrt{n}} + \sqrt{\frac{2t}{\pi}}\, e^{\omega\, t} + \frac{\omega\, t}{\sqrt{n}}\, \exp\left(\omega\, e^{\omega/n}\, t\right)\right)&\varphi_n(u)\ .\end{aligned}$$

□

From the classical theory of the cosine functions (see [68] and [50, Chapter II] for more details) we have that the unique solution of the following second-order Cauchy problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = A^2 u(t,x) \; , & t \in \mathbb{R} \; ; \\ u(0,x) = u_0(x) \; , & x \in \mathbb{R} \; ; \\ \dfrac{\partial}{\partial t} u(t,x)|_{t=0} = u_1(x) \; , & x \in \mathbb{R} \; , \end{cases} \tag{5.1.7}$$

with $u_0, \; u_1 \in D$, is given by

$$\begin{aligned} u(t,x) &= C(t)u_0(x) + \int_0^t C(v)u_1(x)\, dv \qquad (5.1.8) \\ &= \frac{1}{2} \lim_{n\to\infty} \left(L_n^{[n\,t]} u_0 + M_n^{[n\,t]} u_0 + \int_0^t \left(L_n^{[n\,v]} u_1 + M_n^{[n\,v]} u_1 \right) dv \right) \; , \end{aligned}$$

for every $t \in \mathbb{R}$ and $x \in \mathbb{R}$ (observe that the sequences $(L_n^{[n\,v]} u_1)_{n\geq 1}$ and $(M_n^{[n\,v]} u_1)_{n\geq 1}$ are equibounded for $v \in [0,t]$ and this makes possible to apply the Lebesgue dominated convergence theorem).

Remark 5.1.3 Observe that if (5.1.2) and (5.1.3) hold in a dense subspaces of D, we obtain the validity of (5.1.5) in the same subspace, provided that a quantitative Voronovskaja's formula is satisfied on the larger subspace D. $\square$

5.2 Quantitative estimate of the resolvent

The next result is concerned with the approximation of the resolvent operator of the generator A^2, which will be denoted by $R(\lambda^2, A^2)$ for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$.

For the sake of simplicity we shall assume that E is a complex Banach space, otherwise we can replace it with its complexification.

We recall that (see [50, p. 30])

$$R(\lambda^2, A^2)u := \frac{1}{\lambda}\int_0^{+\infty} e^{-\lambda\, t} C(t)u\, dt\,, \qquad u \in E\,. \tag{5.2.1}$$

Hence, for every $n \geq 1$ we can define the linear operator $R_{\lambda,n} : E \to E$ as follows

$$R_{\lambda,n}u := \frac{1}{2\lambda}\int_0^{+\infty} e^{-\lambda\, t}\left(L_n^{[n\, t]}u + M_n^{[n\, t]}u\right)\, dt\,, \qquad u \in E\,.$$

We have the following quantitative estimate on the approximation of the resolvent operator.

Theorem 5.2.1 *Consider the same assumptions of Theorem 5.1.1. Then for every $n \geq 1$, $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega\, e^{\omega/n}$ and $u \in D$, we have*

$$\begin{aligned} \|R(\lambda^2, A^2)u - R_{\lambda,n}u\| \leq{}& \frac{M^2}{\operatorname{Re}\lambda(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2}\,\psi_n(u) \\ &+ \frac{M}{\operatorname{Re}\lambda\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})}\right. \\ &\left.+\frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda - \omega)^{3/2}} + \frac{\omega}{\sqrt{n}\,(\operatorname{Re}\lambda - \omega\, e^{\omega/n})^2}\right)\varphi_n(u)\,. \end{aligned} \tag{5.2.2}$$

In particular, the sequence $(R_{\lambda,n})_{n\geq 1}$ strongly converges to $R(\lambda^2, A^2)$.

Proof. As in the proof of Theorem 5.1.1, consider the C_0-semigroup $(T_+(t))_{t\geq 0}$ generated by the closure of A and the C_0-semigroup $(T_-(t))_{t\geq 0}$ generated by he closure of $-A$ and denote by $R(\lambda, A)$ and $R(\lambda, -A)$ their resolvent operators which satisfy, for every $u \in E$ and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \omega$,

$$R(\lambda, A)u = \int_0^{+\infty} e^{-\lambda\, t} T_+(t)u\, dt\,, \qquad R(\lambda, -A)u = \int_0^{+\infty} e^{-\lambda\, t} T_-(t)u\, dt\,,$$

moreover

$$R(\lambda^2, A^2) = \frac{1}{\lambda}\int_0^{\infty} e^{-\lambda t}\frac{1}{2}\left(T_+(t) + T_-(t)\right) = \frac{1}{2\lambda}\left(R(\lambda, A) + R(\lambda, -A)\right)\,. \tag{5.2.3}$$

Then, we can apply Theorem 1.2.1 to the resolvent operators $R(\lambda, A)$ and $R(\lambda, -A)$ and taking into account the definition of the operators $R_{\lambda,n}$ we obtain completely the proof. □

5.3 Approximation processes for resolvent operators

In this section we introduce some sequences of linear operators which approximate the resolvent operators associated with the generator of a cosine function. The main aim is the possibility of describing the resolvent operators in terms of classical convolution approximation processes.

Let $(a_n)_{n\geq 1}$ be a sequence of positive integers tending to $+\infty$ and for every $n \geq 1$ consider the linear operator $P_{\lambda,a_n,n} : E \to E$ defined by

$$P_{\lambda,a_n,n}u := \frac{1}{2\lambda n}\sum_{k=0}^{a_n} e^{-\lambda k/n}(L_n^k u + M_n^k u) , \qquad u \in E . \tag{5.3.1}$$

Theorem 5.3.1 *If the sequence* $(a_n)_{n\geq 1}$ *satisfies*

$$\lim_{n\to+\infty} \frac{a_n}{n} = +\infty , \tag{5.3.2}$$

then $\lim_{n\to+\infty} P_{\lambda,a_n,n}u = R(\lambda^2, A^2)u$ *for every* $u \in E$.

Proof. First, we observe that the closure of (A, D) generates a C_0-semigroup $(T_+(t))_{t\geq 0}$ in E satisfying $\|T_+(t)\| \leq M\, e^{\omega t}$ and $\lim_{n\to+\infty} L_n^{[nt]} = T_+(t)$ strongly on E for every $t \geq 0$; analogously the closure of $(-A, D)$ generates a C_0-semigroup $(T_-(t))_{t\geq 0}$ in E satisfying $\|T_-(t)\| \leq M\, e^{\omega t}$ and $\lim_{n\to+\infty} M_n^{[nt]} = T_-(t)$ strongly on E (see Theorem II.1.1).

Since $\mathrm{Re}\,\lambda > \omega$ we have $\left\|e^{-\lambda/n}L_n\right\| \leq Me^{-(\mathrm{Re}\,\lambda-\omega)/n}$ and $\left\|e^{-\lambda/n}M_n\right\| \leq Me^{-(\mathrm{Re}\,\lambda-\omega)/n}$ and consequently $\|P_{\lambda,a_n,n}\| \leq M/\left(\mathrm{Re}\,\lambda(1-e^{-(\mathrm{Re}\,\lambda-\omega)})\right)$. This shows that $(P_{\lambda,a_n,n})_{n\geq 1}$ is equibounded and we can establish the convergence property on the dense subspace D. Let $u \in D$; we have

$$\left\|P_{\lambda,a_n,n}u - R(\lambda^2, A^2)u\right\| \leq \|P_{\lambda,a_n,n}u - R_{\lambda,n}u\| + \left\|R_{\lambda,n}u - R(\lambda^2, A^2)u\right\| . \tag{5.3.3}$$

The second term converges to zero from Theorem 5.2.1. As regards to the first term we preliminary observe that

$$R_{\lambda,n}u = \frac{1}{2\lambda n}\sum_{k=0}^{\infty} e^{-\lambda k/n}\frac{1-e^{\lambda/n}}{\lambda/n}\left(L_n^k + M_n^k\right) ,$$

since L_n^k and M_n^k are constant on each interval $[k/n,(k+1)/n[$. Hence

$$\begin{aligned}
&\|P_{\lambda,a_n,n}u - R_{\lambda,n}u\| \qquad (5.3.4)\\
&\quad = \Bigg\| \frac{1}{2\lambda n} \\
&\qquad \times \left(\sum_{k=0}^{a_n} e^{-\lambda k/n}(L_n^k u + M_n^k u) - \sum_{k=0}^{\infty} e^{-\lambda k/n}\, \frac{1-e^{-\lambda/n}}{\lambda/n}\,(L_n^k u + M_n^k u)\right)\Bigg\| \\
&\quad \le \frac{M}{\operatorname{Re}\lambda\, n}\,\|u\| \sum_{k=a_n+1}^{+\infty} e^{-(\operatorname{Re}\lambda-\omega)k/n} \\
&\qquad + \frac{M}{\operatorname{Re}\lambda\, n}\|u\| \left|1-\frac{1-e^{-\lambda/n}}{\lambda/n}\right| \sum_{k=0}^{+\infty} e^{-(\operatorname{Re}\lambda-\omega)k/n} \\
&\quad \le \frac{M}{\operatorname{Re}\lambda\, n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right)}\,\|u\| \\
&\qquad \times \left(e^{-(\operatorname{Re}\lambda-\omega)(a_n+1)/n} + \left|1-\frac{1-e^{-\lambda/n}}{\lambda/n}\right|\right)\ ;
\end{aligned}$$

since $\lim_{n\to+\infty} n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right) = \operatorname{Re}\lambda-\omega$, the assumption (5.3.2) ensures that the first term in (5.3.3) tends to 0. □

Our next aim is to provide a quantitative estimate of the convergence in Theorem 5.3.1.

Theorem 5.3.2 *Assume that* (5.1.2) *and* (5.1.3) *hold.*

Then, for every $n > \omega/\log(\operatorname{Re}\lambda/\omega)$ *(take* $n \ge 1$ *if* $\omega = 0$*) and* $u \in D$*, we have*

$$\begin{aligned}
\|P_{\lambda,a_n,n}u - R(\lambda^2,A^2)u\| &\le \frac{M^2}{\operatorname{Re}\lambda(\operatorname{Re}\lambda-\omega\, e^{\omega/n})^2}\,\psi_n(u) \\
&+\frac{M}{\operatorname{Re}\lambda\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda-\omega\, e^{\omega/n})}\right. \\
&+\left.\frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda-\omega)^{3/2}} + \frac{\omega}{\sqrt{n}\,(\operatorname{Re}\lambda-\omega\, e^{\omega/n})^2}\right)\,\varphi_n(u) \\
&+\frac{M\left(e^{-(\operatorname{Re}\lambda-\omega)a_n/n} + \frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}\right)}{\operatorname{Re}\lambda(\operatorname{Re}\lambda-\omega)\left(1-\frac{\operatorname{Re}\lambda-\omega}{n}\right)}\,\|u\|\ . \qquad (5.3.5)
\end{aligned}$$

Proof. We estimate the two terms at the righthand side of (5.3.3). The estimate of the second term is provided by Theorem 5.2.1 and we have only to estimate the first term.

To this end, we use (1.2.11) and (1.2.12)

$$\left|1-\frac{1-e^{-\lambda/n}}{\lambda/n}\right| \le \frac{|\lambda|^{3/2}}{n\left|\operatorname{Re}\sqrt{\lambda}\right|}\ ,$$

$$\frac{1}{n\left(1-e^{-(\operatorname{Re}\lambda-\omega)/n}\right)} \le \frac{1}{(\operatorname{Re}\lambda-\omega)\,(1-(\operatorname{Re}\lambda-\omega)/n)}\ ,$$

and from (5.3.4) we get the following estimate of the first term in (5.3.3)

$$\|P_{\lambda,a_n,n}u - R_{\lambda,n}u\| \le \frac{M\left(e^{-(\operatorname{Re}\lambda-\omega)a_n/n} + \frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}\right)}{\operatorname{Re}\lambda(\operatorname{Re}\lambda-\omega)\,(1-(\operatorname{Re}\lambda-\omega)/n)}\,\|u\|\ , \qquad (5.3.6)$$

which completes the proof of (5.3.5). □

Taking $a_n \ge [n\log n/\operatorname{Re}\lambda]$, estimate (5.3.5) becomes

$$\|P_{\lambda,a_n,n}u - R(\lambda,A)u\| \le C_1(\lambda)\,\psi_n(u) + \frac{C_2(\lambda)}{\sqrt{n}}\,\varphi_n(u) + \frac{C_3(\lambda)}{n}\,\|u\|\ , \quad (5.3.7)$$

for every $\omega \ge 0$, $u \in D$ and $n > \omega/\log(\operatorname{Re}\lambda/\omega)$, where $C_i(\lambda)$, $i = 1, 2, 3$, are suitable constants depending only on λ.

5.4 Applications to Rogosinski operators

Denote by $C_{2\pi}$ the space of all 2π-periodic continuous real functions on $\mathbb{R}$ and put $\Pi := \{\pi + 2k\pi \mid k \in \mathbb{Z}\}$. Moreover, let $a \in C_{2\pi} \cap C^1(\mathbb{R} \setminus \Pi)$ be such that $a \neq 0$ in $]-\pi, \pi[$ and consider the first-order differential operator $(A, D(A))$ defined by

$$Au := au', \qquad u \in D(A) := \left\{u \in C_{2\pi} \cap C^1(\mathbb{R} \setminus \Pi) \mid Au \in C_{2\pi}\right\} .$$

In order to consider the generation of cosine functions, we also consider the operator A^2 on the following domain

$$D(A^2) := \left\{u \in C_{2\pi} \cap C^2(]-\pi, \pi[) \mid a(au')' \in C_{2\pi}\right\} .$$

It is well-known (see e.g. [35, Theorem 1.1]) that $(A^2, D(A^2))$ generates a cosine functions $(C(t))_{t\geq 0}$ in $C_{2\pi}$ if and only if

$$\frac{1}{a} \in L^1(-\pi, 0) , \qquad \frac{1}{a} \in L^1(0, \pi) \tag{5.4.1}$$

or alternatively

$$\frac{1}{a} \notin L^1(-\pi, 0) , \qquad \frac{1}{a} \notin L^1(0, \pi) . \tag{5.4.2}$$

Now, we consider the Rogosinski kernel defined by setting, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$r_n(x) := 1 + 2\sum_{k=1}^{n} \cos\left(\frac{k\pi}{2n+1}\right) \cos(kx) ,$$

and the corresponding n-th *Rogosinski operator* $R_n : C_{2\pi} \to C_{2\pi}$ given by

$$R_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-v) r_n(v)\, dv , \qquad f \in C_{2\pi} ,\ x \in \mathbb{R} .$$

The n-th generalized Rogosinski operator $R_{a,n} : C_{2\pi} \to C_{2\pi}$ introduced in [35] is defined by putting

$$R_{a,n} f(x) = R_n f\left(x + \frac{2\pi}{2n+1}\, a(x)\right) , \qquad f \in C_{2\pi} , \quad x \in \mathbb{R} .$$

From [35, Theorem 2.1] the sequence $(\|R_{a,n}\|)_{n\in\mathbb{N}}$ is equibounded and moreover $\|R_{a,n}^k\| \leq 2\pi$ for every $n, k \geq 1$. Further, there exists a positive constant $C > 0$ such that

$$\|R_{a,n} f - f\| \leq C\ \omega\left(f; \frac{1}{n}\right) , \qquad f \in C_{2\pi} . \tag{5.4.3}$$

Our next aim is to establish a quantitative estimate in order to apply Theorems 5.1.1 and 5.3.2.

Lemma 5.4.1 *Let $0 < \alpha \leq 1$. Then, for every $f \in C^{1,\alpha}_{2\pi}$,*

$$\left\| \frac{2n+1}{2\pi} (R_{a,n} f - f) - Af \right\| \leq 24 \, (\|a\| + 1) \left(\frac{2\pi}{2n+1} \right)^{\alpha} .$$

Proof. For every $f \in C^{1,\alpha}_{2\pi}$ we have

$$\left\| \frac{2n+1}{2\pi} (R_{a,n} f - f) - Af \right\| = \left\| \frac{2n+1}{2\pi} (R_{a,n} f - R_n f) - Af \right\| + \\ + \left\| \frac{2n+1}{2\pi} (R_n f - f) \right\| . \qquad (5.4.4)$$

As regards to the first term at the righthand side of (5.4.4), from Lagrange's theorem we can write

$$f(y+t) - f(y) = f'(y) t + \big(f'(\xi) - f'(y)\big) \, t \, , \qquad y, t \in \mathbb{R}$$

where $\xi \in]y, y+t[$. For every $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} &\frac{2n+1}{2\pi} (R_{a,n} f(x) - R_n f(x)) - a(x) f'(x) \\ &= \frac{2n+1}{2\pi} \, \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f\left(x - v + \frac{2\pi}{2n+1} \, a(x) \right) - f(x-v) \right) r_n(v) \, dv \\ &\quad -a(x) f'(x) \\ &= \frac{2n+1}{2\pi} \, \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x-v) \frac{2\pi}{2n+1} \, a(x) \, r_n(v) dv - a(x) f'(x) \\ &\quad + \frac{2n+1}{2\pi} \, \frac{1}{2\pi} \int_{-\pi}^{\pi} \big(f'(\xi) - f'(x-v)\big) \, \frac{2\pi}{2n+1} a(x) \, r_n(v) \, dv \\ &= a(x) (R_n f'(x) - f'(x)) + a(x) \frac{1}{2\pi} \int_{-\pi}^{\pi} \big(f'(\xi) - f'(x-v)\big) \, r_n(v) \, dv \, , \end{aligned}$$

where $\xi \in]x - v, x - v + 2\pi a(x)/(2n+1)[$.

We recall that (see e.g. [21, Theorem 2.4.8, pag 106])

$$\|R_n g - g\| \leq (2\pi + 1) E_n(g) + 4 \, \omega \left(g; \frac{1}{n} \right) \, , \qquad g \in C_{2\pi} \, ,$$

where $E_n(g)$ is the best approximation of the function g by trigonometric polynomials of degree n and hence, from the classical Jackson Theorem,

$$\|R_n g - g\| \leq 6(2\pi + 1) \, \omega \left(g; \frac{1}{n} \right) + 4 \, \omega \left(g; \frac{1}{n} \right) \leq (12\pi + 10) \omega \left(g; \frac{1}{n} \right) .$$

Applying the above inequality to f' and f we get

$$\begin{aligned} &\left| \frac{2n+1}{2\pi} (R_{a,n} f(x) - R_n f(x)) - a(x) f'(x) \right| \leq \\ &\leq \|a\| \left((12\pi + 10) \omega \left(f'; \frac{1}{n} \right) + \omega \left(f'; \frac{2\pi}{2n+1} \right) \right) \\ &\leq \|a\| (12\pi + 11) \, \omega \left(f'; \frac{2\pi}{2n+1} \right) \, , \end{aligned}$$

and consequently

$$\left\| \frac{2n+1}{2\pi} (R_{a,n}f - f) - Af \right\| \le \|a\|(12\pi + 11)\omega \left(f'; \frac{2\pi}{2n+1} \right)$$
$$+ \frac{2n+1}{2\pi}(12\pi + 10)\omega \left(f; \frac{1}{n} \right) .$$

Since $f \in C^{1,\alpha}_{2\pi}$ we have $\omega(f, \delta) \le \delta^{(1+\alpha)}$ and $\omega(f', \delta) \le \delta^{\alpha}/2$ so we conclude

$$\left\| \frac{2n+1}{2\pi} (R_{a,n}f - f) - Af \right\|$$
$$\le \|a\| \left((6\pi + 5) \left(\frac{2\pi}{2n+1} \right)^{\alpha} \right) + \frac{2n+1}{n\,\pi}(6\pi + 5)\frac{1}{n^{\alpha}}$$
$$\le (6\pi + 5) (\|a\| + 1) \left(\frac{2\pi}{2n+1} \right)^{\alpha} .$$

□

In [35, Theorem 2.7] we have established that besides the generation of the cosine function $(C(t))_{t\ge 0}$, condition (5.4.1) or (5.4.2) also ensures that $C^1_{2\pi} \cap D(A^2)$ is a core for $(A^2, D(A^2))$ and further, for every $t > 0$,

$$C(t) = \frac{1}{2} \lim_{n\to\infty} \left(R^{k(n)}_{a,n} u + R^{k(n)}_{-a,n} u \right) , \tag{5.4.5}$$

where $(k(n))_{n\ge 1}$ is a sequence of positive integers such that $\lim_{n\to+\infty} \frac{2\pi k(n)}{2n+1} = t$.

From Lemma 5.4.1, we can take $M = 2\pi$, $\omega = 0$, $\psi_n(u) = 24 (\|a\| + 1) \left(\frac{2\pi}{2n+1} \right)^{\alpha}$ and $\varphi_n(u) = \psi_n(u) + \|Au\|$ in Theorem 5.1.1 and we directly obtain the following quantitative version of (5.4.5).

Theorem 5.4.2 *Let $a \in C_{2\pi} \cap C^1(\mathbb{R} \setminus \Pi)$. If* (5.4.1) *or alternatively* (5.4.2) *holds, then for every $t \ge 0$ and $u \in C^{1,\alpha} \cap D_{2\pi}(A^2)$*

$$\left\| C(t)u - \frac{1}{2} \left(R^{k(n)}_{a,n} u + R^{k(n)}_{-a,n} u \right) \right\| \le (2\pi)^2\, t\, (\|a\| + 1) \left(\frac{2\pi}{2n+1} \right)^{\alpha}$$
$$+ 2\pi \left(\left| \frac{k(n)}{n} - t \right| + \sqrt{\frac{2}{\pi}}\, \frac{\sqrt{k(n)}}{n} \right)$$
$$\times \left(24 (\|a\| + 1) \left(\frac{2\pi}{2n+1} \right)^{\alpha} + \|Au\| \right) , \tag{5.4.6}$$

where $(k(n))_{n\ge 1}$ is a sequence of positive integers such that $\lim_{n\to+\infty} \frac{2\pi k(n)}{2n+1} = t$.

Now, let $(a_n)_{n\geq 1}$ be a sequence of positive integers tending to $+\infty$. For every $n \geq 1$, we consider the linear operator $P_{\lambda,a_n,n} : C_{2\pi} \to C_{2\pi}$ defined by

$$P_{\lambda,a_n,n}u := \frac{1}{2\lambda}\frac{1}{n}\sum_{k=0}^{a_n} e^{-\lambda k/n}(R_{a,n}^{k(n)}u + R_{-a,n}^{k(n)}u) , \qquad u \in E , \tag{5.4.7}$$

If the sequence $(a_n)_{n\geq 1}$ satisfies $1/a_n = o(1/n)$ as $n \to +\infty$ then from Theorem 5.3.1 we obtain $\lim_{n\to+\infty} P_{\lambda,a_n,n}u = R(\lambda^2, A^2)u$ for every $u \in C_{2\pi}$ and we have the following estimate of the convergence.

Theorem 5.4.3 *Let $a \in C_{2\pi} \cap C^1(\mathbb{R} \setminus \Pi)$. If* (5.4.1) *or alternatively* (5.4.2) *holds, then for every $t \geq 0$ and $u \in C^{1,\alpha} \cap D_{2\pi}(A^2)$ and for every $n \geq 1$ and $u \in D$, we have*

$$\begin{aligned}\|P_{\lambda,a_n,n}u - R(\lambda^2, A^2)u\| &\leq 24\,\frac{(2\pi)^{2+\alpha}(\|a\|+1)}{(2n+1)^\alpha(\operatorname{Re}\lambda)^3}\\ &+\frac{2\pi}{\operatorname{Re}\lambda\sqrt{n}}\left(\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda)}\right.\\ &\left.+\frac{1}{\sqrt{2}\,(\operatorname{Re}\lambda)^{3/2}}+\frac{1}{\sqrt{n}\,(\operatorname{Re}\lambda)^2}\right)(\psi_n(u)+\|Au\|)\\ &+\frac{2\pi\left(e^{-(\operatorname{Re}\lambda)a_n/n}+\frac{|\lambda|^{3/2}}{n\,|\operatorname{Re}\sqrt{\lambda}|}\right)}{(\operatorname{Re}\lambda)^2\left(1-\frac{\operatorname{Re}\lambda}{n}\right)}\,\|u\| .\end{aligned} \tag{5.4.8}$$

Exactly the same procedure can be also applied to other sequences of trigonometric polynomials such as Fejér operators and more general averages of trigonometric interpolating operator considered in [35, 28]. Since in these cases the cosine function is the same, we limit ourselves to observe that (5.4.6) remains still valid when considering these other sequences of trigonometric interpolating operators too.

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Notations and symbols

In this work we shall often make use of the following symbols:

$\lVert \cdot \rVert$	Uniform norm	p. 1
$\lVert \cdot \rVert_w$	Weighted uniform norm	p. 56
$\mathbf{1}$	Constant function of value 1	p. 21
$A(K)$	Space of all affine functions on the compact set K	p. 15
$A_\infty(K)$	Space of all finite products in $A(K)$	p. 16
$C(K)$	Space of all real continuous functions on K	p. 15
$C^{2,\alpha}(K)$	Class of twice differentiable functions with α-Hölder continuous second-order derivative	p. 21
$C(\overline{\mathbb{R}})$	Set of all continuous real functions on $\mathbb{R}$ which admit finite limits at the points $\pm\infty$	p. 40
$C^2(\overline{\mathbb{R}})$	Set of all functions in $C(\overline{\mathbb{R}})$ with second-order derivatives in $C(\overline{\mathbb{R}})$	p. 40
$C_w(\overline{\mathbb{R}})$	Weighted space of $C(\overline{\mathbb{R}})$	P. 56
$C^2_w(\overline{\mathbb{R}})$	Set of all functions in $C_w(\overline{\mathbb{R}})$ with second-order derivatives in $C_w(\overline{\mathbb{R}})$	p. 59
$C_w([0,1])$	Weighted space in the interval $[0,1]$	p. 74
$C^2_w([0,1])$	Weighted space of twice differentiable functions in the interval $[0,1]$	p. 74
$C^{(b)}(\mathbb{R}^2)$	Space of all continuous bounded real functions on $\mathbb{R}^2$	p. 80
$C_0(\mathbb{R}^2)$	Subspace of all continuous functions vanishing at the point at infinity of $\mathbb{R}^2$	p. 80
$C^{(b)}_w(\mathbb{R}^2)$	Bounded weighted space of $C^{(b)}(\mathbb{R}^2)$	p. 80
$C_{0,w}(\mathbb{R}^2)$	Bounded weighted space of $C_0(\mathbb{R}^2)$	p. 86
$C^{2,(b)}_w(\mathbb{R}^2)$	Space of all functions in $C^{(b)}_w(\mathbb{R}^2)$ with bounded second-order partial derivatives.	p. 82
$\Delta_h(f,x)$	Divided difference operator	p. 31
$\Delta^r_h(f,x)$	r-th order divided difference	p. 31
δ_{ij}	Kronecker symbol	p. 25

ε_x	Dirac measure	p. 16
id	Identity function	p. 26
$L^1_{\mathrm{loc}}(\mathbb{R})$	Spaces of all locally integrable functions	p. 39
$L^1_{\mathrm{loc}}(\mathbb{R}^2)$	Spaces of all locally integrable functions on $\mathbb{R}^2$	p. 79
$\mathcal{M}^+(K)$	Positive Radon Measure	p. 15
pr_i	Canonical i-th projection of $\mathbb{R}^n$ onto $\mathbb{R}$	p. 17
$\omega(f,\delta)$	Modulus of continuity	p. 31
$\omega_r(f,\delta)$	Modulus of continuity	p. 31