Introduction

Linear elliptic and parabolic operators with regular and bounded coefficients have nowadays a satisfactory theory including existence, uniqueness and regularity for the solutions to the corresponding equations in several functional spaces, such as L^p spaces, Hölder spaces and so on. Recently, the literature shows an increasing interest towards operators with unbounded or singular coefficients. Motivations come from probability and in particular from stochastic analysis. Indeed, there is a strong connection between second order differential operators and Markov processes. We briefly describe it. Let $\Xi = \{\xi_t\}$ be a Markov process in a probability space $(\Omega, \mathcal{F}, P)$, with state space $\mathbb{R}^N$. The corresponding transition probabilities $p(t, x, B)$, for $t > 0, x \in \mathbb{R}^N$, B Borel set of $\mathbb{R}^N$, represent the probability that Ξ reaches the set B at time t starting from x at $t = 0$. Given the initial distribution μ of Ξ, in order to reconstruct the process it is sufficient to determine the family of measures $p(t, x, \cdot)$ since, by the formula of total probability, one has

$$P(\xi_t \in B) = \int_{\mathbb{R}^N} p(t, x, B)\mu(dx).$$

Setting $(U(t)\mu)(B) := \int_{\mathbb{R}^N} p(t, x, B)\mu(dx)$, one obtains a semigroup in the space of all positive finite Borel measures in $\mathbb{R}^N$. This fact leads to look for an equation satisfied by $p(t, x, B)$. Such an equation actually exists and it is known as Kolmogorov backward equation. Unfortunately, it requires strong regularity to the function $p(t, x, B)$. This is the reason why it is more convenient to consider the adjoint semigroup $(T(t))$ in the space of all bounded continuous functions in $\mathbb{R}^N$. Under suitable assumptions on the process Ξ, it turns out that the generator of $(T(t))$ is a second order differential operator A with unbounded coefficients. By means of A, we can reconstruct the semigroup $(T(t))$ and therefore, by duality, the transition probabilities $p(t, x, B)$. The prototype of differential operators with unbounded coefficients is the Ornstein-Uhlenbeck operator $\mathcal{A}u = \mathrm{Tr}\,(QD^2u) + \langle Bx, Du\rangle$, where Q is a real, symmetric and nonnegative matrix and B is a real, nonzero matrix. The associated Markov semigroup $(T(t))$ has an explicit representation formula, due to Kolmogorov (see [16]).

For such a class of operators, it is not obvious to derive existence, uniqueness or regularity results similar to the classical ones. The well-known Calderon Zygmund estimate shows that the Laplacian Δ, endowed with domain $W^{2,p}(\mathbb{R}^N)$, generates a strongly continuous analytic semigroup in $L^p(\mathbb{R}^N)$, for every $p \in\,]1, \infty[$. By adding an unbounded lower order term, the picture of the situation changes radically, since the new term cannot be treated as a small perturbation of the Laplacian. To be definite, we mention two quite meaningful cases. Let V be a nonnegative function in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and consider the Schrödinger operator $A = \Delta - V$. By making use of the theory of quadratic forms, one can show that A generates a strongly continuous analytic semigroup in $L^2(\mathbb{R}^N)$, which can be extended to $L^p(\mathbb{R}^N)$, for every $1 \le p < \infty$. A natural question is whether the domain in $L^p(\mathbb{R}^N)$, when $p > 1$, coincides with the intersection of the domains of Δ and V, i.e. $W^{2,p}(\mathbb{R}^N) \cap D(V)$, where $D(V) = \{u \in L^p(\mathbb{R}^N) \mid Vu \in L^p(\mathbb{R}^N)\}$. This further information is not automatic as in the classical case, where V is bounded. In order to get it one needs to require an additional assumption on V, namely, the oscillation condition

$|DV| \leq \gamma V^{3/2}$, where γ is a sufficiently small positive constant (see [43], [41]). We remark that even for $p = 2$ the domain of A as generator can be strictly larger than $W^{2,2}(\mathbb{R}^N) \cap D(V)$ if in the previous condition the constant γ is too big (see [41]). On the other hand, the potential $V(x, y) = x^2y^2$ does not satisfy $|DV| \leq \gamma V^{3/2}$, for any γ, nevertheless the domain of $\Delta - V$ is $W^{2,2}(\mathbb{R}^N) \cap D(V)$ (see [42]). Surprisingly enough, the situation is much better in $L^1(\mathbb{R}^N)$ where the domain of $\Delta - V$ is always the intersection of the domains.

Now, let us consider the case when the Laplacian is perturbed by adding a first order term. For simplicity, we consider the Ornstein Uhlenbeck operator in one dimension, $Au = u'' + xu'$. It is readily seen that, if $1 < \alpha p \leq p + 1$, the function $u(x) = (x^2 + 1)^{-\frac{\alpha}{2}} \sin x$ belongs to $W^{2,p}(\mathbb{R}^N)$ but xu' is not in $L^p(\mathbb{R}^N)$. Therefore $W^{2,p}(\mathbb{R}^N)$, which is the domain of the Laplacian, is strictly larger than $\{u \in W^{2,p}(\mathbb{R}) \mid xu' \in L^p(\mathbb{R})\}$, which is the domain on which A generates a strongly continuous semigroup. The same one dimensional operator is also a counterexample to analyticity (see [40]).

We remark that also second order operators in the complete form, namely with both first and zero order terms, are object of investigation. For instance, the operator $\Delta - \langle D\Phi, D\rangle$ in the weighted space $L^p(\mathbb{R}^N, e^{-\Phi}dx)$ is isometric to a complete second order operator in the unweighted space $L^p(\mathbb{R}^N)$. Hence, several properties for the former can be deduced by studying the latter.

In this thesis, we focus our attention on regularity properties of solutions to partial differential equations involving second order elliptic operators with regular, (possibly) unbounded coefficients. Even though stochastic calculus is an useful tool to treat such operators, our approach is purely analytic. We cite the recent book of S. Cerrai [13] for an exhaustive analysis of what can be proved by stochastic methods.

We start in Chapter 1 by considering the following elliptic operator in divergence form

$$A = \sum_{i,j=1}^{N} D_i(q_{ij}D_j) + \langle F, D\rangle - V, \tag{0.0.3}$$

in $L^p(\mathbb{R}^N)$, with $1 < p < \infty$. The coefficients are always supposed to be real valued. If, in addition, q_{ij} are in $C^1_b(\mathbb{R}^N)$ and F_i, V are measurable and bounded, then it is well known that $(A, W^{2,p}(\mathbb{R}^N))$ generates a strongly continuous analytic semigroup. As a consequence, one obtains optimal regularity for the solutions to the resolvent equation $\lambda u - Au = f$, when λ is sufficiently large. This means that assuming only $u, \lambda u - Au \in L^p(\mathbb{R}^N)$, one deduces $u \in W^{2,p}(\mathbb{R}^N)$. Our first aim is to generalize such a result to the case where the lower order coefficients of the operator are unbounded. More precisely, we look for conditions on q_{ij}, F_i, V which allow to prove that the operator A endowed with its *natural* domain generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$. We consider as *natural* the domain given by the intersection of the domains of each addend of A, i.e. $\{u \in W^{2,p}(\mathbb{R}^N) \mid \langle F, Du\rangle,\ Vu \in L^p(\mathbb{R}^N)\}$. We have pointed out that such a domain may be strictly contained in $W^{2,p}(\mathbb{R}^N)$.

There are several approaches to show that elliptic operators with unbounded coefficients generate strongly continuous semigroups in L^p (see [11], [12], [19], [35], [37], [41] and the list of references therein), but only some of them give a precise description of the domain. Besides, in some cases the problem is investigated only for $p = 2$ (see [17], [18] and in [50]). Our work gets inspiration essentially from [37] and [41]. In [37] the case $V = 0$ and F globally Lipschitz continuous is considered. Under the further assumption $\langle F, Dq_{ij}\rangle \in L^\infty(\mathbb{R}^N)$, $i, j = 1, ..., N$, it is proved that the corresponding operator A, endowed with the domain $\{u \in W^{2,p}(\mathbb{R}^N)\colon\ \langle F, Du\rangle \in L^p(\mathbb{R}^N)\}$, generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$, for every $1 < p < \infty$. Here, the characterization of the domain follows from regularity results for the solution to the non homogenous Cauchy problem associated with A.

In [41], a second order operator in the complete form is considered and the description of the domain of the generator in $L^p(\mathbb{R}^N)$ is given assuming that V is strictly positive and that the following conditions hold: $|DV| \le \gamma V^{3/2}$, $|F| \le \kappa V^{1/2}$ and $\mathrm{div} F + \beta V \ge 0$, where γ, κ, β are sufficiently small constants. We observe that the first two assumptions are the same of Cannarsa and Vespri in [12], whereas the last one replaces an additional bound on the constant κ assumed in [12]. In [41], with a more direct approach, it is proved that A generates a strongly continuous analytic semigroup in $L^p(\mathbb{R}^N)$, $(1 < p < \infty)$, with domain $\{u \in W^{2,p}(\mathbb{R}^N) : Vu \in L^p(\mathbb{R}^N)\}$. An interpolatory estimate allows to control the L^p norm of $\langle F, Du\rangle$ by the L^p norms of Vu and D^2u. The assumption $|DV| \le \gamma V^{3/2}$ is the essential ingredient to determine the domain and, as observed at the beginning in the case of Schrödinger operators, it is optimal. The condition $|F| \le \kappa V^{1/2}$ is the best possible to yield analyticity. Finally, we observe that the cases $p = 1$ and $p = \infty$ are also considered.

We formulate new conditions on F, V and their first order derivatives to show that $(A, \mathcal{D}_p)$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$ $(1 < p < \infty)$, where $\mathcal{D}_p$ is defined as

$$\mathcal{D}_p := \{u \in W^{2,p}(\mathbb{R}^N) : \langle F, Du\rangle \in L^p(\mathbb{R}^N), Vu \in L^p(\mathbb{R}^N)\}.$$

We observe that for suitable choices of the parameters involved, our framework covers [37] or [41]. Thus, our results can be seen as a continuous interpolation between them. We also cover new cases. For instance, we allow the conditions $|F| \le \theta V$, $|DV| \le \alpha V$, $|DF| \le \beta V$.

The first step to achieve our aim consists of proving a priori estimates of the form

$$\|u\|_{W^{2,p}(\mathbb{R}^N)} + \|\langle F, Du\rangle\|_{L^p(\mathbb{R}^N)} + \|Vu\|_{L^p(\mathbb{R}^N)} \le C(\|u\|_{L^p(\mathbb{R}^N)} + \|Au\|_{L^p(\mathbb{R}^N)}), \tag{0.0.4}$$

for every $u \in \mathcal{D}_p$ and for some constant $C > 0$ independent of u. For every test function u we prove the corresponding estimates for $\|Vu\|_{L^p(\mathbb{R}^N)}$ and $\|Du\|_{L^p(\mathbb{R}^N)}$ using the variational method, which relies on suitable integrations by parts and other elementary tools. The same technique yields the estimate of the second order derivatives when $p = 2$, too, and therefore (0.0.4) is completely proved, since the last term $\|\langle F, Du\rangle\|_{L^2(\mathbb{R}^N)}$ can be estimated by difference. Of course, the method fails for $p \ne 2$. The Calderon Zygmund estimate cannot be proved by means of integrations by parts. Thus a different method has to be used, but it requires stronger assumptions. This is the reason why we treat the cases $p = 2$ and $p \ne 2$ separately. When $p \ne 2$, the idea is to get first local estimates. To this aim, we localize the equation $Au = f$ by multiplying it by cutoff functions supported in certain balls $B(x_0, r(x_0))$, and then we make a change of variables, which is determined by the potential. This technique produces a family of new operators $\{A_{x_0}\}$ which satisfy the assumptions of [37], up to a bounded perturbation. Then, to each operator A_{x_0} we can apply the a priori estimate for the second order derivatives proved in [37], so that in the original setting we find out the following local estimates

$$\int_{B(x_0,r(x_0))} |D^2u|^p \le C \int_{B(x_0,2r(x_0))} |u|^p + |Au|^p + |Vu|^p + |V^{1/2}Du|^p \tag{0.0.5}$$

A crucial point is to make the dependence of the constant C precise. In particular, in order to apply a covering argument and to obtain a global estimate starting with (0.0.5), we need C to be independent of x_0. In this way we deduce that

$$\int_{\mathbb{R}^N} |D^2u|^p \le C \int_{\mathbb{R}^N} |u|^p + |Au|^p + |Vu|^p + |V^{1/2}Du|^p,$$

and then, using known results

$$\|D^2u\|_{L^p(\mathbb{R}^N)} \le C(\|u\|_{L^p(\mathbb{R}^N)} + \|Au\|_{L^p(\mathbb{R}^N)}),$$

as required. The last estimate among (0.0.4), namely the one for $\langle F, Du\rangle$, follows by difference. Hence (0.0.4) are verified for every test function u. By density, they can be extended to $\mathcal{D}_p$ and this yields, without any further effort, the closedness of $(A, \mathcal{D}_p)$ in $L^p(\mathbb{R}^N)$. It is also an easy task to prove that $(A, \mathcal{D}_p)$ is quasi dissipative.

The second step of our procedure consists of proving the surjectivity of $\lambda - A$ from $\mathcal{D}_p$ onto $L^p(\mathbb{R}^N)$, for sufficiently large λ. This is done, once again, differently when $p = 2$ or $p \neq 2$. In the first case, we find the solution of the equation $\lambda u - Au = f$ in $\mathcal{D}_2$ as the limit of a sequence of solutions of the same equation in balls with increasing radii and Dirichlet boundary conditions. This argument does not work for $p \neq 2$. In this case, we check the surjectivity of $\lambda - A$ approximating A by a family of operators which belong to the class studied in [37]. At this point, we can apply the Hille Yosida generation theorem and we show that $(A, \mathcal{D}_p)$ generates a strongly continuous semigroup, which is positive, but not analytic in general.

The generation result just proved holds whenever $1 < p < \infty$. If $p = \infty$, in spaces of continuous functions, the situation is more delicate and the explicit description of the domain is more complicated. However, useful information can be obtained if *gradient estimates* hold. To be definite, in the second chapter, we consider the second order differential operator in non divergence form

$$\mathcal{A} = \sum_{i,j^1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} F_i D_i - V, \tag{0.0.6}$$

in a smooth open unbounded subset Ω of $\mathbb{R}^N$. Ω may be the whole space $\mathbb{R}^N$, but in this case several results are already known. We deal with the Cauchy-Neumann problem

$$\begin{cases} u_t(t,x) - \mathcal{A}u(t,x) = 0 & t > 0,\ x \in \overline{\Omega}, \\ \dfrac{\partial u}{\partial \eta}(t,x) = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = f(x) & x \in \overline{\Omega}, \end{cases} \tag{0.0.7}$$

where f is a continuous and bounded function in $\overline{\Omega}$ and η is the outward unit normal vector to $\partial\Omega$. Our aim is to determine conditions on the coefficients of $\mathcal{A}$ and on the domain Ω such that (0.0.7) admits a unique bounded classical solution u, whose spatial gradient verifies the following estimate

$$|Du(t,x)| \leq \frac{C_T}{\sqrt{t}} \|f\|_\infty \tag{0.0.8}$$

$0 < t \leq T, x \in \overline{\Omega}$. Estimate (0.0.8) has been deeply investigated in the literature, expecially by means of probabilistic tools. Our approach is purely analytic and allows to treat unbounded domains which do not coincide with the whole space. We proceed as follows. We consider the solutions u_n of Cauchy-Neumann problems with initial datum f, in a nested sequence of bounded regular domains $\{\Omega_n\}$, whose union is Ω. Since Neumann boundary conditions do not imply monotonicity (unlike Dirichlet boundary conditions), the main tool to prove the convergence of (u_n) is given by the classical Schauder estimates together with a compactness argument. The limit function u is not yet the classical solution to (0.0.7), since the continuity at $(0, x)$, when $x \in \partial\Omega$, is not ensured. To solve this problem we prove sharp estimates for the gradient of u_n. More precisely, we consider the function $z_n = u_n^2 + at|Du_n|^2$ and we prove that the differential inequality $(D_t - \mathcal{A})z_n \leq 0$ holds for a suitable choice of the parameter a independent of n. To do this, we assume a dissipativity condition on the drift F, a bound from below for V and that V grows at most exponentially. Moreover, assuming that Ω is convex and choosing all the domains Ω_n to be convex, we deduce that z_n has nonpositive normal derivative on $\partial\Omega_n$. This is the crucial point of our procedure. The classical maximum principle implies that $|Du_n| \leq C_T t^{-1/2} \|f\|_\infty$, where C_T is a constant independent of n. This estimate leads to the continuity of u in $\{0\} \times \partial\Omega$ as well as to the gradient estimate (0.0.8), as soon as n tends to ∞. The method used is known as *Bernstein's method*. It was used by A.Lunardi in [34] to prove (0.0.8) in the whole $\mathbb{R}^N$, whereas the same result is proved in [13] by means of probabilistic tools. A Liapunov type condition ensures that a maximum principle holds, hence the function u, produced by the previous approximation argument, is the *unique* bounded classical solution to (0.0.7).

Setting $(P_t f)(x) = u(t,x)$, we obtain a semigroup of linear bounded operators in $C_b(\overline{\Omega})$. Such a semigroup is not strongly continuous, in general, hence we cannot consider the generator in the classical sense, but only the so called weak generator. We also note that (P_t) is neither analytic in $C_b(\overline{\Omega})$, otherwise estimate (0.0.8) could be deduced from the analyticity estimate $\|\mathcal{A}T(t)f\|_\infty \le C\,t^{-1}\|f\|_\infty$ by an interpolation argument. We show that, in our situation, the weak generator coincides with the realization of $\mathcal{A}$ in $C_b(\overline{\Omega})$ with homogeneous Neumann boundary conditions, i.e. with the operator $\mathcal{A}$ endowed with the domain

$$D(\mathcal{A}) = \left\{ u \in C_b(\overline{\Omega}) \cap \bigcap_{1\le p<\infty} W^{2,p}(\Omega \cap B_R) \text{ for all } R>0 : \ \mathcal{A}u \in C_b(\overline{\Omega}), \frac{\partial u}{\partial \eta}\Big|_{\partial\Omega} = 0 \right\}.$$

The weak generator shares several properties with the generators of strongly continuous semigroups. In particular, since we assume $V \ge 0$, we have that $(0,+\infty) \subset \rho(\mathcal{A})$. Therefore, for every $f \in C_b(\overline{\Omega})$ and $\lambda > 0$ there exists a unique solution in $D(\mathcal{A})$ of the elliptic problem

$$\begin{cases} \lambda u(x) - \mathcal{A}u(x) = f(x) & x \in \Omega, \\ \dfrac{\partial u}{\partial \eta}(x) = 0 & x \in \partial\Omega. \end{cases}$$

A consequence of (0.0.8) is that the domain of $\mathcal{A}$ is contained in $C_b^1(\overline{\Omega})$. This can be interpreted as a partial regularity result for the solutions of the elliptic problem above.

Assuming that $V \equiv 0$, we derive further gradient estimates of pointwise type. More precisely, if $p > 1$ and $f \in C_b^1(\overline{\Omega})$ with $\partial f/\partial\eta = 0$ on $\partial\Omega$, then

$$|DP_t f(x)|^p \le e^{\sigma_p t} P_t(|Df|^p)(x), \tag{0.0.9}$$

where σ_p is a real constant depending on the coefficients of $\mathcal{A}$, N and p. If $q_{ij} = \delta_{ij}$, then the previous estimate is true also when $p = 1$. This case is almost optimal, since in [58] it is proved that (0.0.9) with $p = 1$ holds in $\mathbb{R}^N$ if and only if $D_k q_{ij} + D_j q_{ki} + D_i q_{kj} = 0$, for every i, j, k. Following the ideas of [7] for $p = 2$, we deduce that

$$|DP_t f(x)|^p \le \left(\frac{\sigma_2 \nu_0^{-1}}{2(1 - e^{-\sigma_2 t})} \right)^{\frac{p}{2}} P_t(|f|^p)(x), \tag{0.0.10}$$

for all $p \ge 2$, where ν_0 is the ellipticity constant of $\mathcal{A}$. An analogous estimate holds when $1 < p < 2$. Also in this case the thesis fails if $p = 1$, even for the heat semigroup. The previous estimate with $p = 2$ improves the first global gradient estimate (0.0.8), which can be now reformulated in the form

$$\|DP_t f\|_\infty \le \left(\frac{\nu_0^{-1}\sigma_2}{2(1 - e^{-\sigma_2 t})} \right)^{\frac{1}{2}} \|f\|_\infty.$$

Therefore, if $\sigma_2 \le 0$, a Liouville type theorem for the operator $\mathcal{A}$ holds and it implies that if $f \in D(\mathcal{A})$ and $\mathcal{A}f = 0$, then f is constant. Other interesting consequences can be deduced if an invariant measure exists. We say that a probability Borel measure μ is invariant for (P_t) if for every bounded Borel function f and every $t \ge 0$

$$\int_\Omega P_t f \, d\mu = \int_\Omega f \, d\mu \, .$$

In this case, (P_t) can be extended to a strongly continuous semigroup in $L^p(\Omega, \mu)$ for every $1 \le p < \infty$ and, integrating estimate (0.0.10) with respect to μ, one gets an analogous estimate in the L^p norm. This implies that the domain of the generator of (P_t) in $L^p(\Omega, \mu)$ is continuously embedded in $W^{1,p}(\Omega, \mu)$.

Moreover, one can derive the hypercontractivity of P_t in the space $L^2(\Omega, \mu)$ and logarithmic Sobolev inequalities (this is the well known Bakry-Émery criterion). Finally, (0.0.9) with $p = 2$ and $\sigma_2 < 0$ implies the Poincaré inequality in $W^{1,2}(\Omega, \mu)$ and the spectral gap for the generator A_2 of (P_t) in $L^2(\Omega, \mu)$, which means that $\sigma(A_2) \setminus \{0\} \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \le -C\}$, for some $C > 0$.

In the case $\Omega = \mathbb{R}^N$, estimates (0.0.9) and (0.0.10) with $p = 2$ and $q_{ij} = \delta_{ij}$ were proved respectively in [6] and [7] in the setting of abstract Markov generators, for functions belonging to a suitable algebra of smooth functions which is required to be invariant under the generator. Estimate (0.0.9) was proved also in [56] by probabilistic methods. A probabilistic approach is used in [49] too, for establishing estimate (0.0.9) in the case of a compact Riemannian manifold with convex boundary or of a complete manifold without boundary.

Dissipativity conditions are of crucial importance to get gradient estimates. Indeed, we give a counterexample to estimate (0.0.8) for an operator $\mathcal{A} = \Delta + \sum F_i D_i$ where F is not dissipative.

In the third chapter we deal with Cauchy-Dirichlet problems

$$\begin{cases} u_t(t,x) - \mathcal{A}u(t,x) = 0 & t > 0,\ x \in \Omega, \\ u(t,x) = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = f(x) & x \in \Omega, \end{cases} \tag{0.0.11}$$

where $\mathcal{A}$ is defined by (0.0.6) and f is continuous and bounded in Ω. Our aim is again to derive gradient estimates for bounded classical solutions to (0.0.11). Our approach is slightly different from the previous case. Indeed, if (u_n) is a sequence of solutions of Cauchy-Dirichlet problems in bounded domains, whose union is Ω, then it is not difficult to show that (u_n) converges to a solution of (0.0.11). But, if we set $z_n = u_n^2 + at|Du_n|^2$ and we try to apply the maximum principle to z_n, it is not clear what happens to z_n at $\partial\Omega$, even when Ω is a halfspace. To overcome this difficulty, we proceed in the following way. The existence of a bounded classical solution u to (0.0.11) can be proved completely by approximation. We note that in this case we do not expect that the solution is continuous at $(0, x)$, for $x \in \partial\Omega$. The uniqueness follows once again from a generalized version of the classical maximum principle. Afterwards, we observe that since $u = 0$ on $\partial\Omega$, only the normal derivative of u can be different from zero on $\partial\Omega$. This suggests us the comparison with certain one dimensional operators. In fact, following this idea and assuming a suitable control on F near to the boundary of Ω, we can prove the following estimate for Du on $\partial\Omega$

$$|Du(t,\xi)| \le \frac{C_T}{\sqrt{t}} \|f\|_\infty, \quad 0 < t \le T,\ \xi \in \partial\Omega.$$

Taking such an estimate into account, we can apply the maximum principle to the function $z = u^2 + at|Du|^2$ and we obtain

$$|Du(t,x)| \le \frac{C_T}{\sqrt{t}} \|f\|_\infty \quad 0 < t \le T,\ x \in \Omega. \tag{0.0.12}$$

We note that the technique allows to have a precise control of the constant C_T. Unfortunately, the last step works if one already knows that u is smooth enough. This is not our case, even though the initial datum f is smooth. Therefore, we use a trick, which consists in introducing an auxiliary potential W. We take W big enough to control the growth of the drift term F and then we consider the perturbed operators $A_\varepsilon = A - \varepsilon W$. We show that the realization of

A_ε with homogeneous Dirichlet boundary conditions generates a strongly continuous analytic semigroup $(T_{p,\varepsilon}(t))$ in $L^p(\Omega)$, for $p \geq 2$ and we characterize the domain. Choosing a large p and using Sobolev embeddings we prove that $u_\varepsilon(t,x) = T_{p,\varepsilon}(t)f(x)$ is the bounded classical solution of (0.0.11), with A replaced by A_ε and f smooth. Moreover, we are allowed to apply the previous gradient estimate to each u_ε, obtaining

$$|Du_\varepsilon(t,x)| \leq \frac{C_T}{\sqrt{t}}\|f\|_\infty, \quad 0 < t \leq T, \; x \in \Omega,$$

with C_T independent of ε. A suitable extracted sequence of u_ε converges to the bounded classical solution u of (0.0.11) and estimate (0.0.12) follows by taking the limit. Finally, by a standard approximation argument we prove estimate (0.0.12) for every continuous and bounded function f in Ω. We point out that we do not need convexity assumptions on Ω to carry out this program. At the moment, the same procedure seems to be useful to remove the convexity of Ω in the case where Neumann boundary condition is considered. But this is a work in progress. As far as local gradient estimates for (3.0.1) are concerned, we mention [54], which establishes them in the Riemannian setting, and [15], [53] for the case when Ω is an open subset of a Hilbert space and A is an Ornstein-Uhlenbeck operator. Moreover in [57], see also [31], connections between estimates (0.0.12) and some isoperimetric inequalities are investigated.

In Chapter 4 we study the second order ordinary differential operator $Au = au'' + bu'$ and we characterize the domains on which A generates semigroups in $C_b(\mathbb{R})$ and in $C(\overline{\mathbb{R}})$, the space of continuous functions having finite limits at $\pm\infty$. Unfortunately, the technique used cannot be extended to higher dimensions. We just cite [41], where a complete description of the domain is given in $C_0(\mathbb{R}^N)$ when the operator contains a potential term which balances the growth of the drift coefficient. We refer to [34] for the case of Hölder spaces.

Minimal assumptions on the coefficients of A guarantee that A endowed with the domain

$$D_{\max}(A) := \{u \in C_b(\mathbb{R}) \cap C^2(\mathbb{R}) \mid Au \in C_b(\mathbb{R})\}$$

generates a semigroup in $C_b(\mathbb{R})$, which is not strongly continuous in general and A with domain

$$D_{\mathrm{m}}(A) := \{u \in C(\overline{\mathbb{R}}) \cap C^2(\mathbb{R}) \mid Au \in C(\overline{\mathbb{R}})\}$$

generates a strongly continuous semigroup in $C(\overline{\mathbb{R}})$. Hence, we have only to describe explicitly such domains. Our aim is to show that under suitable assumptions on a and b

$$D_{\max}(A) = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}$$

and

$$D_{\mathrm{m}}(A) = \{u \in C^2(\overline{\mathbb{R}}) \mid bu' \in C(\overline{\mathbb{R}})\}.$$

As a consequence, we derive optimal regularity for the solutions to the elliptic equations $\lambda u - Au = f$ both in $C_b(\mathbb{R})$ and in $C(\overline{\mathbb{R}})$. Let us consider the first case. Set $D = \{u \in C_b^2(\mathbb{R}) \mid au'', bu' \in C_b(\mathbb{R})\}$. Since $D \subset D_{\max}(A)$ and $\lambda - A$ is bijective from $D_{\max}(A)$ onto $C_b(\mathbb{R})$, to prove the statement it is sufficient to show that $\lambda - A$ is surjective from D onto $C_b(\mathbb{R})$. Once again, a crucial point is represented by a priori estimates. More precisely, assuming that $ab' \leq c_1 + c_2 b^2$, we prove that for every $u \in C^2([-\alpha,\alpha])$ with $u'(\pm\alpha) = 0$, we have

$$\|bu'\|_{C([-\alpha,\alpha])} \leq C(\|Au\|_{C([-\alpha,\alpha])} + \|u\|_{C([-\alpha,\alpha])}), \tag{0.0.13}$$

with C independent of α. Then, we construct a solution $u \in D$ of the equation $\lambda u - Au = f$, for $f \in C_b(\mathbb{R})$, by approximation, considering the solutions of the equation $\lambda u - Au = f$ in $[-n, n]$

with Neumann boundary conditions and using (0.0.13). In a similar way, but requiring slightly stronger assumptions on a, b, we prove the statement in $C(\overline{\mathbb{R}})$.

The last chapter is devoted to the collection of some known results concerning invariant measures for Feller semigroups in $C_b(\mathbb{R}^N)$. We present this argument in a quite general context, which is not the most general possible, but is close to the main concrete situation where this concept arises, namely the theory of second order differential operators with unbounded coefficients. In the last section we study the operator

$$B = \operatorname{div}(qD) - \langle qD\Phi, D\rangle + \langle G, D\rangle$$

in the space $L^p(\mathbb{R}^N, \mu)$, $1 < p < \infty$, where $d\mu = e^{-\Phi}dx$. Via the transformation $v = e^{-\frac{\Phi}{p}}u$, the operator B on $L^p(\mathbb{R}^N, \mu)$ is similar to an operator A of the form (0.0.3) in the unweighted space $L^p(\mathbb{R}^N)$. Suitable assumptions on the coefficients q, Φ, G allow to apply the generation results of Chapter 1 to the transformed operator so that, via the inverse transformation, we can deduce that B, endowed with the domain

$$\mathcal{D}_\mu = \{u \in W^{2,p}(\mathbb{R}^N, \mu) \mid \langle G, Du\rangle \in L^p(\mathbb{R}^N, \mu)\} \tag{0.0.14}$$

generates a strongly continuous semigroup $(T(t))$ on $L^p(\mathbb{R}^N, \mu)$. We note that, in particular, the measure μ can be the invariant measure of $(T(t))$.

Lecce, 16 april 2004