Chapter 8

Generalized composition of aggregation operators

Let $\mathcal{A}$ be the class of binary aggregation operators (=agops). In this section, we denote by Θ the class of all increasing functions $f : [0,1] \to [0,1]$. Given f_1, f_2, g_1 and g_2 in Θ and a binary operation H on $[0,1]$, let F be the mapping defined on $[0,1]^2$ by

$$F(x,y) := H\left(A(f_1(x), g_1(y)), B(f_2(x), g_2(y))\right), \tag{8.1}$$

for all A and B in $\mathcal{A}$. The function F is called *generalized composition* of (A, B) with respect to the 5–ple (f_1, g_1, f_2, g_2, H), which is called *generating system*. The prefix "generalized" is used here to distinguish the function F from the classical composition that is obtained when $f_1 = g_1 = f_2 = g_2 = \mathrm{id}_{[0,1]}$, and already studied for agops (see, for instance, [10, 90]).

This chapter aims to establish which conditions on the generating system ensure that, for every choice of A and B in a given subset $\mathcal{B} \subseteq \mathcal{A}$ (for instance, $\mathcal{B}$ is the set of copulas, semicopulas, etc.), F is also an agop belonging to $\mathcal{B}$. Thus, in section 8.1 we analyse the case of agops and sections 8.2, 8.3 and 8.4 are devoted, respectively, to the study of generalized composition in the class of semicopulas, 1–Lipschitz and 2–increasing agops. The case of copulas is considered in section 8.5, where several examples are given together with an interesting application of this method.

The results of this chapter can be also found in [35, 38, 37].

8.1 Composition of agops

As above, given a generating system (f_1, g_1, f_2, g_2, H), for all agops A and B, let F be the mapping defined by (8.1). If H is an agop, then F is increasing, because it is a composition of increasing functions. Moreover, in order to ensure that

$$F(0,0) = H\left(A(f_1(0), g_1(0)), B(f_2(0), g_2(0))\right) = 0,$$

one among the following conditions is sufficient:

$$f_1(0) = g_1(0) = 0 \quad \text{and} \quad f_2(0) = g_2(0) = 0, \tag{8.2}$$

$$f_1(0) = g_1(0) = 0 \quad \text{and} \quad H(0,b) = 0, \text{ for every } b \in [0,1], \tag{8.3}$$

$$f_2(0) = g_2(0) = 0 \quad \text{and} \quad H(a,0) = 0, \text{ for every } a \in [0,1]. \tag{8.4}$$

Analogously, in order to obtain

$$F(1,1) = H\left(A(f_1(1), g_1(1)), B(f_2(1), g_2(1))\right) = 1,$$

one among the following conditions is sufficient:

$$f_1(1) = g_1(1) = 1 \quad \text{and} \quad f_2(1) = g_2(1) = 1, \tag{8.5}$$

$$f_1(1) = g_1(1) = 1 \quad \text{and} \quad H(1,b) = 1, \text{ for every } b \in [0,1], \tag{8.6}$$

$$f_2(1) = g_2(1) = 1 \quad \text{and} \quad H(a,1) = 1, \text{ for every } a \in [0,1]. \tag{8.7}$$

In the sequel, we suppose that a generating system *satisfies one condition among* (8.2)–(8.4) *and another one among* (8.5)–(8.7).

Proposition 8.1.1. *Let* $(A, B) \in \mathcal{A} \times \mathcal{A}$ *and* (f_1, g_1, f_2, g_2, H) *be a generating system. Then the function* F *given by* (8.1) *is an agop.*

The very general form of composition (8.1) allows a great flexibility in constructing new agops and, in particular, the new method includes well-known procedures (see [10] for more details about them), as the following examples show.

Example 8.1.1. Let (f_1, g_1, f_2, g_2, H) be a generating system such that, for every $(x, y) \in [0,1]^2$, $H(x, y) = x$. For every A and B in $\mathcal{A}$, the function F given in (8.1) is equal to the agop $A(f_1(x), g_1(y))$. In particular, if f_1 and g_1 are greater than $\mathrm{id}_{[0,1]}$, this transformation was used for augmenting the output given by A (see [92]).

Example 8.1.2. If $f_1 = g_1 = f_2 = g_2 = \mathrm{id}_{[0,1]}$ and H is an agop, then the generating system (f_1, g_1, f_2, g_2, H) generates, for all agops A and B, an agop F that is the classical composition in $\mathcal{A}$, which includes as special cases the weighted arithmetic mean of agops, by taking $H(x, y) = \lambda x + (1 - \lambda) y$ ($\lambda \in [0,1]$), and the weighted geometric mean of agops, by taking $H(x, y) = x^{\lambda} \cdot y^{1-\lambda}$ ($\lambda \in [0,1]$). In particular, if A is a t–norm and B is a t–conorm, F is a *triangular norm–based compensatory operator*, a special agop introduced as a means for providing compensation between the small and the large degrees of memberships when we combine fuzzy sets (see [92] and the references therein).

Example 8.1.3. Let $A, B \in \mathcal{A}$ and let (f_1, g_1, f_2, g_2, H) be a generating system of the function F given by (8.1). If $f_1 = g_1$ is a bijection and $H(x, y) := f_1^{-1}(x)$, then $F(x, y) = f_1^{-1}(A(f_1(x), f_1(y)))$ is the *transformation of* A by f_1, considered in chapter 9.

Example 8.1.4. Let $A, B \in \mathcal{A}$ and let (f_1, g_1, f_2, g_2, H) be a generating system of the function F given by (8.1). If $f_1 = g_1 = f_2 = g_2 = \mathrm{id}_{[0,1]}$, take $H(x, y) = \min\{1, x + \beta y\}$ $(\beta \in [0, 1])$. Then F is the *augmentation* of A. Similarly, taking $H(x, y) = \max\{0, x - \beta(1 - y)\}$ $(\beta \in [0, 1])$, the corresponding F is the *reduction* of A. This is another method proposed in [31] for augmenting (reducing) the outputs.

Remark 8.1.1. Given two associative agops A and B and a generating system (f_1, g_1, f_2, g_2, H) with $f_1 = g_1$ and $f_2 = g_2$, the function F defined by (8.1) is called a *quasi–associative operator* (see [158]).

8.2 Composition of semicopulas

Now, we give some sufficient conditions for the generalized composition of semicopulas.

Proposition 8.2.1. *Let* (A, B) *be in* $\mathcal{S} \times \mathcal{S}$ *and let* (f_1, g_1, f_2, g_2, H) *be a generating system satisfying* (8.5) *and one condition among* (8.2)–(8.4)*. The function* F *given by* (8.1) *is a semicopula if, and only if,*

$$H(f_1(x), f_2(x)) = x \quad \text{and} \quad H(g_1(x), g_2(x)) = x \qquad \text{for every } x \in [0, 1]. \tag{8.8}$$

Proof. In view of Proposition 8.1.1 it suffices to show that F has neutral element equal to 1. Let x be in $[0, 1]$. We have

$$F(x, 1) = H(A(f_1(x), g_1(1)), B(f_2(x), g_2(1))) = H(f_1(x), f_2(x)) = x,$$

and, analogously, $F(1, x) = x$. □

Example 8.2.1. For every $a \geq 1$, we consider the following generating system:

$$f_1(x) = g_1(x) = \min\{ax, 1\}, \quad f_2(x) = g_2(x) = x, \quad H = \min\{x, y\}.$$

For all $A, B \in \mathcal{S}$, the semicopula F defined by (8.1) is given by

$$F(x, y) := \begin{cases} \min\{A(ax, ay), B(x, y)\}, & \text{if } (x, y) \in [0, 1/a[^2; \\ B(x, y), & \text{otherwise.} \end{cases}$$

Example 8.2.2. For every $a \geq 1$, we consider the following generating system:

$$f_1(x) = g_1(x) = \max\{ax + (1 - a), 0\}, \quad f_2(x) = g_2(x) = x, \quad H = \max\{x, y\}.$$

For all $A, B \in \mathcal{S}$, the semicopula F defined by (8.1) is given by

$$F(x,y) := \begin{cases} B(x,y), & \text{if } (x,y) \in \left[0, \dfrac{a-1}{a}\right]^2; \\ \max\left\{A(ax+(1-a), ay+(1-a)), B(x,y)\right\}, & \text{otherwise.} \end{cases}$$

Example 8.2.3. For all $\alpha, \beta > 0$, we consider the following generating system:

$$f_1(x) = x^{\alpha}, \quad g_1(x) = x^{\beta}, \quad f_2(x) = x^{1-\alpha}, \quad g_2(x) = x^{1-\beta}, \quad H = \Pi.$$

For all $A, B \in \mathcal{S}$, the semicopula F defined by (8.1) is given by

$$F(x,y) := A(x^{\alpha}, y^{\beta}) \cdot B(x^{1-\alpha}, y^{1-\beta}),$$

which is a non–symmetric agop for $\alpha \neq \beta$.

In the case of semicopulas, we can give a full characterization of the classical composition. To this end, first, we give a technical result.

Lemma 8.2.1. *Let s_1, s_2 and t be points in $[0,1[$ with $s_1 \leq s_2$. Then there exist two semicopulas A and B and two points (x_1, y_1) and (x_2, y_2) in $[0,1]^2$, with $x_1 \leq x_2$ and $y_1 \leq y_2$ such that*

$$A(x_1, y_1) = s_1 \qquad \textit{and} \qquad A(x_2, y_2) = s_2, \qquad B(x_1, y_1) = t = B(x_2, y_2).$$

Proof. Three cases will be considered.
Case 1: $t \leq s_1 \leq s_2$. Let A be the ordinal sum given by

$$A = \left(\langle s_i, s_{i+1}, Z\rangle\right)_{i \in I},$$

with $I = \{0,1,2,3\}$ and $s_0 = 0$, $s_3 = 1$, so that

$$A(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0, s_1[^2; \\ s_1, & \text{if } (x,y) \in [s_1, s_2[^2; \\ s_2, & \text{if } (x,y) \in [s_2, 1[^2; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

and let B be the ordinal sum given by $B = (\langle 0, t, Z\rangle, \langle t, 1, Z\rangle)$, so that

$$B(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0, t[^2; \\ t, & \text{if } (x,y) \in [t, 1[^2; \\ x \wedge y, & \text{otherwise.} \end{cases}$$

Then

$$A(s_1, s_1) = s_1, \qquad A(s_2, s_2) = s_2, \qquad B(s_1, s_1) = t = B(s_2, s_2).$$

Case 2: $s_1 \leq t \leq s_2$. Choose B as in the previous case and let A be the frame semicopula defined by

$$A(x,y) := \begin{cases} 0, & (x,y) \in [0,1[^2 \setminus [s_1,1[^2\,, \\ s_1, & (x,y) \in [s_1,1[^2 \setminus]t,1[^2\,, \\ t, & (x,y) \in]t,1[^2 \setminus [s_2,1[^2\,, \\ s_2, & (x,y) \in [s_2,1[^2\,, \\ x \wedge y, & x \vee y = 1. \end{cases}$$

Then

$$A(t,t) = s_1, \qquad A(s_2,s_2) = s_2 \qquad \text{and} \qquad B(t,t) = B(s_2,s_2) = t.$$

Case 3: $s_1 \leq s_2 \leq t$. Choose B as in the two previous cases and let A be the frame semicopula

$$A(x,y) := \begin{cases} 0, & (x,y) \in [0,1[^2 \setminus [t,1[^2\,, \\ s_1, & (x,y) \in [t,1[^2 \setminus [x_1,1[^2\,, \\ s_2, & (x,y) \in [x_1,1[^2\,, \\ x \wedge y, & x \vee y = 1, \end{cases}$$

where the point x_1 belongs to $]t,1]$. Then we have

$$A(t,t) = s_1, \qquad A(x_1,x_1) = s_2, \qquad B(x_1,x_1) = B(t,t) = t,$$

which proves the assertion. □

Theorem 8.2.1. *Let A and B be semicopulas and let H be a binary operation on $[0,1]$. Let $F(x,y) := H\left(A(x,y), B(x,y)\right)$. The following statements are equivalent:*

(a) *for all semicopulas A and B, F is a semicopula;*

(b) *H is an idempotent agop.*

Proof. (a) $\Longrightarrow$ (b): If F is a semicopula, then for every $x \in [0,1]$

$$x = F(x,1) = H\left(A(x,1), B(x,1)\right) = H(x,x).$$

Let s_1, s_2 and t be in $[0,1[$ with $s_1 \leq s_2$. Hence, because of Lemma 8.2.1, there are two points (x_1,y_1) and (x_2,y_2) in $[0,1]^2$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ such that $A(x_1,y_1) = s_1$, $A(x_2,y_2) = s_2$ and $B(x_1,y_1) = B(x_2,y_2) = t$. Therefore

$$\begin{aligned} H(s_1,t) &= H\left(A(x_1,y_1), B(x_1,y_1)\right) = F(x_1,y_1) \\ &\leq F(x_2,y_2) = H\left(A(x_2,y_2), B(x_2,y_2)\right) = H(s_2,t). \end{aligned}$$

In an analogous manner, we prove that, for all $s \in [0,1[$, the function $t \mapsto H(s,t)$ is increasing. Thus H is an idempotent agop.

The converse implication, (b) $\Longrightarrow$ (a), is a consequence of Proposition 8.2.1. □

8.3 Composition of 1–Lipschitz agops

The following result gives a sufficient condition for the generalized composition of 1–Lipschitz agops, whose class is denoted by $\mathcal{A}_1$.

Theorem 8.3.1. *Let (A, B) be in $\mathcal{A}_1 \times \mathcal{A}_1$ and let (f_1, g_1, f_2, g_2, H) be a generating system. Let F be the function defined by (8.1). If H has the kernel property and f_i and g_i are 1–Lipschitz $(i = 1, 2)$, then F is in $\mathcal{A}_1$.*

Proof. Set $\widetilde{A}(x, y) := A(f_1(x), g_1(y))$ and $\widetilde{B}(x, y) := B(f_2(x), g_2(y))$. For every x, x', y, y' in $[0, 1]$ we have

$$\begin{aligned}
&|F(x,y) - F(x',y')| = |H\left(\widetilde{A}(x,y), \widetilde{B}(x,y)\right) - H\left(\widetilde{A}(x',y'), \widetilde{B}(x',y')\right)| \\
&\le \max\{|\widetilde{A}(x,y) - \widetilde{A}(x',y')|, |\widetilde{B}(x,y) - \widetilde{B}(x',y')|\} \\
&\le \max\{|f_1(x) - f_1(x')| + |g_1(y) - g_1(y')|, |f_2(x) - f_2(x')| + |g_2(y) - g_2(y')|\} \\
&\le |x - x'| + |y - y'|,
\end{aligned}$$

which concludes the proof. □

Example 8.3.1. Let (A, B) be in $\mathcal{A}_1 \times \mathcal{A}_1$ and let $(\mathrm{id}_{[0,1]}, \mathrm{id}_{[0,1]}, \mathrm{id}_{[0,1]}, \mathrm{id}_{[0,1]}, H_a)$ be a generating system where, for every $a \in [0, 1]$, $H_a(x, y) = \mathrm{med}(x, y, a)$ is the median among x, y and a. Then the corresponding 1–Lipschitz agop F_a defined by (8.1) is

$$F_a(x, y) = \mathrm{med}\left(A(x, y), B(x, y), a\right).$$

In particular, if $a = 0$ (resp. $a = 1$), then we obtain that the minimum (resp. maximum) of two 1–Lipschitz agops is a 1–Lipschitz agop.

Example 8.3.2. Let (A, B) be in $\mathcal{A}_1 \times \mathcal{A}_1$ and let $(f_a, f_a, \mathrm{id}_{[0,1]}, \mathrm{id}_{[0,1]}, H)$ be a generating system where $a \in\,]0, 1[$ and

$$f_a(x) = \frac{ax}{a + (1 - a)x}, \quad H(x, y) = \min\{x, y\}.$$

Then the corresponding 1–Lipschitz agop F_a defined by (8.1) is

$$F_a(x, y) = \min\left\{A\left(\frac{ax}{a + (1 - a)x}, \frac{ay}{a + (1 - a)y}\right), B(x, y)\right\}.$$

Notice that the range of f_a is not the whole $[0, 1]$.

Corollary 8.3.1. *Let A and B be quasi–copulas and let (f_1, g_1, f_2, g_2, H) be a generating system satisfying the assumptions of Proposition 8.2.1. Let F be the function defined by (8.1). If one of the following statementes holds:*

(a) *$f_i = g_i = id_{[0,1]}$ $(i = 1, 2)$ and H is a kernel agop;*

(b) $f_1 = g_1 = id_{[0,1]}$, f_2 *and* g_2 *are* 1*–Lipschitz and* $H(x, y) = \min\{x, y\}$*;*

(c) f_1 *and* g_1 *are* 1*–Lipschitz,* $f_2 = g_2 = id_{[0,1]}$ *and* $H(x, y) = \min\{x, y\}$*;*

then F *is a quasi–copula.*

Proof. The assertion follows easily from both Theorem 8.3.1 and Proposition 8.2.1 because the function $\mathrm{id}_{[0,1]}$ is 1–Lipschitz and the function $H(x, y) = \min\{x, y\}$ is a kernel agop. □

The characterization of the classical composition of quasi–copulas was given in [90] and it is reproduced here.

Proposition 8.3.1. *Let* H *be a binary operation on* $[0, 1]$ *and denote by* Ω *the subset of the unit square defined by*

$$\Omega := \left\{ (u, v) \in [0, 1]^2 : v \in \left[\max\{2u - 1, 0\}, \frac{u + 1}{2} \right] \right\}.$$

The following statements are equivalent:

(a) *for all quasi–copulas* A *and* B*,* $H(A(x, y), B(x, y))$ *is a quasi–copula;*

(b) H *is an agop which satisfies the kernel property on* Ω*.*

8.4 Composition of 2–increasing agops

We denote by $\mathcal{A}_2$ the class of 2–increasing agops.

Theorem 8.4.1. *Let* A *and* B *be* 2*–increasing agops and let* (f_1, g_1, f_2, g_2, H) *be a generating system. If* H *is* P*–increasing, then the function* F *defined by* (8.1) *is a* 2*–increasing agop.*

Proof. Set $\widetilde{A}(x, y) := A(f_1(x), g_1(y))$ and $\widetilde{B}(x, y) := B(f_2(x), g_2(y))$. The function F given by (8.1) satisfies the 2–increasing property if, and only if, for all x, x', y, y' in $[0, 1]$, $x \le x'$ and $y \le y'$,

$$\begin{aligned} &F(x', y') - F(x', y) - F(x, y') + F(x, y) \\ &= H(\widetilde{A}(x', y'), \widetilde{B}(x', y')) - H(\widetilde{A}(x', y), \widetilde{B}(x', y)) \\ &\quad - H(\widetilde{A}(x, y'), \widetilde{B}(x, y')) + H(\widetilde{A}(x, y), \widetilde{B}(x, y)) \ge 0. \end{aligned}$$

Now, take

$$\begin{aligned} &s_1 = \widetilde{A}(x, y),\ s_2 = \widetilde{A}(x', y),\ s_3 = \widetilde{A}(x, y'),\ s_4 = \widetilde{A}(x', y') \\ &t_1 = \widetilde{B}(x, y),\ t_2 = \widetilde{B}(x', y),\ t_3 = \widetilde{B}(x, y'),\ t_4 = \widetilde{B}(x', y'). \end{aligned}$$

The functions $\widetilde{A}$ and $\widetilde{B}$ are increasing in each place and 2–increasing (in view of Proposition 3.2.1). Therefore the points s_i and t_i ($i \in \{1, 2, 3, 4\}$) satisfy (7.3) and (7.4) and, because H is P–increasing, it follows that F is 2-increasing. □

Notice that the assumptions of Theorem 8.4.1 are only sufficient: for particular agops A and B, in fact, they could be weakened, as the following example shows.

Example 8.4.1. Let A_S be the smallest agop. Consider $f_i = g_i = \mathrm{id}_{[0,1]}$ $(i = 1, 2)$ and let B be an agop in $\mathcal{A}_2$. For every P–increasing agop H, the composition F of A_S and B given by (8.1) is equal to $H(0, B(x, y))$ for every $(x, y) \neq (1, 1)$. Therefore, in order to ensure that F is 2–increasing, it is sufficient to give conditions only on the vertical section $y \mapsto H(0, y)$, and no other assumption on the values of H on $[0, 1]^2$ is required.

The classical composition of 2–increasing agops is characterized here.

Theorem 8.4.2. *Let H be an agop. The following statements are equivalent:*

(a) *H is P–increasing;*

(b) *for every $(A, B) \in \mathcal{A}_2 \times \mathcal{A}_2$, $F(x, y) = H(A(x, y), B(x, y))$ is a 2–increasing agop.*

Proof. Part $(a) \Longrightarrow (b)$ is a particular case of Theorem 8.4.1. Conversely, let $s_i, t_i \in [0, 1]$ $(i \in \{1, 2, 3, 4\})$ such that (7.3) and (7.4) hold, namely

$$s_1 \leq s_2 \wedge s_3 \leq s_2 \vee s_3 \leq s_4, \qquad t_1 \leq t_2 \wedge t_3 \leq t_2 \vee t_3 \leq t_4, \tag{8.9}$$

$$s_1 + s_4 \geq s_2 + s_3, \qquad t_1 + t_4 \geq t_2 + t_3. \tag{8.10}$$

Define the following agops:

$$A(x,y) := \begin{cases} 0, & \text{if } \min\{x,y\} = 0; \\ s_1; & \text{if } (x,y) \in]0,1/2] \times]0,1/2]; \\ s_2; & \text{if } (x,y) \in]0,1/2] \times]1/2,1]; \\ s_3; & \text{if } (x,y) \in]1/2,1] \times]0,1/2]; \\ s_4; & \text{if } (x,y) \in]1/2,1[\times]1/2,1[; \\ 1; & \text{if } (x,y) = (1,1); \end{cases}$$

$$B(x,y) := \begin{cases} 0, & \text{if } \min\{x,y\} = 0; \\ t_1; & \text{if } (x,y) \in]0,1/2] \times]0,1/2]; \\ t_2; & \text{if } (x,y) \in]0,1/2] \times]1/2,1]; \\ t_3; & \text{if } (x,y) \in]1/2,1] \times]0,1/2]; \\ t_4; & \text{if } (x,y) \in]1/2,1[\times]1/2,1[; \\ 1; & \text{if } (x,y) = (1,1). \end{cases}$$

Let $F(x, y) = H(A(x, y), B(x, y))$ be the composition of A and B. Then

$$V_H\left([1/3, 2/3]^2\right) = H(s_1, t_1) + H(s_4, t_4) - H(s_2, t_2) - H(s_3, t_3) \geq 0,$$

viz. H is P–increasing. □

Corollary 8.4.1. *Let $f : [0,1] \to [0,1]$ be such that $f(0) = 0$ and $f(1) = 1$. The following statements are equivalent:*

(a) *f is convex and increasing;*

(b) *for every $(A, B) \in \mathcal{A}_2 \times \mathcal{A}_2$, $F(x, y) = f(A(x, y))$ is a 2–increasing agop.*

Proof. It suffices to apply the above Theorem to the function $H(x, y) = f(x)$, which is P–increasing because of Theorem 7.1.2. □

8.5 Composition of copulas

The following result on the generalized composition of copulas is a direct consequence of Theorem 8.4.1 and Proposition 8.2.1.

Proposition 8.5.1. *Let (A, B) be in $\mathcal{C} \times \mathcal{C}$ and let (f_1, g_1, f_2, g_2, H) be a generating system satisfying the assumptions of Proposition 8.2.1. If H is P–increasing, then the function F defined by (8.1) is a copula.*

Example 8.5.1. Consider, for all $0 < \alpha < \beta < 1$,

$$f_1(x) = \frac{\beta x}{(\beta - \alpha)x + \alpha} \quad \text{and} \quad f_2(x) = \frac{(\beta - \alpha)x + \alpha}{\beta};$$

for every $\gamma \in [0, 1]$,

$$g_1(x) = x^\gamma \quad \text{and} \quad g_2(x) = x^{1-\gamma};$$

and $H = \Pi$. For all copulas A and B, in view of Proposition 8.5.1 we have the following family of copulas:

$$C_{\alpha,\beta,\gamma}(x, y) = A\left(\frac{\beta x}{(\beta - \alpha)x + \alpha}, y^\gamma\right) \cdot B\left(\frac{(\beta - \alpha)x + \alpha}{\beta}, y^{1-\gamma}\right).$$

Example 8.5.2. Consider, for all α and β in $]0, 1]$,

$$\begin{aligned} f_1(x) &= \alpha x + (1 - \alpha), \qquad & f_2(x) &= (1 - \alpha)x + \alpha, \\ g_1(x) &= \beta x + (1 - \beta), \qquad & f_2(x) &= (1 - \beta)x + \beta, \end{aligned}$$

and $H = W$. For all copulas A and B, in view of Proposition 8.5.1 we obtain the following family of copulas:

$$C_{\alpha,\beta}(x, y) = \max\left(A(\alpha x + \overline{\alpha}, \beta x + \overline{\beta}) + B(\overline{\alpha}x + \alpha, \overline{\beta}x + \beta) - 1, 0\right),$$

where $\overline{\alpha} := 1 - \alpha$ and $\overline{\beta} := 1 - \beta$.

Remark 8.5.1. For particular copulas A and B, the conditions of the previous proposition are only sufficient. In fact, let (A, B) be in $\mathcal{C}\times\mathcal{C}$ with $B(x, y) = \min\{x, y\}$ and let (f, g, H) be the generating triple defined, for every $\lambda \geq 1$, by

$$f(x) = \min\{\lambda x, 1\}, \quad g(x) = x, \quad H(x, y) = \min\{x/\lambda, y\}.$$

Thus H is not P–increasing (the horizontal section of H is concave), but, for every copula A, the function F given in (8.1) is

$$F(x, y) = \begin{cases} \dfrac{1}{\lambda} A(\lambda x, \lambda y), & \text{if } (x, y) \in \left[0, \frac{1}{\lambda}\right]^2; \\ \min\{x, y\}, & \text{otherwise;} \end{cases}$$

and F is the ordinal sum $(\langle 0, 1/\lambda, A\rangle)$ and, hence, it is a copula.

In Proposition 8.5.1, when either $f_1 \neq g_1$ or $f_2 \neq g_2$, we generate a family of non–symmetric copulas. In fact, the idea of this kind of composition arises from the paper [61] where the following mechanism is given.

Proposition 8.5.2 (Khoudraji, 1995)**.** *Let C be a symmetric copula, $C \neq \Pi$. A family of non–symmetric copulas $C_{\alpha,\beta}$ with parameters $0 < \alpha, \beta < 1$ $(\alpha \neq \beta)$ that includes C as a limiting case is defined by*

$$C_{\alpha,\beta}(x, y) := x^{1-\alpha} y^{1-\beta} C(x^\alpha, y^\beta).$$

Proof. It suffices to apply Proposition 8.5.1 with $H = \Pi$, $f_2(t) = t^\alpha$, $f_1 = f_2^{-1}$, $g_2(t) = t^\beta$, $g_1 = g_2^{-1}$. Then $C_{\alpha,\beta}$ is the generalized composition of (Π, C) with respect to the generating system $(f_1, f_2, g_1, g_2, \Pi)$. □

In the same manner, we prove:

Proposition 8.5.3. *Let A and B be symmetric copulas. A family of non–symmetric copulas $C_{\alpha,\beta}$ with parameter $0 < \alpha, \beta < 1$, $\alpha \neq 1/2$, is defined by*

$$C_{\alpha,\beta}(x, y) := A(x^\alpha, y^\beta) \cdot B(x^{1-\alpha}, y^{1-\beta}). \tag{8.11}$$

An interesting statistical interpretation can be given for this family. Let U_1, V_1, U_2 and V_2 be random variables uniformly distributed on $[0, 1]$. If A is the connecting copula of (U_1, V_1) and B is the connecting copula of (U_2, V_2) and the pairs (U_1, V_1) and (U_2, V_2) are independent, then $C_{\alpha,\beta}$ is the joint d.f. of

$$U = \max\{U_1^{1/\alpha}, U_2^{1/(1-\alpha)}\} \quad \text{and} \quad V = \max\{V_1^{1/\beta}, V_2^{1/(1-\beta)}\}.$$

Example 8.5.3. In the recent paper [96], a generalization of the bivariate survival d.f. of type Marshall–Olkin was considered. This function is given, for every $x, y \geq 0$, by

$$S^*(x, y) = S(x, y) \exp(-\lambda_{12} \max\{x, y\}),$$

where S is a bivariate survival d.f. with continuous survival marginal d.f.'s $F(x) = e^{-\lambda x}$, $\lambda > 0$ and $\lambda_{12} > 0$. If A is the copula of S, it is an easy computation to obtain that the copula of S^* is of the type (8.11), where $A = C$, $B = M$ and $\alpha = \beta = \lambda/(\lambda + \lambda_{12})$.

Example 8.5.4. Let A and B two Archimedean copulas generated, respectively, by φ and ϕ. In view of Proposition 8.5.3, for every α and β in $[0,1]$ the following functions are copulas

$$C_{\alpha,\beta}(x,y) := \varphi^{[-1]}(\varphi(x^{\alpha}) + \varphi(y^{\beta})) \cdot \phi^{[-1]}(\phi(x^{1-\alpha}) + \phi(y^{1-\beta})). \tag{8.12}$$

In particular, if $\varphi(t) = \phi(t) = (-\ln t)^{\gamma}$ $(\gamma \geq 1)$, then A and B are the members of the so–called *Gumbel–Hougaard* family of copulas. By considering (8.12), we obtain a three–parameter family of non–symmetric copulas,

$$C_{\alpha,\beta,\gamma}(x,y) := \exp\left(- [(-\alpha \ln x)^{\gamma} + (-\beta \ln y)^{\gamma}]^{1/\gamma} - \left[(-\overline{\alpha} \ln x)^{\gamma} + (-\overline{\beta} \ln y)^{\gamma}\right]^{1/\gamma} \right),$$

where $\overline{\alpha} := 1 - \alpha$ and $\overline{\beta} := 1 - \beta$, which can be considered a non–symmetric generalization of the Gumbel–Hougaard family.

The importance of having at disposal families of asymmetric copulas is crucial in copula modelling. In applications, in fact, we have a (bivariate) data set and we have interested in the joint d.f. H that is the best–possible approximation to our data. Thanks to Sklar's theorem, this problem can be decomposed into two steps: the modelling of the marginal d.f.'s and the estimating of a copula that summarizes the dependence between the margins. In several practical cases, we select a large family of copulas C_{θ}, where $\theta = (\theta_1, \dots, \theta_n)$ is a multiparameter belonging to a subset $J^n \subseteq \mathbb{R}^n$, and we choose $\widehat{\theta} \in J^n$ such that $C_{\widehat{\theta}}$ optimaly fits our data (see [60] for more details on the copula modelling). A suitable family C_{θ} could have a simple representation (like the Archimedean copulas), or a simple way to computing it by numerical procedure (like the normal copula), and a sufficiently large dependence structure. In particular, and this is often neglected, *no assumptions on the simmetry of the copulas should be made*, unless it is explicitly required by the problem at hand. In fact, if the copula C is symmetric and the marginals d.f.'s F_1 and F_2 are continuous and both equal to a d.f. F, then the joint d.f. $H = C(F,F)$ is *exchangeable* and, therefore, it is not suitable to describe situations in which the appropriateness of this symmetry condition is doubtful.