Introduction

The official history of the concept of *copula* began with the following words, contained in the seminal paper by Abe Sklar ([149]):

Nous appelerons copule (à n dimensions) toute fonction C_n continue et non décroissante (au sense employé pour une fonction de répartition à n dimensions) définie sur le produit Cartésien de n intervalles fermés $[0, 1]$ *et satisfaisante aux conditions:*

$$C_n(0, 0, \ldots, 0) = 0, \qquad C_n(1, \ldots, 1, \alpha, 1, \ldots, 1) = \alpha.$$

Copulas have been introduced in order to answer a question posed by M. Fréchet on the determining of the classes of multidimensional probability distribution functions with given margins. This problem had occupied several researchers for some years (see, for example, [55, 53, 22]) and the proposed solution states in the following result, since then called *Sklar's Theorem*.

If G is an n-dimensional distribution function with margins F_1, ..., F_n, then there exists a copula C_n such that

$$G(x_1, \ldots, x_n) = C_n(F_1(x_1), \ldots, F_n(x_n)),$$

and, if each F_i is continuous, then C is unique. Conversely, given the univariate distribution functions F_1, ..., F_n, and a copula C_n, the function G, defined as above, is an n–dimensional distribution function.

Therefore, the Fréchet problem can be reduced to the study of the class of copulas.

At the beginning, many results on copulas were obtained in connection with problems arising in the theory of probabilistic metric spaces, a promising research field developed by B. Schweizer and A. Sklar following the original idea of K. Menger ([106, 141]). As explicitly said by B. Schweizer ([138]), in those years there were no "*ideas of possible statistical applications of copulas and the statistical community took little note of this new concept*".

The initial poor diffusion of this new concept is testified by the fact that, since 1959, copulas appeared implicitely, and under different names, in the works of several authors. In 1960, M. Sibuya considered a *dependence function* associated with a pair of random variables ([148]). In 1975, G. Kilmedorf and A.R. Sampson introduced the *uniform representation* and studied it as a tool to define various dependence notions ([77, 78]). Succesively, analogous concepts were introduced by P. Deheuvels, J. Galambos, D.S. Moore and M.C. Spruill (see [138] for more details). It is also important here to mention that a concept similar to that one of copula was introduced in a paper of W. Hoeffding published in 1940, but unknown largely for many years (see [138, 54]).

The situation changed after the paper [142], in which B. Schweizer and E.F. Wolff announced their first results on the use of copulas for defining a rank–based measure of dependence among random variables. This work, published after some years in the *Annals of Statistics* ([143]), gave the input to a large development of copulas in the study of dependence. In fact, as noted by B. Schweizer and E.F. Wolff,

"*it is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strictly increasing transformations. Hence the study of rank statistics – insofar as it is the study of properties invariant under such transformations – may be characterized as the study of copulas and copula–invariant properties*".

Some years later, only to make few examples, M. Scarsini showed the importance of copulas in the definition of a measure of concordance between random variables ([135]); C. Genest and J. Mac Kay studied the so–called *Archimedean* copulas, which can be easily constructed and simulated ([62, 63]); W.F. Darsow *et al.* used the copulas in the study of Markov processes (see [24, 125] and also [144]).

An important help to the diffusion of the copula concept has been given by the international conferences devoted to this idea: Rome (1990), Seattle (1993), Prague (1996), Barcelona (2000), Québec (2004); and by their published proceedings ([23, 133, 8, 19]). But, one should also mention the books by B. Schweizer and A. Sklar ([141]), by H. Joe ([74]) and by R.B. Nelsen ([114]), the most cited references in all the papers concerning this topic. A complete history of the development of this field is given in the papers by B. Schweizer ([138]) and by A. Sklar ([151]).

But, it is precisely in the last five years that the theory of copulas is growing into a central topic in the multivariate models and in the study of the dependence concepts. The explosion of the interest in copulas is testified by the fact that the number of papers reviewed by *Mathematical Reviews* since 2000 and mentioning anywhere the word "copula" is greater than the analogous number of papers in the first "40 years of the life" of the copula notion!

Such growing importance is due mainly to the fact that the copula function has been discovered by many researchers working in different areas of applied mathematics: for instance, in actuarial science ([58, 61]), finance ([51, 15]) and hydrology ([134]).

Nowadays, there are many results on copulas and many applications of them in the real problems. Paraphrasing the words of R.B. Nelsen in the introduction of his book, we could say that "*the study of copulas is a subject still in its youth*".

In this dissertation we present, mainly, several new results in the theory of copulas. However, a great attention is also given to some concepts that are a direct extension of the copula function (e.g., triangular norm, quasi–copula, semicopula, aggregation operator) and which have been introduced in other fields, such as probabilistic metric spaces, semigroup theory, reliability and fuzzy theory: an introduction to these notions is presented in chapter 1.

Taking into account the origin of the problems that spurred the investigations here presented, this dissertation can be divided into three parts, which overlap in several points and which are written in a mixed sequence.

The first part is devoted to the construction of new families of bivariate probability distribution functions. This problem has received great attention in the years ([73]) and, as written by N.I. Fisher in the *Encyclopedia of Statistical Sciences* ([54]), it is one of the main reasons of the interest to statisticians in copulas.

In chapter 4 we study a family of copulas that depend on a univariate function. Specifically, we give necessary and sufficient conditions on a function $f : [0,1] \to [0,1]$ that ensure that the mapping $C_f(x,y) := \min\{x,y\} f(\max\{x,y\})$ is a copula. This method provides several examples and, among others, it is shown that the Cuadras–Augé copulas belong to this class. Such a C_f is suitable to describe the positive dependence between random variables (namely, it is positively quadrant dependent) and, moreover, it has also an interesting probabilistic interpretation.

In chapter 5 we characterize the copulas that can be constructed beginning from their diagonal sections. Note that, if C is the copula associated with two random variables X and Y, then the diagonal section of C, namely $\delta_C(t) := C(t,t)$, expresses the behaviour of the maximum between X and Y. Constructions of this type have been already considered in [56, 57]; in particular, our class is a distinguished subset of the *Bertino class of copulas*, formed by those copulas satisfying a functional equation studied, in the class of triangular norms, by G. Mayor and J. Torrens ([105]).

The study of a generalization of the Archimedean class of copulas is, instead, the topic of chapter 6. This class is larger than the two other ones presented in chapters 4 and 5 and might include both singular and absolutely continuous copulas. Although, as in the Archimedean case, no probabilistic interpretation is given, their simple form and flexibility makes this class suitable to be used in the statistical modelling.

Finally, in chapter 7 we characterize a binary operation on the class of bivariate distribution functions. Such an operation was considered, in the univariate case, by C. Alsina *et al.* ([4]), but their extension to the bivariate case is a bit intricate and stimulate us to introduce the new concept of P–increasing function. Some considerations about bivariate distribution functions with fixed marginal d.f.'s and the convergence of distribution functions are then investigated.

The second part of this dissertation is directly inspired by the work of B. Bassan and F. Spizzichino ([7]). In their investigations on multivariate aging through the analysis of the Schur–concavity of the survival distribution functions, they introduced the concept of *semicopula*, which generalizes the copula function, and studied some of its properties. Following these ideas, we investigate the class of semicopulas (chapter 2) and study a transformation method for copulas, also used in other contexts (chapter 9). Moreover, we notice that semicopulas have an interest of their own in fuzzy logic, where it can considered as a generalization of the boolean conjunction from the set $\{0, 1\}$ to the interval $[0, 1]$, and in fuzzy measures. Chapter 10 is, instead, devoted to the study of Schur–concavity of copulas, which allows us to make some considerations about the properties of associative copulas.

The third part of this dissertation is connected with the theory of aggregation operators. Aggregation or fusion of several inputs into a single output is a basic problem in many practical applications and various categories and several approaches have been proposed and investigated. In particular, this field is especially useful for researchers interested in artificial intelligence and multicriteria decision making, where the aggregation of several inputs is the most difficult and controversial problem. In particular, the aggregation of a finite number of real inputs involves functions already known in a mathematical context as triangular norms, quasi–copulas, copulas and, by now, semicopulas. Through all the dissertation, we often present the results in this most general form, and this point of view is especially underlined in chapter 3, where the class of binary aggregation operator sharing the 2–increasing property is analized in details, and in chapter 8, where another kind of composition is introduced for special subclasses of aggregation operators (semicopulas, quasi–copulas, etc). In particular, this last method is applied to copulas, where it gives a valuable method to construct non–symmetric families.