Chapter 3

Inverse methods

A general *boundary value problem* of elastostatics for a body $\mathcal{B}$ consists in finding a motion $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X})$ that satisfies $\boldsymbol{a}(\boldsymbol{X}, t) = \boldsymbol{0}$ for all particles $\boldsymbol{X}$ of B_r and for all times t. Recalling equation (1.24), this means that the motion must satisfy the equilibrium equation

$$\mathrm{Div}\boldsymbol{T_R} + \rho_r \boldsymbol{b}_r = \boldsymbol{0}, \tag{3.1}$$

everywhere in B_r, and the boundary conditions of surface tractions (1.22) and place,

$$\boldsymbol{t_N} = \boldsymbol{T_R}\boldsymbol{N}, \quad \text{prescribed on } \partial B_r^1, \tag{3.2}$$

$$\boldsymbol{X} = \bar{\boldsymbol{X}}, \quad \text{prescribed on } \partial B_r^2, \tag{3.3}$$

respectively, where ∂B_r^1 and ∂B_r^2 are disjoint parts of ∂B_r such that $\partial B_r = \partial B_r^1 \cup \partial B_r^2$.

The boundary value problem is expressed in terms of material description because, as we have just emphasized in the first chapter, the geometry of the deformed body generally is unknown a priori (otherwise equations (1.18) - (1.20) may be used). From a theoretical point of view, a given rubberlike material can be characterized by an appropriate constitutive equation that will enable us to predict its response to specified loading and displacement boundary conditions. We assume, as a first approximation, that a certain rubber material may be modeled as either a compressible or an incompressible, homogeneous isotropic hyperelastic material such that (1.36) or (1.40) applies. These represention formulae are useful to understand how the given material may be distinguished from another one on grounds of the response functions β_i only. But very little can be said *a priori* about these response functions unless some helpful experiment is made. Then, because measurements can be done only on the boundary of the test specimen, it would better to know *a priori* the kind of deformation that we want to reproduce experimentally in order to know what quantities can be effectively measured. To this end, Beatty [9] says

> *It is clear, in particular, that the experimenter must know a priori the class of deformations that actually may be produced in every compressible or incompressible, homogeneous and isotropic, hyperelastic material by the application of surface loading alone. Also, the surface loads*

> *needed to effect them must be known in order to select the kinds of loading devices that may be used.*

Theoretical results which fit this program are so-called *universal* results. A deformation, or a motion, which satisfies the balance equations with zero body force and which, in equilibrium, is supported by suitable surface tractions alone, is called a *controllable solution*. A controllable solution which is the same for all materials in a given class is a *universal solution* (when the solution is controllable for a specific subclass of material, it is called a *relative universal solution*). Besides universal solutions, other kind of universal results exist, which involve not only the strain but also the stress. For a given deformation or motion, *a local universal relation* is an equation relating the stress components and the position vector which holds at any point of the body and which is the same for any material in a given class.

From the mathematical point of view, the analytical solution for the (3.1) - (3.3) problem may be very hard to attain, even in the simplest boundary value problem, because the set of equations forms a non-trivial system of nonlinear, partial differential equations generating often nonunique solutions. To solve the resulting boundary-value problems, *inverse techniques* can be used to provide simple solutions and to suggest experimental programs for the determination of response functions. Two powerful methods for inverse investigations are the so-called *inverse method* and the *semi-inverse method*. They have been used in elasticity theory as well as in all fields of the mechanics of continua. For example, it is quoted in the book [77] that

> *In the inverse method, a known solution of the displacement is assumed with the aid of which strain and stress states are determined. Finally, using the boundary conditions, the body itself and its load and reactions are determined.*
>
> *In the semi-inverse method, part of unknowns is given, and the missing quantities are determined in such a way that the differential equations and boundary conditions are being satisfied.*

Similarly, Carlson [19] states

> *In the inverse method, we start with a given deformation (i.e., guess an $\boldsymbol{F}$), calculate the corresponding stress from the constitutive equation, and check to see if the stress satisfies equilibrium (generally for zero body force). If equilibrium is not satisfied, then the deformation is discarderd. However, if equilibrium is met, then we attempt to interpret the deformation and stress in a physically meaningful setting. I. e., we consider various shapes for the (deformed) body, calculate the corresponding surface tractions, anh hope to get something of physical interest.*
>
> *The semi-inverse method is just the same, except that in the deformation one includes some arbitrary parameters of functions that can be adjusted so that equilibrium is met or the boundary data comes out to be more interesting.*

Here is what Neményi [90] says about such methods in a general framework of continuum mechanics:

> *We shall call inverse an investigation of a partial differential equation of physics if in it the boundary conditions (or certain other supplementary conditions) are not prescribed at the outset. Instead, the solution is defined by the differential equation, and certain additional analytical, geometrical, kinematical, or physical properties of the field. In the semi-inverse method some of the boundary conditions are prescribed at the outset, whereas others are left open and obtained indirectly through certain simplifying assumption concerning the properties of the fields.*

The true power of the inverse methods is that they can reduce in most cases a system of differential equations in three independent variables to a system having only two, or even one, independent variable(s) which may, or may not, admit an exact solution in closed form. If this reduced system can be solved in closed form, then it is possible to obtain some exact solutions to boundary value problems, that hopefully are meaningful within the framework of the theory that is being employed[1]. Of course, even if it cannot be solved exactly, the semi-inverse method leads to a simpler set of equations that can be resolved numerically. When the use of inverse methods does not lead to new solutions, it may nonetheless yield a negative result in certain cases; that is, the nonexistence of certain types of solutions may be established. Inverse and semi-inverse procedures have been implemented in all fields of the mechanics of continua, and the number of results obtained is very large indeed (see [90] to have an idea of their applications).

3.1 Inverse Method

In order to find exact solutions to the problem (3.1) - (3.3) by the inverse method, the starting point is to assume a suitable form for the deformation, then find the stress fields associated to this deformation by making use of the constitutive equations, and finally verify whether the equilibrium equations are satisfied. In the positive case, one may deduce the surface tractions necessary to maintain the deformation, some of which are of considerable importance experimentally. Let us consider some examples, starting with some *homogeneous deformations*, with zero body forces.

3.1.1 Homogeneous deformations

The most general homogeneous deformation is described by the following form

$$\boldsymbol{x} = \boldsymbol{F}\boldsymbol{X} + \boldsymbol{c}, \tag{3.4}$$

where $\boldsymbol{X}$ and $\boldsymbol{x}$ are the the Cartesian coordinates in the reference and in the current configurations, respectively, $\boldsymbol{F}$ is a constant tensor and $\boldsymbol{c}$ is a constant vector.

[1] This is not always the case, as it is well known in the framework of the Navier-Stokes theory where the exact solutions found by the semi-inverse method are often not compatible with the canonical no-slip boundary conditions.

From (1.35) we deduce that the Cauchy stress $\boldsymbol{T}$ is also constant throughout a compressible material. It follows that the equilibrium equations are satisfied only when the body force $\boldsymbol{b}$ is zero and these deformations therefore may be produced by surface tractions alone[2]. For incompressible materials we deduce from (1.39) that if the hydrostatic pressure p is constant, then the above results also apply.

Let us consider *pure homogeneous deformations*, described by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{3.5}$$

where (X_1, X_2, X_3) and (x_1, x_2, x_3) are the the Cartesian coordinates in the reference and in the current configurations, respectively, and $\lambda_1, \lambda_2, \lambda_3$, are positive constants. The physical components of $\boldsymbol{B}$ and of his inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix}, \tag{3.6}$$

respectively, and the first three principal invariants are given by

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned} \tag{3.7}$$

By formula (1.36) the stress components for a compressible material are

$$\begin{aligned} T_{11} &= \beta_0 + \beta_1\lambda_1^2 + \beta_{-1}\lambda_1^{-2}, & T_{22} &= \beta_0 + \beta_1\lambda_2^2 + \beta_{-1}\lambda_2^{-2}, \\ T_{33} &= \beta_0 + \beta_1\lambda_3^2 + \beta_{-1}\lambda_3^{-2}, & T_{ij} &= 0 \ (i \neq j). \end{aligned} \tag{3.8}$$

In the incompressible case the deformation (3.5) must satisfy the constraint $I_3 = 1$, that is

$$\lambda_1\lambda_2\lambda_3 = 1, \tag{3.9}$$

so that, in contrast to the compressible case, only two of the constants $\lambda_1, \lambda_2, \lambda_3$ are independent. By formula (1.40), and because we are considering zero body force, the equilibrium equations are satisfied only if $p = p_0$ with p_0 constant. The stress components for an incompressible material are

$$\begin{aligned} T_{11} &= -p_0 + 2W_1\lambda_1^2 - 2W_2\lambda_1^{-2}, & T_{22} &= -p_0 + 2W_1\lambda_2^2 - 2W_2\lambda_2^{-2}, \\ T_{33} &= -p_0 + 2W_1\lambda_3^2 - 2W_2\lambda_3^{-2}, & T_{ij} &= 0 \ (i \neq j). \end{aligned} \tag{3.10}$$

In both cases only normal stresses are present on surfaces parallel to the coordinate planes. The incompressible case (3.10) differs from (3.8) by an arbitrary constant p_0. The appearance of this term is one of the reasons why constrained materials are easier to deal with mathematically than unconstrained ones.

[2]This result is important physically since it is relatively easy to apply forces to a boundary, see Beatty [9].

Dilatation

In the special case where $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ (with $\lambda > 0$ because $J > 0$), the deformation (3.5) is called *uniform dilatation.* The stress components (3.8) then become

$$T_{ij} = (\beta_0 + \beta_1\lambda^2 + \beta_{-1}\lambda^{-2})\delta_{ij}, \tag{3.11}$$

where δ is the Kronecker symbol. The term $-(\beta_0 + \beta_1\lambda^2 + \beta_{-1}\lambda^{-2})$ corresponds therefore to a hydrostatic pressure that we denote by $P(\lambda^2)$. When $P > 0$ the body is subjected to a hydrostatic pressure, while for $P < 0$ it is subjected to a hydrostatic tension. By (1.18) we obtain the stress vector $\boldsymbol{t}$ in the form

$$\boldsymbol{t_n} = -P(\lambda^2)\boldsymbol{n}, \tag{3.12}$$

where $\boldsymbol{n}$ is a unit vector normal to the surface. Hence, to maintain this deformation, the stress vector must be normal to the surface at each point of the boundary. In general one would expect that the volume of a compressible material held in equilibrium under the action of a uniform pressure should be less than its volume before deformation, that under the action of a uniform tension, the volume should be greater than its initial volume, and that when no traction is applied on the boundary, the volume remains unchanged. Since the variation of volume is $J = \lambda^3$, an equivalent statement is that

$$\begin{aligned} &\lambda < 1 \text{ when } P > 0,\\ &\lambda > 1 \text{ when } P < 0,\\ &\lambda = 1 \text{ when } P = 0. \end{aligned} \tag{3.13}$$

Furthermore, P should be a monotonic decreasing function of λ in order to increase the volume when the applied pressure is increased and viceversa, so that

$$\frac{\mathrm{d}P}{\mathrm{d}\lambda} < 0. \tag{3.14}$$

In linear elasticity this constraint is equivalent to require that the bulk modulus κ is positive (see $(1.54)_2$). In view of the definition of P, the relation (3.14) places some restrictions on the response functions β_i. In the incompressible case, the constraint (3.9) requires that $\lambda = 1$ so that there is no deformation.

Simple extension

When $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = \bar{\lambda}$, the deformation (3.5) is called *uniform extension* (when $\lambda > 1$) or *contraction* (when $\lambda < 1$) in the X_1-direction, together with equal extension or contraction in the lateral X_2- and X_3-directions. For compressible materials we deduce from (3.8) that the stress components are

$$\begin{aligned} &T_{11} = \beta_0 + \beta_1\lambda^2 + \beta_{-1}\lambda^{-2}, \qquad T_{22} = T_{33} = \beta_0 + \beta_1\bar{\lambda}^2 + \beta_{-1}\bar{\lambda}^{-2},\\ &T_{ij} = 0 \ (i \neq j). \end{aligned} \tag{3.15}$$

The simplest stress system arises when (if possible), $T_{22} = T_{33} = 0$ in order to have traction free lateral sides. This gives

$$\beta_0 + \beta_1 \bar{\lambda}^2 + \beta_{-1} \bar{\lambda}^{-2} = 0. \tag{3.16}$$

Equation (3.16) depends on λ^2 and $\bar{\lambda}^2$ and for a given λ, it is not obvious that it should have a single positive root $\bar{\lambda}^2$. If (3.16) has no root then uniform extension cannot be effected by applying a tension T_{11} alone, i.e. others surface tractions are necessary. If (3.16) has more than one root, then there are more than one tensile sress which produce a given extension with the remaining faces traction-free. When it is possible to apply a tension in the X_1-direction with the other stresses being zero, the extension is called *simple*, and we expect the specimen to increase in length in this direction, whereas when we apply a pressure the length should decrease. Also, if no tension is applied on the boundary, then the length should remain unchanged. Finally when T_{11} is increased, the extension should increase and vice-versa. Hence T_{11} should verify

$$\frac{\mathrm{d}T_{11}}{\mathrm{d}\lambda} > 0. \tag{3.17}$$

This inequality places a further restriction on the response functions β_i. For incompressible materials, the constraint (3.9) implies $\bar{\lambda} = 1/\sqrt{\lambda}$. Here the stress components (3.10) become

$$\begin{aligned} T_{11} &= -p_0 + 2W_1\lambda^2 - 2W_2\lambda^{-2}, \qquad T_{ij} = 0 \ (i \neq j), \\ T_{22} = T_{33} &= -p_0 + 2W_1\lambda^{-1} - 2W_2\lambda. \end{aligned} \tag{3.18}$$

The principal difference with the compressible case is that the boundary conditions $T_{22} = T_{33} = 0$ appropriate to the block subject to a tension T_{11} can always be satisfied on setting

$$p_0 = 2W_1\lambda^{-1} - 2W_2\lambda. \tag{3.19}$$

The uniaxial tension necessary to maintain this deformation (see also (2.6)) is

$$T_{11} = 2(\lambda^2 - \lambda^{-1})W_1 + 2(\lambda - \lambda^{-2})W_2. \tag{3.20}$$

Simple shear

Let us consider the homogeneous deformation of *simple shear*,

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{3.21}$$

where k is a constant parameter representing the *amount of shear*. We consider the shearing by applied surface tractions alone of a block with faces initially parallel to the coordinates planes. This deformation is quite difficult to produce experimentally because of the complex surface tractions needed to maintain it (as we see below). However, it is probably the simplest example illustrating that large deformations are different from infinitesimal deformations described by linear elasticity, not only in magnitude but also in the novel effects they produce.

It easy to determine the physical components of the left Cauchy-Green deformation tensor $\boldsymbol{B}$ and of its inverse $\boldsymbol{B}^{-1}$ as

$$\begin{pmatrix} 1+k^2 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -k & 0 \\ -k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.22}$$

respectively, so that the first three principal invariants of $\boldsymbol{B}$ are $I_1 = I_2 = 3 + k^2$, $I_3 = 1$. The stress components for this deformation for compressible materials (see (1.36)) are given by

$$\begin{aligned} T_{11} &= \beta_0 + \beta_1(1+k^2) + \beta_{-1}, & T_{12} &= k(\beta_1 - \beta_{-1}), \\ T_{22} &= \beta_0 + \beta_1 + \beta_{-1}(1+k^2), & T_{13} &= 0, \\ T_{33} &= \beta_0 + \beta_1 + \beta_{-1}, & T_{23} &= 0. \end{aligned} \tag{3.23}$$

Thus we see that *both* normal and shear stresses are present on surfaces parallel to the coordinate planes and that, as in the previous deformation, the stress components are constants.

To consider the relation between the shear stress and the amount of shear, we define

$$\mu(k^2) = \beta_1(3+k^2, 3+k^2, 1) - \beta_{-1}(3+k^2, 3+k^2, 1), \tag{3.24}$$

so that

$$T_{12} = \mu(k^2)k. \tag{3.25}$$

The quantity $\mu(k^2)$ is called the *generalized shear modulus*. Its value $\mu \equiv \mu(0)$ in the natural state is the initial shear modulus. Physically we would expect the shear stress acting on a surface with normal in the $2-$direction to be in the direction in which the surface has been displaced, so that we expect

$$\mu(k^2) > 0. \tag{3.26}$$

By (3.24) we can see how the empirical inequalities (1.46) are sufficient to establish (3.26). Because the shear stress T_{12} is an odd function of the amount of shear, the shear stress is therefore in the direction of the amount of shear (since the normal stresses are even functions of the amount of shear, they do not depend on its direction). In linear elasticity (making use of (1.51)), simple shear can be maintained by applying only shear stresses on the faces of specimen. A similar situation does not arise with a finite deformation, unless a degenerate material is considered for which a simple shear can be produced in the absence of all stress. In fact, imposing $T_{11} = T_{22} = T_{33} = 0$ gives the conditions

$$\begin{aligned} &\beta_0 + \beta_1(1+k^2) + \beta_{-1} = 0, \\ &\beta_0 + \beta_1 + \beta_{-1}(1+k^2) = 0, \\ &\beta_0 + \beta_1 + \beta_{-1} = 0, \end{aligned} \tag{3.27}$$

from which it follows that $\beta_0 = \beta_1 = \beta_{-1} = 0$ and in this case, T_{12} is also zero. We therefore conclude that, for all materials exhibiting physically reasonable response, the simple shear (3.21) cannot be produced by applying only shear stresses on surfaces parallel to the cordinate planes: normal stresses are also necessary[3].

From (3.23) we obtain

$$\begin{aligned} &\beta_1 k^2 = T_{11} - T_{33}, \quad \beta_{-1} k^2 = T_{22} - T_{33}, \\ &\beta_0 k^2 = (2 + k^2) T_{33} - (T_{11} + T_{22}). \end{aligned} \tag{3.28}$$

This is an example of how the inverse method can be applied to find universal solutions and of how it is possible to use this kind of solutions to design an experimental test to determine the β_i's. However, we note that an experiment based on simple shear only is too restrictive to determine completely the response functions. In fact this deformation allows exploration of what happens only along the line $I_3 = 1,\ I_1 = I_2$ in the space of invariants, made of $I_1 > 0,\ I_2 > 0$ and $I_3 > 0$.

From (3.23) it is also possible to derive the relations

$$T_{13} = T_{23} = 0, \quad kT_{12} = T_{11} - T_{22}. \tag{3.29}$$

These relations provide links between the stress components and the amount of shear k which do not depend on the particular elastic isotropic material. They are *universal relations*. They are important because for example if one finds experimentally that $(3.29)_2$ is not satisfied then one may conclude that the material under investigation is not an isotropic elastic material. Also, except in the case of a degenerate material, T_{11} cannot be equal to[4] T_{22}. By $(3.29)_2$ we deduce also that the knowledge of the behavior of the shear stress in simple shear gives no information about the normal stresses. This intuition is present in linear elasticity, where simple shear alone cannot determine the normal stresses while by $(3.29)_2$ the normal stresses characterize the simple shear.

The unit normal $\boldsymbol{n}$ and the unit tangent $\boldsymbol{\tau}$ on the inclined faces have the components

$$\boldsymbol{n} = (1, -k, 0) \,/ \sqrt{(1 + k^2)}, \qquad \boldsymbol{\tau} = (k, 1, 0) \,/ \sqrt{(1 + k^2)}, \tag{3.30}$$

so that we may calculate the normal stress N and the shear stress T which have to be applied to the inclined faces of the deformed specimen in order to maintain the simple shear deformation. By use of (1.18), they are

$$N = \boldsymbol{t_n} \cdot \boldsymbol{n}, \quad T = \boldsymbol{t_n} \cdot \boldsymbol{\tau}. \tag{3.31}$$

and we therefore obtain the following relationships

$$\begin{aligned} &(1 + k^2) N = T_{11} + k^2 T_{22} - 2k T_{12}, \\ &(1 + k^2) T = k(T_{11} - T_{22}) + (1 - k^2) T_{12}. \end{aligned} \tag{3.32}$$

[3] When normal stresses are not applied, the material tends to contract or expand. This result was apparently conjectured by Kelvin and is often called the *Kelvin effect* [123].

[4] In 1909 Poynting noticed a similar phenomenon and performed a series of torsion experiments to illustrate the lengthening of a metal wire when no normal force was applied. The existence of unequal normal stresses is often referred to as the *Poynting effect* [102].

Using the universal relation $(3.29)_2$ we deduce

$$\begin{aligned} T_{12} &= (1+k^2)T, \\ kT_{12} &= (1+k^2)(T_{22}-N), \\ N &= T_{22} - kT. \end{aligned} \tag{3.33}$$

We deduce some interesting consequences from these relationships. First, we can see that $|T| < |T_{12}|$ and $|N| < |T_{22}|$. Hence if T_{22} is negative, so is N, i.e. if the normal traction on the shearing planes is a pressure, then so is the normal traction on the inclined faces. Since N is different from T_{22} (otherwise this would again imply that $\mu(k^2) = 0$), the Poynting effect still holds when referred to the current faces of the sheared block. Finally by $(3.33)_2$ it follows that there may be special elastic materials such that $N = 0$ for all shears k.

Many of the results for compressible materials are still valid for incompressible bodies. In this case, by (1.40) the stress components are given as

$$\begin{aligned} T_{11} &= -p_0 + 2(1+k^2)W_1 - 2W_2, \\ T_{22} &= -p_0 + 2W_1 - 2(1+k^2)W_2, \\ T_{33} &= -p_0 + 2W_1 - 2W_2, \\ T_{12} &= k\mu(k^2), \quad T_{13} = T_{23} = 0, \end{aligned} \tag{3.34}$$

where p_0 is a constant to be determined by the prescribed boundary conditions, and $\mu(k^2) = 2(W_1 + W_2)$ is obtained from (3.24) by replacing β_1 with $2W_1$ and β_{-1} with $-2W_2$. As in the compressible case, the remarks concerning the behaviour of normal stress and shear stresses when the direction of shear is reversed are still valid, as are the results (3.29), (3.32) and (3.33) from which we deduce that T_{11} and T_{22} cannot be equal, and that the Poynting effect is still present. In constrast with the compressible case, it is possible to make any one of the normal stresses vanish by an appropriate choice of p_0. For example we may choose p_0 such that $T_{33} = 0$ and in this case we see that $T_{11} > 0$ and $T_{22} < 0$ if and only if the empirical inequalities hold. In this particular case ($T_{33} = 0$) we obtain

$$p_0 = 2(W_1 - W_2), \tag{3.35}$$

and

$$T_{11} = 2k^2 W_1, \quad T_{22} = -2k^2 W_2, \tag{3.36}$$

$$T_{12} = k\mu(k^2), \quad T_{33} = T_{13} = T_{23} = 0, \tag{3.37}$$

showing that the normal stress on the shearing planes is always a pressure since $T_{22} < 0$, from which we deduce (as in the compressible case) that $T < 0$. If these pressures are not applied in addition to the shear forces, then we would expect the material to stretch in the 1− and 2−directions and hence to contract in the 3−direction (because of the incompressibility constraint). In other words, one form of the Poynting effect is observed.

Rivlin was one of the first authors to use inverse procedures to construct some examples of exact solutions of physical interest to both analysts and experimenters.

His work is very interesting also because it marked the birth in 1948 of the modern theory of finite elasticity (see Rivlin [111] for the collected works). Later, Ericksen [33, 34] introduced a different and more general approach to the investigation of inverse solutions, and such results provide the kinds of tools requested by experimenters.

3.1.2 Universal solutions

We know from the previous section that homogeneous deformations, which play a fundamental role in the theory of finite elastic deformations, can be maintained in all homogeneous bodies under the action of surface forces alone, because the stress corresponding to (3.4) is a constant tensor, and the balance equations are then trivially satisfied in absence of body forces. They therefore represent a set of universal solutions for all homogeneous materials. Ericksen [34] proved in 1955 that they are the only controllable deformations possible in every *compressible*, homogeneous and isotropic hyperelastic material. This result is known as *Ericksen's theorem*.

For *incompressible* materials, the definite answer is still lacking in the search for all universal solutions. So far, five families of universal solutions have been found in addition to homogeneous deformations. All solutions are such that appropriate physical components of stress are constants on each member of a family of parallel planes, coaxial cylinders, or concentric spheres. Let us start by looking at how some restrictions on the physical components of the stress can help to simplify the problem, in general by reducing a partial differential system to an ordinary one with less unknowns. To this end we consider the case of cylindrical coordinate (r, θ, z) only. A similar discussion can be conducted for Cartesian and spherical coordinates.

The equilibrium equations, in the absence of body force (1.20), read

$$\begin{aligned} &\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0, \\ &\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2}{r}T_{r\theta} = 0, \\ &\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} = 0. \end{aligned} \tag{3.38}$$

If we assume that $\boldsymbol{T} + p\boldsymbol{I}$ depends on r only (such assumption is often made when the problem has cylindrical symmetry), the partial differential system (3.38) simplifies as

$$\begin{aligned} &\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0, \\ &\frac{1}{r^2}\frac{\partial\left(r^2 T_{r\theta}\right)}{\partial r} - \frac{1}{r}\frac{\partial p}{\partial \theta} = 0, \\ &\frac{1}{r}\frac{\partial\left(r T_{rz}\right)}{\partial r} - \frac{\partial p}{\partial z} = 0. \end{aligned} \tag{3.39}$$

By the further assumptions that

$$T_{r\theta} = 0, \quad T_{rz} = 0, \tag{3.40}$$

it follows from (3.39) that p depends only on r and consequently that the T_{rr} component depends on r only. Under these strong assumptions, the partial differential system (3.38) is reduced to an ordinary differential system that is easier to solve. The T_{rr} and $T_{\theta\theta}$ components are given by

$$T_{rr} = -\int \frac{T_{rr} - T_{\theta\theta}}{r}\,\mathrm{d}r, \quad T_{\theta\theta} = \frac{\mathrm{d}\,(rT_{rr})}{\mathrm{d}r}. \tag{3.41}$$

Let us consider the first family of universal solutions (in the literature, these solutions are classified in "families"). It is given by the following deformation

Family 1:

$$r = \sqrt{2AX}, \quad \theta = BY, \quad z = \frac{Z}{AB} - BCY, \tag{3.42}$$

which describes *bending, stretching and shearing of a rectangular block*. Here (X, Y, Z) and (r, θ, z) are the cartesian and cylindrical coordinates in the reference and in the current configuration, respectively, and A, B, C are constants with $AB \neq 0$. If $C = 0$ the deformation describes *pure bending* and carries the parallepipedic block bounded by the planes $X = X_1,\ X = X_2,\ Y = \pm Y_0,\ Z = \pm Z_0$ into the circular annular wedge bounded by the cylinders $r = r_1 = \sqrt{2AX_1}$, $r = r_2 = \sqrt{2AX_2}$, and the planes $\theta = \pm\theta_0 = \pm BY_0,\ z = \pm z_0 = \pm Z_0/(AB)$. When B is prescribed, then the arbitrary axial stretch $1/(AB)$ is allowed, and the radial stretch is so adjusted as to render the deformation isochoric[5]. The physical components of $\boldsymbol{B}$ and of its inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} \dfrac{A^2}{r^2} & 0 & 0 \\ 0 & B^2r^2 & -B^2Cr \\ 0 & -B^2Cr & B^2C^2 + \dfrac{1}{A^2B^2} \end{bmatrix}, \tag{3.43}$$

and

$$\begin{bmatrix} \dfrac{r^2}{A^2} & 0 & 0 \\ 0 & \dfrac{1}{B^2r^2} + \dfrac{A^2B^2C^2}{r^2} & \dfrac{A^2B^2C}{r} \\ 0 & \dfrac{A^2B^2C}{r} & A^2B^2 \end{bmatrix}, \tag{3.44}$$

respectively. The first two principal strain invariants are

$$I_1 = \frac{A^2}{r^2} + B^2r^2 + B^2C^2 + \frac{1}{A^2B^2}, \tag{3.45}$$
$$I_2 = \frac{r^2}{A^2} + \frac{1}{r^2}\left(\frac{1}{B^2} + A^2B^2C^2\right) + A^2B^2,$$

[5] In the general case, the deformation may be effected in two steps, the first of which is the bending and axial stretch, while the second is a homogeneous strain which carries the body into the solid bounded by the cylindrical surfaces $r = r_1$ and $r = r_2$, the planes $\theta = \pm\theta_0$, and the helicoidal surfaces $z + C\theta = \pm z_0$.

respectively, and $I_3 = 1$ in agreement with incompressibility. From (1.40) we see that the physical components of $\boldsymbol{T} + p\boldsymbol{I}$ are functions of r only and that (3.40) is satisfied. By $(3.41)_1$

$$\begin{aligned} T_{rr} &= -\int \left[2\frac{\partial W}{\partial I_1}\left(\frac{A^2}{r^3} - B^2 r\right) - 2\frac{\partial W}{\partial I_2}\left(\frac{r}{A^2} - \frac{1}{r^3}\left(\frac{1}{B^2} + A^2B^2C^2\right)\right)\right] \mathrm{d}r \\ &= \int \left(\frac{\partial W}{\partial I_1}\frac{\mathrm{d}I_1}{\mathrm{d}r} - \frac{\partial W}{\partial I_2}\frac{\mathrm{d}I_2}{\mathrm{d}r}\right) \mathrm{d}r, \end{aligned} \tag{3.46}$$

from which we obtain the other components of the stress.

$$\begin{aligned} T_{\theta\theta} &= T_{rr} + 2\left[B^2r^2 - \frac{A^2}{r^2}\right]\frac{\partial W}{\partial I_1} - 2\left[\frac{1}{r^2}\left(\frac{1}{B^2} + A^2B^2C^2\right) - \frac{r^2}{A^2}\right]\frac{\partial W}{\partial I_2}, \\ T_{zz} &= T_{rr} + 2\left[B^2C^2 + \frac{1}{A^2B^2} - \frac{A^2}{r^2}\right]\frac{\partial W}{\partial I_1} - 2\left[A^2B^2 - \frac{r^2}{A^2}\right]\frac{\partial W}{\partial I_2}, \\ T_{\theta z} &= -2B^2Cr\frac{\partial W}{\partial I_1} - 2\frac{A^2B^2C}{r}\frac{\partial W}{\partial I_2}. \end{aligned} \tag{3.47}$$

To obtain the unknown p we make use of (1.40),

$$p = -T_{rr} + 2W_1(\boldsymbol{B})_{11} - 2W_2(\boldsymbol{B}^{-1})_{11}, \tag{3.48}$$

and therefore by (3.43), (3.44) and (3.46),

$$p = -\int \left(\frac{\partial W}{\partial I_1}\frac{\mathrm{d}I_1}{\mathrm{d}r} - \frac{\partial W}{\partial I_2}\frac{\mathrm{d}I_2}{\mathrm{d}r}\right) \mathrm{d}r + 2\left(\frac{A^2}{r^2}W_1 - \frac{r^2}{A^2}W_2\right). \tag{3.49}$$

It is possible to choose the constants in (3.46) in order to have the cylinder $r = r_1$ free of traction. To have the cylinder $r = r_2$ also free of traction, it is then necessary that

$$\int_{r_1}^{r_2} \left(\frac{\partial W}{\partial I_1}\frac{\mathrm{d}I_1}{\mathrm{d}r} - \frac{\partial W}{\partial I_2}\frac{\mathrm{d}I_2}{\mathrm{d}r}\right) \mathrm{d}r = 0, \tag{3.50}$$

and when this is verified, a particular relation among the constants A, B, C applies. Independently of whether or not (3.50) can be satisfied, the helicoidal faces $z + C\theta = \pm z_0$ (with unit normal $\boldsymbol{n} = (0, C/r, 1)/\sqrt{1 + C^2/r^2}$) cannot be free of traction in order to maintain the deformation. The normal and tangential tractions N and T, respectively, are

$$\begin{aligned} N &= \frac{1}{1 + C^2/r^2}\left[T_{zz} + 2\frac{C}{r}T_{\theta z} + \frac{C^2}{r^2}T_{\theta\theta}\right], \\ T &= \frac{1}{1 + C^2/r^2}\left[\left(1 - \frac{C^2}{r^2}\right)T_{\theta z} + \frac{C}{r}\left(T_{\theta\theta} - T_{zz}\right)\right], \end{aligned} \tag{3.51}$$

respectively. Only when pure bending is considered (i.e. $C = 0$) we can deduce from (3.47) and (3.51) that $T = 0$ and $N = T_{zz}$. In general, both normal and tangential tractions must be applied. The presence of these tractions gives rise to the Poynting effect for bending, similar to the Poynting effect discussed for the simple shear deformation. We thus underline how the Poynting effet is in general inevitable in nonlinear elasticity.

A similar discussion can be made for the other four families, which are described by the following deformations.

Family 2: Straightening, stretching and shearing of a sector of a hollow cylinder,

$$x = \frac{1}{2}AB^2R^2, \quad y = \frac{\Theta}{AB}, \quad z = \frac{Z}{B} + \frac{C\Theta}{AB}. \tag{3.52}$$

Family 3: Inflation, bending, torsion, extension and shearing of an annular wedge,

$$r = \sqrt{AR^2 + B}, \quad \theta = C\Theta + DZ, \quad z = E\Theta + FZ, \tag{3.53}$$

with $A(CF - DE) = 1$.

Family 4: Inflation or eversion of a sector of a spherical shell,

$$r = \left(\pm R^3 + A\right)^{1/3}, \quad \theta = \pm\Theta, \quad \varphi = \Phi. \tag{3.54}$$

Family 5: Inflation, bending, extension, and azimuthal shearing, of an annular wedge,

$$r = \sqrt{A}R, \quad \theta = D\ln(BR) + C\Theta, \quad z = FZ, \tag{3.55}$$

with $ACF = 1$.

Here A, B, C, D, E, F are constants. It seems that the class of static deformations that are possible in all homogeneous, isotropic, incompressible elastic bodies subject to surface tractions only is likely to be exhausted by these cases. Some progress toward determining other deformations may be made if we replace the purely inverse method by a semi-inverse one, considering a family of deformations involving one or more arbitrary functions which may be determined so as to render the deformation possible for a particular material.

3.2 Semi-inverse method

In elasticity the first application of the semi-inverse method is due to Saint-Venant [5, 6] in 1855. He was the first to study the problem of linear elastostatics for a right long cylinder free from volume forces and loaded only at the bases by unspecified tractions. This problem was later on called the problem of Saint-Venant (*Saint-Venantsche Problem*) by Clebsch [23]. The starting point of the application of the semi-inverse method in order to solve the problem is that some components of the stress vanish. In particular, it is assumed that the normal tension on every section parallel to X_3, the axis of the cylinder, be zero:

$$T_{11} = T_{12} = T_{22} = 0. \tag{3.56}$$

When this assumption is made, it is possible to find a closed-form solution of the problem by the use of the linear equilibrium equations (1.50), of the linear constitutive equations (1.48), of the compatibility Beltrami conditions (1.63), and of the prescribed boundary conditions. The displacement field for the points of the cylinder turns out to depend linearly on four constants; these represent kinematic parameters to be specified at one base of the cylinder. Each of them characterizes a simple mode of deformation of the cylinder: extension, bending, torsion, and flexure, and it may be shown that the four kinematic parameters are linear functions of the resultant actions on the bases.

The semi-inverse assumption on the field of stress is of fundamental importance to find the approximate analytical solution. Although this assumption is suggested by the geometrical and boundary surface tractions, it is justified afterwards by the existence of the solution found. The Kirchhoff principle shows then the uniquess of the solution.

> KIRCHHOFF, 1859 *If either the surface displacement or the surface tractions are given, the solution of the problem of equilibrium of an elastic body is unique in the sense that the state of stress (and strain) is determinate without ambiguity, provided that the magnitude of the stress (and strain) is so small that the strain energy function exists and remains positive definite.*

Several applications of the semi-inverse method can be found in the literature on nonlinear elasticity. Of course it is not possible to list all such results because a survey aiming at completeness would require a whole book. Here we present some representative examples in order to underline some aspects of the semi-inverse method, and other useful examples for our discussion are given in the next chapters. First, we recall something just discussed on *simple extension*, but here we modify the problem a little bit. Then we discuss a problem of *anti-plane shear* (see [59, 61, 63, 113] for more details). Finally, some others remarks are discussed for the *radial deformation* problem.

3.2.1 Simple uniaxial extension

Let us consider the uniaxial extension of a rod by prescribing some boundary conditions at the outset: we ask that our model gives traction-free lateral surfaces. In this case, coming back to (3.5), we set $\lambda_1 = \lambda$ to denote the uniaxial stretch, while λ_2 and λ_3 denote the lateral stretches.

In the compressible case, making use of the (3.8) relations and the fact that the stress components are constants, we have to set

$$\begin{aligned} \beta_0 + \beta_1\lambda_2^2 + \beta_{-1}\lambda_2^{-2} &= 0, \\ \beta_0 + \beta_1\lambda_3^2 + \beta_{-1}\lambda_3^{-2} &= 0. \end{aligned} \tag{3.57}$$

Forming the difference, we obtain

$$(\lambda_2^2 - \lambda_3^2)\left(\beta_1 - \frac{1}{\lambda_2^2\lambda_3^2}\beta_{-1}\right) = 0. \tag{3.58}$$

Applying the empirical inequalities (1.46) to (3.58), we obtain a necessary condition: $\lambda_2 = \lambda_3$ to be satisfied. The same condition is also discussed in the previous chapter by using Batra's Theorem [7]. But now we see how the arbitrary parameters λ_2 and λ_3 are adjusted a posteriori to meet the boundary conditions. In this case, by equations (3.57), we may solve uniquely as

$$\lambda_3 = \lambda_2 = \lambda_2(\lambda), \tag{3.59}$$

and obtain a simple extension under a tensile stress

$$T_{11}(\lambda) = (\lambda^2 - \lambda_2^2)\left(\beta_1 - \frac{1}{\lambda^2\lambda_2^2}\beta_{-1}\right). \tag{3.60}$$

In the incompressible case, equation (3.60) is remplaced by (3.20). The Poisson function $\nu(\lambda)$ given by the expression (2.34) for an incompressible material reads here as

$$\nu(\lambda) = \frac{1}{\sqrt{\lambda}(\sqrt{\lambda}+1)}. \tag{3.61}$$

In the natural state of an incompressible material, the Poisson ratio has the value $\nu = \nu(1) = 1/2$, otherwise (3.61) is a monotone decreasing function of the amount of uniaxial stretch. In a simple tension experiment, we see that (3.61) can be used to evaluate whether the material is incompressible[6]. Indeed equation (3.61) is universal for any isotropic uniform elastic material which is incompressible. In the case of compressibility, equation (2.34) is not universal since $\lambda_2(\lambda)$ depends on the special material we are considering. For example, for a general Blatz-Ko material (2.38), we know from (2.36) that

$$\lambda_2(\lambda) = \lambda^{(n-1)/2}, \tag{3.62}$$

where n is parameter characterizing a particular Blatz-Ko model. By formula (2.34) it is clear that the Poisson function,

$$\nu(\lambda) = \frac{1-\lambda^{(n-1)/2}}{\lambda - 1}, \tag{3.63}$$

depends now on the Blatz-Ko model used.

3.2.2 Anti-plane shear deformation

Let us consider the following deformation written in Cartesian coordinates

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + w(X_1, X_2), \tag{3.64}$$

representing an *anti-plane shear deformation*, where $\boldsymbol{X}$ denote the reference coordinates and $\boldsymbol{x}$ the current coordinates of the body. The displacement is therefore described by a single smooth scalar function (the *out-of-plane* displacement) $w \equiv w(X_1, X_2)$. By the semi-inverse procedure, in the search of static solutions with zero body force, we must verify if the balance equations $\text{div}\boldsymbol{T} = \boldsymbol{0}$ are satisfied for some specified w and/or for some specific class of materials.

Let us consider the incompressible case. Before analysing the general case (3.64) we suppose that our body has axial symmetry (cylindrical body) and we assume that the anti-plane shear problem may be solved by considering an axisymmetric deformation of the form

$$w(X_1, X_2) = w(X_1^2 + X_2^2). \tag{3.65}$$

[6]Beatty and Stalnaker [12] show that although the Poisson function of every incompressible material has the universal constant, natural limit value 1/2, the converse is generally false.

In cylindrical coordinates, the deformation (3.64) can be rewritten as

$$r = R, \quad \theta = \Theta, \quad z = Z + w(R), \tag{3.66}$$

where $w(R)$ is the axial displacement. Such a deformation is also called *telescopic shear*. The physical components of $\boldsymbol{B}$ and its inverse $\boldsymbol{B}^{-1}$ are given by

$$\begin{bmatrix} 1 & 0 & w' \\ 0 & 1 & 0 \\ w' & 0 & 1+w'^2 \end{bmatrix}, \quad \begin{bmatrix} 1+w'^2 & 0 & -w' \\ 0 & 1 & 0 \\ -w' & 0 & 1 \end{bmatrix}, \tag{3.67}$$

respectively, where the prime denotes differentation respect to R, and the first three principal strain invariants are

$$I_1 = I_2 = 3 + w'(R)^2, \tag{3.68}$$

and $I_3 = 1$ in agreement with the incompressibility constraint. By formula (1.40) we obtain the physical components of the Cauchy stress tensor as

$$\begin{aligned} T_{rr} &= -p + 2W_1 - 2(1+w'^2)W_2, & T_{r\theta} &= 0, \\ T_{\theta\theta} &= -p + 2W_1 - 2W_2, & T_{rz} &= 2(W_1 + W_2)w', \\ T_{zz} &= -p + 2(1+w'^2)W_1 - 2W_2, & T_{\theta z} &= 0. \end{aligned} \tag{3.69}$$

Finally, the equilibrium equations reduce to equations (3.39) but now $T_{r\theta}$ only is zero, showing that $p = p(r, z)$. On using the expressions of T_{rr} and $T_{\theta\theta}$ in $(3.39)_1$ we obtain

$$p(r, z) = 2W_1 - 2W_2(1+w'^2) - \int \frac{2}{r} W_2 w'^2 \, \mathrm{d}r + g(z), \tag{3.70}$$

where g is an arbitrary function of z. By virtue of (3.70) and the expression of T_{rz}, we can rewrite $(3.39)_3$ as

$$\frac{\mathrm{d}}{\mathrm{d}r}(rT_{rz}) = \lambda r, \tag{3.71}$$

where $\lambda = \mathrm{d}g(z)/\mathrm{d}z$. This equation is a second-order nonlinear ordinary differential equation for $w(R)$, with an immediate first integral in the form of a first-order differential equation for $w(R)$, namely

$$2(W_1 + W_2)w' = \frac{\lambda R}{2} + \frac{C_1}{R}, \tag{3.72}$$

where C_1 is a constant of integration. The problem may be completely solved once the strain energy function W is specified.

The important issue to emphasize here is that the system of partial differential equations $\mathrm{div}\boldsymbol{T} = \boldsymbol{0}$ is a compatible system that may have an analytical solution when W is given. The telescopic shear is a special anti-plane shear problem and even though a solution to this problem may be found, in general we are not able to get any information on the general anti-plane problem (3.64). In fact, reconsidering (3.64), we show that the previous favourable situation is not verified now. This means that in the search for solutions of the balance equations by a semi-inverse method we are not always lucky; in some cases, the semi-inverse method may be used in a negative sense, by showing the nonexistence of solutions. For example, as in the general antiplane shear (3.64), it may happen that the balance equations reduce to an overdetermined set of differential equations which are not compatible, showing therefore that a pure antiplane shear is not always possible.

The physical components of $\boldsymbol{B}$ and of its inverse $\boldsymbol{B}^{-1}$ for the general anti-plane shear deformation (3.64), are given by

$$\begin{bmatrix} 1 & 0 & w_1 \\ 0 & 1 & w_2 \\ w_1 & w_2 & 1+k^2 \end{bmatrix}, \quad \begin{bmatrix} 1+w_1^2 & w_1 w_2 & -w_1 \\ w_1 w_2 & 1+w_2^2 & -w_2 \\ -w_1 & -w_2 & 1 \end{bmatrix}, \tag{3.73}$$

respectively, and the first three principal invariants are $I_1 = I_2 = 3 + k^2,\ I_3 = 1$, where $k = |\nabla w|$ and $w_i\ (i = 1, 2)$ are the derivatives of w with respect to $X_i,\ (i = 1, 2)$. Following (1.40), the Cauchy stress components are given by

$$\begin{aligned} T_{11} &= -p + 2\left(W_1 - (1+w_1^2)W_2\right), & T_{12} &= -2w_1 w_2 W_2, \\ T_{22} &= -p + 2\left(W_1 - (1+w_2^2)W_2\right), & T_{13} &= 2(W_1 + W_2)w_1, \\ T_{33} &= -p + 2W_1(1+k^2) - 2W_2, & T_{23} &= 2(W_1 + W_2)w_2. \end{aligned} \tag{3.74}$$

It is easy to check that now the balance equations form a system of three differential equations in the two unknowns $p(X_1, X_2, X_3)$ and $w(X_1, X_2)$, i.e.

$$\begin{aligned} &p_{,1} - 2[W_1 - (1+w_1^2)W_2]_{,1} + 2[w_1 w_2 W_2]_{,2} = 0, \\ &p_{,2} - 2[W_1 - (1+w_2^2)W_2]_{,2} + 2[w_1 w_2 W_2]_{,1} = 0, \\ &p_{,3} - 2[(W_1 + W_2)w_1]_{,1} - 2[(W_1 + W_2)w_2]_{,2} = 0, \end{aligned} \tag{3.75}$$

where the subscripts 1 and 2 stand for differentiation with respect to X_1 and X_2, respectively, and where

$$W_i = \left.\frac{\partial W}{\partial I_i}\right|_{I_1=I_2=3+k^2, I_3=1}. \tag{3.76}$$

Since w_1 and w_2 are independent of X_3, so are I_1 and I_2. From $(3.75)_3$, we deduce that $p_{,3}$ has the same property. Thus p is linear in X_3:

$$p(X_1, X_2, X_3) = cX_3 + \bar{p}(X_1, X_2), \tag{3.77}$$

where c is a constant (called here the *axial pressure gradient*) and $\bar{p} = \bar{p}(X_1, X_2)$ is an undetermined function. A further reduction of the first two equilibrium equations $(3.75)_1$ and $(3.75)_2$ may be obtained by eliminating $p_{,12}$ by appropriate cross-differentation. In the end, we obtain an overdetermined differential system in which the unknown w must satisfy simultaneously the following two nonlinear ordinary differential equations,

$$\begin{aligned} &\left[\left(w_1^2 - w_2^2\right) W_2\right]_{,12} = [w_1 w_2 W_2]_{,11} - [w_1 w_2 W_2]_{,22}, \\ &[(W_1 + W_2)w_1]_{,1} + [(W_1 + W_2)w_2]_{,2} - \frac{c}{2} = 0. \end{aligned} \tag{3.78}$$

It is possible to show that the overdetermined differential system (3.78) is compatible only for particular choices of the strain energy function and only for special classes of materials. Knowles [72] gives necessary and sufficient condition in terms of the strain energy function for a homogeneous, isotropic, incompressible material to admit nontrivial states of anti-plane shear. For example, in the case of the following *rectilinear shear deformation*

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + w(X_1), \tag{3.79}$$

the system (3.78) reduces to a single second-order differential equation for $w(X_1)$:

$$[(W_1 + W_2)w_1]_{,1} - \frac{c}{2} = 0, \tag{3.80}$$

where the subscript 1 stands for differentation with respect the argument X_1. In this case a formal solution of the balance equations is possible. This situation is similar to the situation discussed earlier for a telescopic shear deformation. Another example where positive results may occur is that of the *generalized neo-Hookean materials* (2.12). Here the overdetermined system (3.78) reduces to a single quasi-linear second-order partial differential equation

$$[(W_1)w_1]_{,1} + [(W_1)w_2]_{,2} - \frac{c}{2} = 0, \tag{3.81}$$

and then a formal solution of the balance equations is also possible.

From a mathematical point of view, the fact that a *pure* antiplane shear deformation cannot be sustained in an elastic material means that the overdetermined differential system (3.78), corresponding to the strain energy function we are using to model *real* materials, do not have common solutions. Therefore Mathematics says that the geometry and load condition of the problem does not allow a pure antiplane shear deformation. On the other hand, it may be possible to have a pure antiplane shear deformation coupled to *secondary* deformations (see [63]). For example, by coupling an *in-plane* deformation to the antiplane one, as

$$\begin{aligned} x_1 &= X_1 + u(X_1, X_2), \\ x_2 &= X_2 + v(X_1, X_2), \\ x_3 &= X_3 + w(X_1, X_2), \end{aligned} \tag{3.82}$$

where u, v are the *in-plane* smooth displacement functions. For every incompressible elastic material, the balance equations $\mathrm{div}\boldsymbol{T} = \boldsymbol{0}$ now reduce to a determined system of partial differential equations. This does not mean that, for a generic material, it is not possible to deform the body as prescribed by our geometry and load condition, but it emphasizes that by semi-inverse methods it is not easy to understand when the equations lead to a deformation field that is more complex than an anti-plane shear. For generalized neo-Hookean materials, we have the following expressions for the Cauchy stress components

$$\begin{aligned} T_{11} &= -p + 2W_1[(1 + u_1)^2 + u_2^2], \\ T_{22} &= -p + 2W_1[v_1^2 + (1 + v_2)^2], \\ T_{33} &= -p + 2W_1[w_1^2 + w_2^2 + 1], \end{aligned} \tag{3.83}$$

$$T_{12} = 2W_1[(1+u_1)v_1 + u_2(1+v_2)],$$
$$T_{13} = 2W_1[(1+u_1)w_1 + u_2 w_2],$$
$$T_{23} = 2W_1[v_1 w_1 + (1+v_2)w_2].$$

It is clear that the stress components T_{13} and T_{23} involve a coupling of in-plane and out-of-plane deformations[7]. The boundary condition of traction may therefore couple the in-plane displacements with the out-of-plane displacement. It is only for special cases (for example of pure displacement boundary conditions) that in-plane and out-of-plane displacements may be decoupled.

3.2.3 Radial deformation

The following example shows how the semi-inverse method may be used to search for exact and analytical solutions which are not universal but relative universal (see Horgan [60]). Let us consider spherical polar coordinates for the radial deformation written as

$$r = r(R), \qquad \theta = \Theta, \qquad \phi = \Phi, \tag{3.84}$$

where (R, Θ, Φ) are the polar coordinates in the reference configuration and (r, θ, ϕ) are the polar coordinates in the current configuration, respectively, and $\mathrm{d}r/\mathrm{d}R > 0$. The polar components of the deformation gradient tensor associated with (3.84) are given by

$$\boldsymbol{F} = \mathrm{diag}(\mathrm{d}r/\mathrm{d}R, r/R, r/R) \tag{3.85}$$

and the principal stretches are thus $\lambda_1 = \mathrm{d}r/\mathrm{d}R$, $\lambda_2 = \lambda_3 = r/R$. Now, the equilibrium equations in the absence of body forces $\mathrm{div}\boldsymbol{T} = \boldsymbol{0}$ can be shown to reduce to the single equation

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2 \hat{W}_1\right) - 2R\hat{W}_2 = 0, \tag{3.86}$$

which is a second-order nonlinear ordinary differential equation for $r(R)$. Six classes of compressible materials have received much attention in the literature; they are all examples of relative universal solutions for the solutions $r(R)$.

$$\text{Class I. } W = f(i_1) + b_1(i_2 - 3) + c_1(i_3 - 1), \quad f''(i_1) \neq 0, \tag{3.87}$$

where f is an arbitrary function of i_1, b_1 and c_1 are arbitrary constants and i_1, i_2, i_3 are the principal invariants of $\boldsymbol{V}$. This class represents the harmonic materials introduced by John [67]. In this case, on using the hypothesis $f''(i_1) \neq 0$, one finds that

$$r(R) = AR + \frac{B}{R^2}, \tag{3.88}$$

where A and B are constants of integration. Abeyaratne and Horgan [1] and Ogden [95] employed the deformation (3.88) to obtain closed-form solutions for pressurized

[7]It is possible to show that the pressure depends on the out-of-plane deformation and therefore that the normal stresses in (3.83) contain all the deformation fields.

hollow spheres composed of harmonic materials. Aboudi and Arnold [2] applied (3.88) to micromechanical modeling of multiphase composites.

$$\text{Class II. } W = a_2(i_1 - 3) + g(i_2) + c_2(i_3 - 1), \quad g''(i_2) \neq 0, \tag{3.89}$$

where g is an arbitrary function of i_2 and a_2 and c_2 are arbitrary constants. Here, one finds that

$$r^2(R) = AR^2 + \frac{B}{R}, \tag{3.90}$$

where A and B are constants of integration. Murphy [88] used the controllable deformation (3.90) to treat the problems of inflation and eversion of hollow spheres of class II materials. Aboudi and Arnold [2] utilized (3.90) in their recent study of micromechanics of multiphase composites.

$$\text{Class III. } W = a_3(i_1 - 3) + b_3(i_2 - 3) + h(i_3), \quad h''(i_3) \neq 0, \tag{3.91}$$

where h is an arbitrary function of i_3 and a_3 and b_3 are arbitrary constants. This class of materials are called *generalized Varga materials* [58]. Here, one finds that

$$r^3(R) = AR^3 + B, \tag{3.92}$$

where A and B are constants of integration. Horgan [58] used the controllable deformation (3.92) to illustrate the phenomenon of *cavitation* for compressible materials in a particularly tractable setting. Aboudi and Arnold [2] utilized (3.92) in their micromechanics analysis of composites undergoing finite deformation. Murphy [87] introduced the next three material classes.

$$\text{Class IV. } W = a_4 i_1 i_2 + b_4 i_1 + c_4 i_2 + d_4 i_3 + e_4, \quad a_4 \neq 0, \tag{3.93}$$

where a_4, b_4, c_4, d_4, e_4 are arbitrary constants. Here, one finds that

$$r^3(R) = \frac{(A + BR^3)^2}{R^3}, \tag{3.94}$$

where A and B are constants of integration.

$$\text{Class V. } W = a_5 i_2 i_3 + b_5 i_1 + c_5 i_2 + d_5 i_3 + e_5, \quad a_5 \neq 0, \tag{3.95}$$

where a_5, b_5, c_5, d_5, e_5 are arbitrary constants. Here, one finds that

$$r^5(R) = \frac{(A + BR^3)^2}{R}, \tag{3.96}$$

where A and B are constants of integration.

$$\text{Class VI. } W = a_6 i_1 i_3 + b_6 i_1 + c_6 i_2 + d_6 i_3 + e_6, \quad a_6 \neq 0, \tag{3.97}$$

where a_6, b_6, c_6, d_6, e_6 are arbitrary constants. Here, one finds that

$$r^2(R) = AR^2 + \frac{B}{R}, \tag{3.98}$$

where A and B are constants of integration. This deformation field is identical to that given for Class II (see (3.89)).

This type of investigation was proposed by Currie and Hayes [25] where the search for exact solutions starts from a different point of view. They search for special solutions by choosing a deformation whose geometry is completely known a priori; in doing so they are solving Ericksen's problems *in miniature*: they are searching all the corresponding relative universal solutions[8].

Other typical applications of semi-inverse investigations are concerned for example in finding, for a given deformation (fixed a priori), the general form of the strain energy for which such deformation is a controllable solution. This is a sort of *inverse problem*: find the elastic materials (i.e. the functional form of the strain energy function) for which a given deformation field is controllable (i.e. for which the deformation is a solution to the equilibrium equations in the absence of body force).

Both problems are very difficult to solve and generally only partial results are available. The influential papers by Knowles [75] and Currie and Hayes [25] have stimulated the developement of a large amount of research on closed-form solutions in nonlinear elasticity. Beatty, Boulanger, Carroll, Chadwick, Hill, Horgan, Murphy, Ogden, Polignone, Rajagopal, Saccomandi, Wineman, and many others have determined a long list of exact solutions for special classes of constitutive equations. We refer to the recent books edited by Fu and Ogden [43] and by Hayes and Saccomandi [55] for an overview of this activity.

Although several authors have used such *inverse procedures*, and many solutions have been derived by using such methods, it is not easy to find a definition describing the true power of these methods. Further, no general mathematical theory can be applied, at least at first, sight, because they are a sort of *heuristics methods*. Only Lie group theory can provide a general, algorithmic, and efficient method for obtaining exaxt solutions of partial differential equations by a reduction method. It shares many similarities with the semi-inverse method. For this reason many authors have tried to find a relationship between Lie's classical method of reduction and the semi-inverse method, but the standard Lie method of symmetry reduction is not always applicable; it has to be generalized to recover all the solutions obtainable via ad hoc reduction methods. Olver and Rosenau [96] introduced the concept of *weak symmetry*, based on the analytic properties of the overdetermined system, and made it clear that a group theory nature is indeed possible for every solution of a given partial differential equation[9]. But it is still not known how to obtain the relevant groups.

[8] The expression "in miniature" is taken from a paper by Knowles [75] where the author tries to find non-homogeneous universal solutions in the family of anti-plane shear deformations.

[9] In [114] Saccomandi, considering the Navier-Stokes equations, shows how it is necessary to resort to the idea of weak symmetries to recover all the solutions found by the semi-inverse method.

Notes

A list of some suitable of inverse methods, useful to solve boundary value problems in elasticity, are given in this chapter. We have also underlined the important contributions of these methods to continuum mechanics in understanding the non-linear behaviour of materials (or fluids in the case of fluid dynamics), overcoming the difficulty in solving boundary value problem by direct methods. By inverse procedure, in addition to those homogeneous, five families of universal solutions are been found (they are listed in Section 3.1.2) where we have not widely discussed families 2-5 to save space but we refer to Section 57 of [127] for more details. The first investigation about universal solutions dates back to 1954, when Ericksen [33] was able to find several families of universal inhomogeneous deformations. However the proof of Ericksen was not complete in two points:

1. when two principal stretches of the deformation are equal and at least one of the principal invariants is not constant;

2. when all the principal invariants of the deformation are constants.

The first point has been completely resolved by Marris and Shiau [80] who showed that if two principal stretches are equal then the universal deformations are homogeneous or enclosed in Family 2. As regards the second point the final answer is still lacking but further developments on this problem are contained in the work on universal solutions for the elastic dielectric by Singh and Pipkin [119]. As a by-product of this research a new family of deformation with constant invariants has been discovered (Family 5).

Although the search for solutions of boundary value problems by use of inverse methods has been important and fundamental, on the other hand we are nonetheless of the opinion that some solutions have been the source of possible confusion in the field, and that some investigations are even incorrect in their use of the semi-inverse method (see [28]). In the next chapter we develop our point of view further, by analysing in detail some inhomogeneous solutions for compressible materials subjected to isochoric deformations.