Abstract

Recently, the biomechanics of soft tissues has become an important topic of research in several engineering, biomedical and mathematical fields. Soft tissues are biological materials that can undergo important deformations (both within physiological and pathological fields) and they clearly display a nonlinear mechanical behaviour. In this case the analysis of the deformations by computational methods (e.g. finite elements) can be complex. Indeed, it is not easy to know exactly the "right" constitutive equations to describe the behaviour of the material, and often the commercial software turns out to be unsuited for dealing with trust the solutions for the corresponding balance equations. The geometrical nonlinearity of the model under investigation makes it very difficult to grasp the true physics of the problem and often the intuition of the engineer can do very little if it is not guided by careful and exact mathematical analysis. To this end the possibility of obtaining easy exact solutions for the field equations is an important and privileged tool, helping us to gain a better understanding of several biomechanics phenomena.

The semi-inverse method is one of few known methods available to obtain exact solutions in the mathematical theory of Continuum Mechanics. The semi-inverse method has been used in a systematic way during the whole history of Continuum Mechanics (for example to derive the celebrated Saint Venant solutions [5, 6]), but unfortunately this use has always happened essentially in a heuristic way, completely disconnected from a general method.

Essentially, the purpose of the semi-inverse method consists in formulating a priori a special ansatz for the unknown fields in a certain theory and in reducing the general balance equations to a simplified subset of equations. Here, by simplifying action, one often means that the balance equations are reduced to an easier system of differential equations (for example passing from a system of partial differential equations to an ordinary differential system, see [90]).

The following Thesis, developed in six chapters, studies several points of view of this method and other connected methodologies. The first chapters are essentially introductory while the others collect the results of research obtained during my PhD ([26, 27, 28]).

The First Chapter is devoted to the definitions, symbols and basic concepts of the theory of nonlinear elasticity. In that chapter we define the kinematics of finite deformation, introducing the concept of material body and of deformation. We introduce the balance laws, the stress and the equations of motion. We also propose constitutive concepts, such as those of frame indifference, material isotropy and hyperelasticity. We analyse the restrictions imposed on the mathematical models, such as the empirical inequalities of Truesdell and Noll, to ensure a reasonable

mechanical behaviour.

The Second Chapter exhibits some special constitutive laws for hyperelastic materials. One of the problems encountered in Continuum Mechanics concerns the choice of models for the strain energy function for a good description of the mechanical behaviour of "real" materials. Here we describe some models (both for compressible and incompressible materials) that are commonly used in the literature, including: the neo-Hookean model, the Mooney-Rivlin model, the generalized neo-Hookean model, the Hadamard model, the Blatz-Ko model, and finally an expansion of the strain energy function with respect to the Green Lagrange strain tensor, used to study small-but-finite deformations.

The Third Chapter introduces a small overview of the use of the semi-inverse method in elasticity. We show some examples which may be considered the most representative and/or meaningful and highlight their strengths and weaknesses. We apply the inverse method by searching universal solutions both in the compressible (where the only admissible deformations are homogeneous [34]) and in the incompressible case (where in addition to homogeneous, five other inhomogeneous "families" have been found in the literature [33, 119]).

The Ericksen result [34] shows that there are no other finite deformations beyond those homogeneous that are controllable for all compressible materials. The impact of that result on the theory of nonlinear elasticity was quite important. For many years there has been "the false impression that the only deformations possible in an elastic body are the universal deformations" [25]. In the same time as the publication of Ericksen's result, there was considerable activity in trying to find solutions for nonlinear elastic materials using the semi-inverse method. And the search of the exact solutions for nonlinear isotropic elastic incompressible materials, thanks to the constraint of incompressibility, has been easier than for the compressible ones. In other words it has been possible to find exact solutions which are not universal.

In recent years, there has been a great interest in the possibility to determine classes of exact solutions for compressible materials as well. One of the strategies used is to take inspiration from the inhomogeneous solutions for nonlinear elastic incompressible materials and to seek similar solutions in compressible materials. The Fourth Chapter focusses on the results obtained for compressible materials using this line of research. The object is to determine which compressible materials can sustain isochoric deformations such as, for example, "pure torsion", "axial pure shear" and "azimuthal pure shear". We believe that these lines of research can be misleading. To illustrate our thesis we have considered small perturbations on some classes of compressible materials capable to sustain a certain isochoric deformation. As a result, although the perturbation is "small", the corresponding volume variation is not negligible. We emphasize that it does not turn out to be of any utility to understand which materials can sustain a simple isochoric torsion, because these materials do not exist, but it is far more important to understand which complex geometrical deformation accompanies the action of a moment twisting for a cylinder. Only in this way, can the results obtained with the semi-inverse method be meaningful.

Among the examples of application of the semi-inverse method, we report the search of solutions for the "anti-plane shear" and "radial" deformation. In the incompressible case we know that, for a general elastic solid, the balance equations are consistent with the anti-plane shear assumption only in the cylindrical symmetry case. We can say nothing when the body geometry is more general, since in that case the equilibrium equations for a generic elastic solid reduce to an overdetermined system that is not always consistent. This means that for general bodies, the anti-plane shear deformation must be coupled with secondary deformations. A complex tensional state is automatically produced in the body.

The Fifth Chapter presents a short overview of the results already obtained in literature on the latent deformations (see [39, 63, 83]). Then we give a new analytical example for the above issue (see also [27]). We consider a complex deformation field for an isotropic incompressible nonlinear elastic cylinder, namely a combination of an axial shear, a torsion and an azimuthal shear. After fixing some boundary conditions, one can show that for the neo-Hookean material, the azimuthal shear is not essential regardless of whether the torsion is present or not. When the material is idealized as a Mooney-Rivlin material, the azimuthal shear cannot vanish when a non-zero amount of twist is considered. Applying the stress field, obtained from the neo-Hookean case, in order to extrude a cork from a bottle of wine, then we conjecture that is more advantageous to accompany the usual vertical axial force by a twisting moment.

The Thesis ends with a Sixth Chapter giving a new application of the semi-inverse method (see also [26]). The celebrated Euler buckling formula gives the critical load for the axial force for the buckling of a slender cylindrical column. Its derivation relies on the assumptions that linear elasticity applies to this problem, and that the slenderness of the cylinder is an infinitesimal quantity. Considering the next order for the slenderness term, we find a first nonlinear correction to the Euler formula. To this end, we specialize the exact solution of non-linear elasticity for the homogeneous compression of a thick cylinder with lubricated ends to the theory of third-order elasticity. This example is especially important because it supposes a general method, even if it is approximated, and it may be applied to several contexts.

These results show again the true complexity of nonlinear elasticity where it is difficult to choose the reasonable reductions. Moreover the results obtained have an important applications in biomechanic, a topic that will be the subject of future research.