FLOCKS OF OVAL CONES AND EXTENSIONS OF THEOREMS OF THAS

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Abstract. *The flocks of oval cones in $PG(3,2^r)$ which have the property that no four planes of the flock share a common point are classified as the translation oval flocks of Thas.*

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1. INTRODUCTION

Recently, there has been considerable interest in flocks of quadric sets in $PG(3,q)$. At least one reason for this is the connection with such objects and other geometric incidence structures. For example, there are associated translation planes with spreads covered by reguli in various ways, generalized quadrangles of type (q^2,q), projective planes of Lenz-Barlotti class II-1 (exactly one incident point-line transitivity) and classes of ovals.

Actually, a generalization of such flocks can be formulated using ovals in Desarguesian projective planes instead of conics and this is done in Thas [8].

Definition 1. *Let O_1 be an oval in a projective plane π_1 embedded in $PG(3,q)$ and let v_o be a point of $PG(3,q)-O_1$. Join the $q+1$ lines of O_1 to v_o and define the points of this set to be the oval cone C_{O_1} determined by O_1 (and v_o). If there is a set of q planes of $PG(3,q), \pi_1, \pi_2, \ldots, \pi_q$ such that the union of the ovals of intersection $\pi_i \cap C_{O_1}, i = 1, 2, \ldots, q$ is $C_{O_1} - \{v_0\}$ then the set of planes $\{\pi_1, \ldots, \pi_1\}$ is a flock of the oval cone C_{O_1}.*

Remark 1. *Note that given any oval in a Desarguesian projective plane π_1, there is a flock of the associated oval cone by planes containing a fixed line. Let Σ be any plane containing the vertex v_o and let L be any line of Σ not containing v_o where $L = \Sigma \cap \pi_1$ (change Σ so that L does not contain v_o- if necessary). Then the set of planes incident with L and not equal to Σ form a flock of the oval cone. We call the flock a linear flock.*

Initially, we were under the impression that except for the conical flocks (of a quadratic cone defined by a conic), the only known flocks of oval cones are the linear flocks. And, we proceeded to determine if there were any such flocks of oval cones. We were able to obtain an infinite class of such flocks of translation oval cones. We present these in this article.

However, these flocks appear in Fisher and Thas [2] (3.11) and are actually due to Thas similarly as the flocks of quadratic cones of Fisher appearing in the same article are due to Fisher.

Thus, the construction parts of this article may be considered an explication of some of the results of Thas. Furthermore, we may provide some extensions of some results of Thas.

In this article, we consider flocks of translation oval cones. Given any translation oval O_T in $PG(2,q)$, we provide an alternative construction of the class of nonlinear flocks of the translation oval cone C_{O_T} mentioned above.

In Fisher and Thas [2] (3.11), the emphasis is more on the structure of the osculating planes of a corresponding $(q+1)$-arc in $PG(3,q)$. However, here we are more interested in the structure of the automorphism group and our approach starts with this direction.

All of these flocks admit automorphism groups admitting a doubly transitive automorphism group on the planes of the flock. When the translation oval is actually a conic, the flocks obtained correspond to the translation planes of Betten (also called the flocks of Fischer-Thas-Walker). However, when the translation oval is not a conic, these flocks may not and probably do not correspond to translation planes.

In Jha-Johnson [4], [5], the authors completely determine the set of flocks of oval cones in $PG(3,q)$ that admit doubly transitive automorphism groups (in $PGL(4,q)$). Hence, we are most interested in the action of the group. In section 2, we provide a group theoretic construction of the flocks.

In section 3, we are able to characterize the flocks via symplectic polarities similarly as in Fisher and Thas. Actually, we are able to provide a bonus and determine the set of possible isomorphism (see section 4). Using this latter approach, it shall become apparent that no four planes of the flock share a common point.

Definition 2. *We shall say that a flock of an oval cone satisfies the no four property if and only if no four planes containing the ovals of the flock share a point.*

Recently, Thas [7] characterized the flocks of quadratic cones in $PG(3,q)$ whose planes have the no four property. That is, no four planes that contain the conics of the flock share a common point.

Theorem 1.1. *(Thas [7] Theorem B).*

Let $F = \{C_1, C_2, \ldots, C_q\}$ be a flock of the quadratic cone K in $PG(3,q), q \geq 4$, with either

(a) q even, or

(b) q odd with $q > 83$ or $q < 17$ or q in \{ 27, 81 \}.

Then F is the flock of Fisher-Thas-Walker if and only if no four of the planes π_i with $C_i \subset \pi_i$ have a point in common.

In this note, we are able to extend the result of Thas to show that the class of flocks of oval cones in $PG(3,q)$ for q even whose planes satisfy the no four property is exactly the class of translation oval flocks of Thas. When the oval is actually a conic, these flocks are the flocks of Fisher-Thas-Walker.

2. THE CONSTRUCTION

As our treatment is slightly different than Fisher and Thas, we first note

Theorem 2.1. *(See Fisher and Thas [2] (3.11)). Let q be even where $q \equiv -1 \bmod 3$ and let $\sigma \epsilon$ Gal $GF(q)$ such that $\{(1,t,t^\sigma),(0,0,1) | t \epsilon GF(q)\} = O_1$ is a translation oval. Let homogenous coordinates (x_0,x_1,x_2,x_3) where $x_i \epsilon GF(q), i = 0,1,2,3$ define the projective 3-space $PG(3,q)$. Embed the oval in the plane $x_3 = 0$ and form the translation oval cone by projecting $(0,0,0,1) = v_0$ to the oval O_1 in $x_3 = 0$.*

Then the following set of planes form a flock of the translation oval cone C_{O_1}:

$$\pi_s \text{ is } s^{\sigma+1}x_o + s^\sigma x_1 + sx_2 + x_3 = 0 \text{ for all } s \epsilon GF(q).$$

Furthermore, this flock admits a double transitive group acting on the planes of the flock *given as ST (semi-direct product of S by T) where*

$$S = <\tau_s = \begin{pmatrix} 1 & s & s^\sigma & s^{\sigma+1} \\ 0 & 1 & 0 & s^\sigma \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} | s \epsilon GF(q) >$$

and

$$T = <\rho_t = \begin{pmatrix} t^{\sigma+1} & 0 & 0 & 0 \\ 0 & t^{\sigma-1} & 0 & 0 \\ 0 & 0 & t^{1-\sigma} & 0 \\ 0 & 0 & 0 & t^{-1-\sigma} \end{pmatrix} | t \epsilon GF(q) - \{0\} >.$$

Proof. We consider the oval O_1 embedded in the plane $x_3 = 0$ with the form $\{(1, t, t^\sigma, 0), (0, 0, 1, 0) | t \epsilon GF(q)\}$ and note that the knot is then $(0, 1, 0, 0)$. We first consider the image of $x_3 = 0$ under S.

We note that τ_s maps $(1, 0, 0, 0) \to (1, s, s^\sigma, s^{\sigma+1})$,
$(0, 1, 0, 0) \to (0, 1, 0, s^\sigma)$ and $(0, 0, 1, 0) \to (0, 0, 1, s)$ so that S maps $x_3 = 0$
onto the set $\pi_s : s^{\sigma+1}x_o + s^\sigma x_1 + s x_2 + x_3 = 0$. Note that τ_t has the following action on $\pi_s : \pi_S \tau_t = \pi_0 \tau_s \tau_t = \pi_0 \tau_{s+t} = \pi_{s+t}$. Moreover, consider the set of lines of the oval cone:

$L_\infty = <(0, 0, 1, 0), (0, 0, 0, 1)>, L_t = <(1, t, t^\sigma, 0), (0, 0, 0, 1)>$ and the line defined by the knot and the vertex $L_K = <(0, 1, 0, 0), (0, 0, 0, 1)>$. Note that S fixes L_∞ and L_K and τ_s maps L_t onto L_{t+s}. Since S also fixes $(0, 0, 0, 1), S$ acts as an automorphism group of the translation oval cone C_{O_1} which acts transitively on the planes π_s for all $s \epsilon GF(q)$.

Now consider the action of T. ρ_t, for $t \neq 0$, fixes π_0, and maps
$\pi_1 = <(0, 0, 1, 1), (0, 1, 0, 1), (1, 1, 1, 1)>$ as follows:
$(0, 0, 1, 1) \to (0, 0, t^{\sigma-1}, t^{-1-\sigma}), (0, 1, 0, 1) \to (0, t^{\sigma-1}, 0, t^{-1-\sigma})$ and
$(1, 1, 1, 1) \to (t^{\sigma+1}, t^{\sigma-1}, t^{1-\sigma}, t^{-1-\sigma})$ and these points are on
$\pi_{t^{-2}} : (t^{-2})^{\sigma+1}x_0 + (t^{-2})^\sigma x_1 + (t^{-1})x_2 + x_3$. Moreover,
$L_{s^{-2}}$ onto $L_{(st)^{-2}}$. Since T fixes (0,0,0,1), it follows that T is an automorphism group of the translation oval cone which fixes one plane and acts transitively on the remaining planes. Hence, ST acts doubly transitively on the set of planes $\{\phi_s | s \epsilon GF(q)\}$.

It remains only to show that the intersections of the planes with the lines of the translation oval cone partition the nonvertex points of the oval cone. However, since ST acts doubly transitively on the planes, we only need to check that $\pi_0 \pi_1 C_{O_1}$ is empty. This is equivalent to showing that $x_0 + x_1 + x_2 = 0$ has no solutions on the translation oval cone $C_{O_1} = \{L_\infty, L_s | s \epsilon GF(q)\}$. Clearly, there is no solution on $L_\infty = <(0, 0, 1, 0), (0, 0, 0, 1)>$ since any solution is of the form $(0, 0, \alpha, \beta)$ for $\alpha\beta \neq 0$ so that $x_0 = x_1 = 0$ which would force $x_2 = 0$.

Assume that there is an intersection on $L_t = <(1, t, t^\sigma, 0), (0, 0, 0, 1)>$ so that $x_0 = \beta, x_1 = \beta t$ and $x_2 = \beta t^\sigma$ for some nonzero β in $GF(q)$. Then this would imply that $\beta(1 + t + t^\sigma) = 0$. So that $1 + t + t^\sigma = 0$. This implies that the trace $(1 + t + t^\sigma) = 0$. So that $1 + t + t^\sigma = 0$. This implies that the trace $(1 + t + t^\sigma) = 0$ but the trace of $(t + t^\sigma) = 0$ and trace $1 = 1$ provided $q = 2^r$ and r is odd. Hence, this proves (2.1).

Definition 3. *The flocks of (2.1) shall be called the translation oval flocks of Thas.*

Remark 2. *Finally, we note that the necessary conditions in Fisher and Thas for construction of the flocks are that* $(2^r - 1, 2^m + 1) = 1$ *where r is odd and m is between 1 and* $r - 1$. *The conditions that we have been using above are simply that* $(r, m) = 1$ *and r is odd.*

However, note that $(2^r - 1, 2^m + 1)$ *divides* $(2^r - 1, 2^{2m} - 1) = (2^{(r,m)} - 1) = 1$ *since* $(r, m) = 1$.

3. A CHARACTERIZATION OF THE TRANSLATION OVAL FLOCKS OF THAS BY SYMPLECTIC POLARITIES

Let $\{(1, t, t^\sigma, t^{\sigma+1}), (0, 0, 0, 1) | t \epsilon GF(q)\}$ be the $q + 1$ arc $C(\sigma)$ where σ induces an automorphism of $GF(q)$ and if $\sigma = 2^m$ and $q = 2^r$ then $(r, m) = 1$. Furthermore, assume r is odd. Lüneburg [6] (44.3) defines the "osculating plane" to a point of $(1, t, t^\sigma, t^{\sigma+1})GF(q)$ of $C(\sigma)$ as the plane with equation $t^{\sigma+1}x_0 + t^\sigma x_1 + tx_2 + x_3 = 0$. Further, the osculating plane of $C(\sigma)$ at $(0, 0, 0, 1)GF(q)$ is $x_0 = 0$. Moreover, by Lüneburg [6] (44.3), there is a unique symplectic polarity β such that P^β is the osculating plane of $C(\sigma)$ in P for all $P \in C(\sigma)$.

Recall that a special unisecant L to a $(q + 1)$-arc is a line incident with a point x of the arc such that any plane containing L intersects the $(q + 1)$-arc in at most one other point $\neq x$. Furthermore, there are exactly two unisecants to any point x of the $(1 + 1)$-arc and these generate a plane called the "contact plane" which shares exactly x with the arc (see Hirschfeld [3] (21.3)). Furthermore, the set of special unisecants form the union of the two ruling classes of lines of a hyperbolic quadric (Hirschfeld [3] (21.3.10)).

The symplectic polarity β defined by Lüneburg is as follows:

$$\beta(x_O, x_1, x_2, x_3, y_O, y_1, y_2, y_3) = x_O y_3 + x_1 y_2 + x_2 y_1 + x_3 y_3.$$

Since there is also a symplectic polarity associated with the hyperbolic quadric of special unisecants, the question is whether these two polarities are the same. Furthermore, we should like to know if the osculating plane defined above is the contact plane. Since we shall require these results later, we note:

Proposition 1. *(1) The osculating plane defined as above is the contact plane.*

(2) The symplectic polarity β *is the polarity afforded by the hyperbolic quadric of special unisecants.*

Proof. Actually, (1) may be found in the proof of (21.3.15) p. 251 of Hirschfeld [3].

To see that (2) is valid, we need to show that line β-invariant lines on any point of the $(q + 1)$-arc are the two special unisecants.

Note there is a group isomorphic to $PGL(2, q)$ acting triply transitvely on the points of the $q + 1$-arc. Since the special unisecants incident with x map under g in $PGL(2, q)$ to the special unisecants incident with xg, it is sufficient to consider the statement of (2) relative to the point $(1, 0, 0, 0,)$. Using β above, it is easy to calculate that the β-invariant lines incident with $(1, 0, 0, 0)$ are the lines $<(1, 0, 0, 0), (0, 1, 0, 0)>$ and $<(1, 0, 0, 0), (0, 0, 1, 0)>$ as note that $\beta(1, 0, 0, 0, y_O, y_1, y_2, y_3) = 0$ if and only if $y_3 = 0$ and $\beta(0, 1, 0, 0, y_O, y_1, y_2, y_3) = 0$ if and only if $y_2 = 0$. Hence, the β-image points are $(y_O, y_1, 0, 0)$.

Now we merely check that any place containing one of these lines intersects any point of the $(q + 1)$-arc in at most one point. The plane $<(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, a, 1)>$ contains

$(0,0,0,1)$ if and only if $a = 0$. In this case, the plane contains $(1, t, t^\alpha, t^{\sigma+1})$ if and only if $t^\sigma = 0$.

If $a \neq 0$ then assume the plane contains $(1, t, t^\sigma, t^{\sigma+1})$ and $(1, s, s^\sigma, s^{\sigma+1})$ for $s \neq t$ and $st \neq 0$. The equation for the plane is $x_2 + ax_3 = 0$. The plane contains the two indicated points if and only if the equation

$$y^\sigma = ay^{\sigma+1} \text{ or rather } y = 1/a$$

is satisfied for $y = t$ or s.

Thus, the plane contains only the point $(1, w, w^\sigma, w^{\sigma+1})$ for $w = 1/a$. This proves the proposition.

Continuing with our discussion, we note that Thas [8] shows that a flock of an oval cone gives rise to a flock of a hyperoval cone. Also, we point out that Casse and Glynn [1] show that any q-arc in $PG(3, q = 2^r)$ may be uniquely extended to a $(q+1)$-arc.

We may characterize the translation (hyper)oval flocks of Thas as follows:

Theorem 3.1. *Let K be a $q+1$ arc in $PG(3,q)$ for $q^r = 2^r$ and for r odd.*

Define a translation hyperoval cone as follows: For P_1, let $O(P_1)$ denote the osculating plane at P_1. For $P_O \neq P_1$, form the lines P_OP for all P in $C - \{P\}$, and construct the q-arc in $O(P_1)$ from the intersection points of the lines. Let Q_1, Q_2 denote the points extending the q-arc to a hyperoval in $O(P_1)$. Construct a translation hyperoval cone $C(K)$ by projecting the hyperoval in $O(P_1)$ from P_O.

Then there exists a unique symplectic polarity β such that the set of osculating planes P^β for all $P \neq P_O$ in K is exactly the translation hyperoval flock of Thas of the translation hyperoval cone $C(K)$.

Proof. Since we have noted that there is a group isomorphic to $PGL(2,q)$ acting triply transitvely on the points of the $q+1$-arc, we take a version of this statement with P_O to be $(0,0,0,1)$, the $q+1$-arc K to have the form $\{(1, t, t^\sigma, t^{\sigma+1}), (0,0,0,1) | t\epsilon GF(q)\}$ and P_1 to be $(1,0,0,0)$ so that with the symplectic polarity β defined as before,

$$\beta(x_O, x_1, x_2, x_3, y_O, y_1, y_2, y_3) = x_Oy_3 + x_1y_2 + x_2y_1 + x_3y_3,$$

we have $O(p_1) = P_1^\beta = (x_3 = 0)$.

Note that no four planes of the flock can share a common point as these are osculating planes to the points of a $q+1$-arc connected by the symplectic polarity β.

The stabilizer of the point P_O is doubly transitive on the points of the $q+1$-arc different from P_O so that this group acts doubly transitively on the set of osculating planes $\{P^\beta | P$ in $K - \{P_O\}\}$.

If follows that we may translate everything back into the setting of (2.1) and employ the associated doubly transitive group. Hence, we obtain a flock in $PG(3,q)$ exactly when $q \equiv -1$ mod 3 so that $q = 2^r$ for r odd.

4. THE ISOMORPHISM QUESTION

Let F_σ and F_τ be flocks of oval cones corresponding to the translation ovals O_σ and O_τ corresponding to the $q+1$ arcs in $PG(3,q)$ with defining automorphisms σ and τ.

Theorem 4.1. *There are exactly $\phi(r)/2$ mutually nonisomorphic translation (hyper)oval flocks of Thas (of the type constructed in (2.1)) in $PG(3,q)$ where $q = 2^r$ and ϕ is the Euler ϕ-function.*

Proof. If the flocks are isomorphic then the dual flocks are isomorphic and hence so are the q-arcs and associated $q+1$-arcs. But, two $q+1$ arcs are isomorphic if and only if $\sigma = \tau$ or τ^{-1} (see Lüneburg [6] (44.2). Hence, there are exactly $\phi(r)/2$ such flocks where $q = 2^r$ and r is odd. This proves the result.

Moreover, we note that the translation ovals are conics exactly when the σ is 2 or 2^{r-1}.

5. THE NO 4-THEOREM

Theorem 5.1. *Let F be a flock of an oval cone in $PG(3,2^r)$ such that no four of the planes of the flock containing the ovals of the cover share a common point.*

Then F is isomorphic to one of the translation oval flocks of Thas.

(Note that this also implies that r is odd).

Furthermore, when the oval is a conic the flock is the flock of Fischer-Thas-Walker.

Proof. We shall structure the proof along the lines of the proof that Thas gives in [7]. The main point here is that this proof also works in this more general setting with appropriate modifications for the fact that we are only assuming that the flocks is defined an oval cone as opposed to a quadratic one.

Let O_1 be an oval in a Desarguesian projective plane π_1. Consider π_1 within $PG(3,2^r)$ and let v_O be a point of $PG(3,2^r) - \pi_1$. Form the oval cone C_{O_1} defined as the points on the lines of $v_O P$ for all $P \epsilon O_1$. If O_1^+ is the hyperoval consisting of O_1 and the knot k_1 of O_1, we may also consider the hyperoval cone $C_{O_1}^+ = C_{O_1} \cup v_O k_1$. It was pointed out in Thas [8] that a flock of a oval cone also produces a flock of the corresponding hyperoval cone. Let $F = \{\pi_1, \pi_2 \ldots, \pi_q\}$ be a set of planes whose intersections with C_{O_1} partition the points $\neq v_O$ of the oval cone. Now assume that no four planes of F share a common point. Now dualize $PG(3,2^r)$ so that the set of lines of the oval cone C_{O_1} becomes a set of lines in a plane π_{v_O} (the dual of v_O) with the property that no three are concurrent. That is, the dual of the set of lines of C_{O_1} becomes a dual oval in a plane π_{v_O}. Similarly, the set of lines of the hyperoval cone $C_{O_1}^+$ when dualized become a dual hyperoval in a plane π_{v_O}. Note that the flock $F = \{\pi_1, \pi_2 \ldots, \pi_q\}$ when dualized becomes a set of points $p_1, p_2, \ldots, p_q$ with the property that the join $p_i p_j$ of two distinct points intersects the plane π_{v_O} is a point p_{ij} not on a line of the dual hyperoval. Let the dual oval be denoted by Γ, the dual hyperoval by Γ^+ and the dual flock by $F^D = \{p_1, p_2, \ldots, p_q\}$.

Now assume that the flock of planes has the no four property. Then this is also true of the dual flock F^D so that this becomes a q-arc in $PG(3,q)$. By Casse and Glynn [1], there is a unique extension to a $q+1$-arc $F^{D,+} = F^D \cup \{z\}$.

This is exactly the same way that Thas sets up the argument in [7] except that the original oval is a conic there. With some modification particularity in the following lemma 3 (which we noted in a previous section), Thas's argument extends essentially directly. Thus, it is really only necessary to read the statements of the following four lemmas, the proof to lemma 3 and the concluding remarks. However, we shall include the details for the benefic of the reader who wishes to ensure that everything works in the more general case.

Lemma 1. *The point z extending the $q+1$ arc F^D is the plane π_{v_O} dual v_O.*

Proof. Assume not! Let p_{ij} denote the point intersecting π_{v_O} and recall that none of these $q(q-1)/2$ points are on a line of the dual hyperoval Γ^+. Let L be a line of Γ^+ and consider the plane $<L,z>$. Note that L contains at most one point of F^D since otherwise some p_ip_j would intersect L. There are q planes on L different from π_{v_O} and these cover the points of $PG(3,1)-\pi_{v_O}$. Hence, it follows that each such plane on L contains exactly one point of F^D. Let $<L,z>$ contain the point say p_1. Recall that a special unisecant to an element x of a k - arc is a line incident with x such that any plane containing this line contains at most one point other point of the k-arc ([3] (21.3). Also, when the k-arc is a $q+1$-arc, there are exactly two special unisecants to a point x ([3] (21.3.4). Furthermore, as mentioned previously, these $2(q+1)$ special unisecants are the generators of a hyperbolic quadric H. At each point y, the "contact plane" generated by the two special unisecants intersects the $q+1$-arc in exactly the point y of the $q+1$-arc.

Let L_1,L_2 denote two special unisecants at z. Form the planes $<L_1,p_1>$ and $<L_2,p_1>$. By [3] (21.3.1), note that zp_1 is a special unisecant to the original q-arc. Hence, any plane containing z and p_1 intersects the dual flock F^D in at most one other point. Since there are $q-1$ planes that contain zp_1 and another point p_i for $i\neq 1$, there are exactly two planes containing the point z and p_1 of the $q+1$-arc and no other points of the $q+1$-arc namely the planes $<L_1,p_1>$ and $<L_2,p_1>$.

Hence, $<L,z>$ contains at least one of the unisecants to z and this is valid for all lines of Γ^+. Let $L_i\pi_{v_O}=x_i\epsilon L$ for $i=1$ or 2. Then there exist $q+2$ lines of γ^+ in π_{v_O} which intersect x_1 or x_2 so one of these is on at least $(q+2)/2\geq 3$ lines of Γ^+ (for $q\geq 4$) which is contrary to the fact that Γ^+ is a dual hyperoval. Hence $z\epsilon\pi_{v_O}$.

Lemma 2. *The contact plane of any point of F^D intersects π_{v_O} in a line of the dual hyperoval Γ^+ - \{ the two special unisecants at z \}. Furthermore, π_{v_O} is the contact plane at z.*

Proof. Since there are $q(q-1)/2$ points p_{ij} for $i\neq j$, and there are exactly $q(q-1)/2$ points which are not on lines of Γ^+ in π_{v_O}, it follows that z is on a line of Γ^+. Since every point of π_{v_O} is on 0 or 2 lines of the dual hyperoval, let M_1,M_2 be the lines of Γ^+ thru z. For any point p_k, $<M_1,p_k>$ and $<M_2,p_k>$ contain z and p_k and since the M_i are lines of Γ^+, these planes share no other points of F^{D+}. As p_k varies over the q-arc, we obtain q planes $<M_1,p_k>$ so that M_1 and M_2 are the two special unisecants to z and hence $\{M_1,M_2\}=\{L_1,L_2\}$. Then, $<L_1,L_2>=\pi_{v_O}$ is the contact plane at z.

Now let N_1,N_2 be the two special unisecants to p_k. Note that $<x,p_k,p_j>$ for $j\neq k$ provides a set of $q-1$ planes that share zp_k and since $<L_1,p_K>$ and $<L_2,p_K>$ share exactly z and p_k with F^{D+}, we may, without loss of generality, take $<N_1,z>=<L_1mp_k>$ and $<N_2,z>=<L_2,p_k>$. Moreover, $N_i\pi_{v_O}=n_i$ is on L_i for $i=1,2$ and since Γ^+ is a dual hyperoval, there are lines $R_i\neq L_i$ for $i=1,2$ incident with n_i and in Γ^+, for $i=1,2$.

Since $<R_1,N_1>$ contains exactly the point p_k, it follows that $<R_1,N_1>=<R_2,N_2>$ $=<N_1,N_2>$ is the contact plane at p_k. Note that this implies that $R_1=R_2=n_1n_2\epsilon\Gamma^+$.

Hence, the contact plane at p_k intersects π_{v_O} in a line of $\Gamma^+-\{L_1,L_2\}$.

Lemma 3. *No three contact planes can share a line.*

Proof. By the proposition in section 2, the osculating plane is the contact plane. No three planes can share a line since there a symplectic polarity β such that P^β for P in F^{d+} is the set of contact planes and otherwise three points of the $(q+1)$-arc would also share a common line.

Lemma 4. *The q contact planes at points of F^D intersect the contact plane of z in the q lines of the dual hyperoval not equal to L_1 or L_2.*

Proof. Simply note that a contact plane at p_i and a contact plane at p_k for i not k cannot share a line on the contact plane of z by lemma 3.

Let M be any line of $\Gamma^+ - \{L_1, L_2\}$ and let p be in F^{D+} such that $p^\beta \pi_{v_o} = M$. Then M^β is a line incident with $\pi_{v_o}^\beta = z$ and with p.

Hence, the $q+2$ lines of $(\Gamma^+)^\beta$ are the lines $\{zp_i$ for $i = 1, 2, \ldots, q\} \cup \{L_1, L_2\}$.

Now take any contact plane p_1^β and intersect the set of lines zp_i, L_1, L_2. This intersection is a hyperoval in p_1^β. Note that if we take the representation as given $z = (0,0,0,1)$ and $p_1 = (1,0,0,0)$ then $p_1^\beta = (x_3 = 0)$ and the intersection with the lines is the hyperoval

$$\{(1,t,t^\sigma,0),(0,0,1,0),(0,1,0,0)\}).$$

By the theorems (2.1) and (3.1), the set of contact planes p_i^β for $i = 1, 2, \ldots, q$ in $PG(3, 2^r)$ forms a flock of a (hyper translation oval) translation oval cone if and only if r is odd (also see below).

However, since $\{p_1, \ldots, p_q\}$ forms the dual flock then $\{p_i^\beta, \ldots, p_q^\beta\}$ is the original flock of planes and the original oval produces the hyperoval cone which when dualized is the dual hyperoval. Hence the set of lines of the original hyperoval cone is the set of lines $\Gamma^{+\beta}$ so that the flock must be one of the translation oval flocks of Thas.

Note that it also follows from the above argument that considering the flock in $PG(3, 2^r)$ that r is odd. In particular, one obtains a flock if and only if $1 + t + t^\sigma \neq 0$ for all $t \in GF(q)$. If $q = 2^r$ for r ever then since $\sigma = 2^m$ and $(r, m) = 1$, it follows that m is odd $= 2s+1$. But, then there is a subfield isomorphic to $GF(4)$ and for t in this subfield, we have $t^{2^m} = t^{2^{2s+1}} = (t^{4^k})^2 = t^2$. Since there exists an element such that $t^2 = t+1$, the construction is valid if and only if $q = 2^r$, for r odd.

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