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# ON $\gamma$-WEAK SOLUTIONS OF ELLIPTIC EQUATIONS

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**Abstract.** *We build on the 'alternative' approach to the Dirichlet problem in weighted Sobolev spaces (cf. A. Kufner & J. Rakosnik, [11]) to formulate sufficient conditions for interior regularity of $\gamma$-weak solutions of the Dirichlet problem for second order elliptic PDEs with Lipschitz coefficients. Next, we prove a maximum principle (for $\gamma$-weak subsolutions). Last, we generalize (from power type weights to a larger class of weights) a trace theorem by J. Necas, [14].*

## 1. REVIEW OF A. KUFNER'S THEORY AND STATEMENT OF MAIN RESULTS

Let $\Omega \subset \mathbf{R}^N$ be a domain with boundary $\partial\Omega$. Let $\gamma(x)$ be a *weight*, i.e. a Lebesgue measurable function on $\Omega$, and $\gamma > 0$ a.e. on $\Omega$. The weighted Sobolev space $W^{1,2}(\Omega, \gamma)$ consists of all $u \in W^1(\Omega)$ so that:

$$\|u\|_{W^{1,2}(\Omega,\gamma)} \equiv \left( \sum_{|\alpha| \le 1} \int_\Omega |D^\alpha u(x)|^2 \gamma(x) dx \right)^{1/2} < \infty \tag{1}$$

where $W^1(\Omega)$ denotes the space of all weakly differentiable functions on $\Omega$. We request (together with [11], p. 187) that:

$$\gamma \in L^1_{loc}(\Omega), \ \gamma^{-1} \in L^1_{loc}(\Omega) \tag{2}$$

Consequently $C_0^\infty(\Omega) \subseteq W^{1,2}(\Omega, \gamma)$ and $W^{1,2}(\Omega, \gamma)$ is complete (in the norm (1)).

A large amount of literature was dedicated in the last thirty years to the study of the structure of weighted Sobolev spaces (cf. R.A. Adams, [1], A. Avantaggiati, [2], A. Kufner, [10], M. Troisi, [17], etc). as well as to their applications in the theory of PDEs (cf. P. Bolley & J. Camus, [4], V. Benci & D. Fortunato, [3], B. Hanouzet, [9], A. Kufner & A.M. Sandig, [12], A. Kufner & J. Voldrich, [13], J. Necas, [14], etc.). The present paper falls into this second category. Let:

$$(Lu)(x) \equiv \sum_{|\alpha|,|\beta| \le 1} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x)) \tag{3}$$

$x \in \Omega$, be a second order linear differential operator with $a_{\alpha\beta} \in L^\infty(\Omega)$, for any $|\alpha| \le 1$, $|\beta| \le 1$. Assume $\gamma \in C^1(\Omega)$. Let $W_0^{1,2}(\Omega, \gamma)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega, \gamma)$ (with respect to the norm (1)). Let:

$$u_0 \in W^{1,2}(\Omega, \gamma)$$

and:

$$F \in [W_0^{1,2}(\Omega, \gamma)]^*$$

be given. Then $u \in W^{1,2}(\Omega, \gamma)$ is a $\gamma$-weak solution of the Dirichlet problem for $L$ if:

$$u - u_0 \in W_0^{1,2}(\Omega, \gamma) \tag{4}$$

$$a_\gamma(u, \phi) = F(\phi) \tag{5}$$

for any $\phi \in W_0^{1,2}(\Omega, \gamma)$. Here $a_\gamma$ is given by:

$$a_\gamma(u, v) = \sum_{|\alpha|, |\beta| \leq 1} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x) D^\alpha (v(x)\gamma(x)) dx \tag{6}$$

A weight $\gamma \in C^1(\Omega)$ is said to possess the property $(P_1)$ (cf. [11], p. 188) if there is $C^* > 0$ so that:

$$|\nabla \gamma(x)| \leq C^* \gamma(x) \tag{7}$$

a.e. on $\Omega$. If $L$ is elliptic, i.e.

$$\sum_{|\alpha| = |\beta| = 1} a_{\alpha\beta}(x) \xi_\alpha \xi_\beta \geq C_0 |\xi|^2 \tag{8}$$

for some $C_0 > 0$ and any $\xi = (\xi_\alpha)_{|\alpha|=1} \in \mathbf{R}^N$, and $a_{\alpha\beta}$ are essentially bounded then there is $C > 0$ so that for any weight $\gamma$ with the property $(P_1)$ and with $C > C^*$, and for any $u_0 \in W^{1,2}(\Omega, \gamma)$ and any $F \in [W_0^{1,2}(\Omega, \gamma)]^*$ there is a unique $\gamma$-weak solution of the Dirichlet problem (4)-(5) for $L$. Cf. Theor. 40.11 in [12], p. 192. The weaker ellipticity assumption:

$$a(\phi, \phi) \geq C_0 \|\phi\|^2_{W^{1,2}(\Omega)} \tag{9}$$

for any $\phi \in W_0^{1,2}(\Omega)$ is actually sufficient, where $a(u, v)$ is given by (6) for $\gamma \equiv 1$. It is our purpose in the present paper to formulate sufficient conditions under which a given $\gamma$-weak solution $u \in W^{1,2}(\Omega, \gamma)$ is regular at the interior of $\Omega$.

A weight $\gamma(x)$ is termed *admissible* if:

i) for any $s = 1, \ldots, N$ and any $h \in \mathbf{R}$ with $\Omega_{s,h} \neq \emptyset$, there is a constant $1 \leq C(h) < \infty$ so that:

$$C(h)^{-1} \gamma(x) \leq \gamma(x + h e_s) \leq C(h) \gamma(x) \tag{10}$$

for any $x \in \Omega_{s,h}$ (except for a set of measure zero),

ii) $\lim_{h \to 0} |h|^{-1}(C(h) - 1)$ exists ($< \infty$), and

iii) for any $\delta > 0$ fixed one has:

$$\sup_{|h| \leq \delta} C(h) < \infty, \ \sup_{|h| \leq \delta} \frac{C(h) - 1}{|h|} < \infty \tag{11}$$

Here $e_s$ is the $s$-th vector in the canonical basis of $\mathbf{R}^N$ and $\Omega_{s,h} = \{x \in \Omega \colon x + h e_s \in \Omega\}$. Besides from restricting ourselves to the case of admissible weights, we assume $a_{00} =$

$0, a_{0\beta} = 0, a_{\alpha 0} = 0$, i.e. (3) is given by $Lu \equiv -\sum_{|\alpha|=|\beta|=1} D^{\alpha}(a_{\alpha\beta}D^{\beta}u)$. We obtain the following:

**Theorem 1.** *Let $u \in W_0^{1,2}(\Omega,\gamma)$ be a $\gamma$-weak solution of the Dirichlet problem (4) - (5) for the elliptic operator L, where $u_0 \equiv 0$ and $F \in [W_0^{1,2}(\Omega,\gamma)]^*$ is given by $F(\phi) = \int_{\Omega}(\sum_{|\alpha|=1} f_{\alpha}D^{\alpha}\phi - f\phi)dx$, with $f_{\alpha}, f \in L^2(\Omega,\gamma^{-1}), |\alpha| = 1$. If i) $\gamma$ is admissible, ii) $a_{\alpha\beta}$ are Lipschitz in $\Omega$, $|\alpha| = |\beta| = 1$, and iii) $f_{\alpha} \in W^{1,2}(\Omega,\gamma^{-1})$, $|\alpha| = 1$, then $u \in W_{loc}^{2,2}(\Omega,\gamma)$ and for any $\Omega' \subset\subset \Omega$ there is a constant $Q = Q(\Omega,\Omega',\gamma) > 0$ so that:*

$$\int_{\Omega'} |D^{\alpha}u|^2\gamma dx \leq Q\int_{\Omega}(|\nabla u|^2\gamma + |\nabla F|^2\gamma^{-1} + |f|^2\gamma^{-1})dx \tag{12}$$

*for any $|\alpha| = 2$. Here $F = (f_{\alpha})_{|\alpha=1}$.*

The proof of our **Theorem 1** relies on the **Lemmas 1** and **2** (cf our Section 3). **Lemma 2** is a straightforward version (for weighted spaces and norms) of Prop. 3.4 in [7], p. 47 (a proof is included for the sake of completeness). The main estimate (12) is performed in Section 4.

Let us fix $u_0 \in W^{1,2}(\Omega,\gamma)$ and a continuous linear functional:

$$F \in [W_0^{1,2}(\Omega,\gamma)]^*$$

Originally (cf. [10], p. 129) a function $u \in W^{1,2}(\Omega,\gamma)$ was termed weak solution of the Dirichlet problem for $L$ if (4) holds while (5) is replaced by:

$$a(u,\phi) = F(\phi) \tag{13}$$

for any $\phi \in C_0^{\infty}(\Omega)$. Moreover, a theorem of existence and uniqueness of such weak solutions in $W^{1,2}(\Omega,\gamma)$ of the Dirichlet problem (4), (13) still holds (cf. Theor. 15.2 in [10], p. 144) when:

$$\gamma(x) = s(d_M(x)) \tag{14}$$

where $s(t)$ is a continuous positive function defined for $t > 0$ and such that either $\lim_{t\to 0} s(t) = 0$ or $\lim_{t\to 0} s(t) = \infty$ and $d_M(x) = dist(x,M)$ for some real $m$-dimensional $(0 \leq m \leq N-1)$ manifold $M \subseteq \partial\Omega$. It is noteworthy that while the classical (unweighted) solution of the Dirichlet problem (cf. Theor. 8.3 in [8], p. 181) relies on the Lax-Milgram theorem, the proof of Theor. 15.2 in [10] requests a generalization (cf. [15], Ch. 6, Sect. 3.1) of the Lax-Milgram theorem (cf. also Lemma 13.6 in [10], p. 131). As pointed out in [11], the advantage of the 'alternative' approach (4)-(5) to the Dirichlet problem for $L$ in $W^{1,2}(\Omega,\gamma)$ (based on the use of the bilinear form $a_{\gamma}(u,v)$ rather than $a(u,v)$) comes from the fact that J. Necas' generalized Lax-Milgram theorem is not needed anylonger (the proof of Theor. 40.11 in [12], p 192, follows from Theor. 39.5, p. 179, alone). Once $a(u,v)$ is replaced by $a_{\gamma}(u,v)$ one may say that $u \in W^{1,2}(\Omega,\gamma)$ satisfies (by definition) $Lu \leq 0 (= 0, \leq 0)$ in $\Omega$ if $a_{\gamma}(u,\phi) \leq 0$ $(= 0, \geq 0)$ for any nonnegative $\phi \in W_0^{1,2}(\Omega,\gamma)$. It is a natural question to ask whether one may establish a (weak) maximum principle for $L$ (cf. Theor. 8.1 in [8], p. 179, for the unweighted case).

Let $\gamma \in C^0(\overline{\Omega}), \gamma > 0$ in $\Omega$. Set:

$$I(x_0) = \{x \in \Omega : |x - x_o|^2 < \gamma(x_0)\}$$

for $x_0 \in \Omega$. We say $\gamma$ is *admissible in the sense of M. Troisi* if the weight $\rho = \gamma^{1/2}$ satisfies the properties I)-II) in [17], p. 50. In particular $I(x_0)$ has the cone property for any $x_0 \in \Omega$, and there is a constant $A_0 > 0$ so that:

$$A_0^{-1}\rho(x_0) \leq \rho(x) \leq A_0\rho(x_0) \tag{15}$$

for any $x_0 \in \Omega$ and any $x \in I(x_0)$. We may state the following:

**Theorem 2.** *Let $N$ be an odd integer $\geq 3$. Let $\Omega \subseteq \mathbf{R}^N$ be a bounded domain and $\gamma \in C^1(\Omega) \cap C^0(\overline{\Omega})$ an admissible (in the sense of M. Troisi) weight. Assume that $a^{ij}, b^i, c^i$ and $d$ satisfy (51) up to (53) and $a^{ij} \in L^\infty(\Omega)$. If $u \in W^{1,2}(\Omega, \gamma)$ is such that $Lu \geq 0$ in $\Omega$ then:*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ \tag{16}$$

In a forthcoming paper we shall study the boundary regularity of $\gamma$-weak solutions of the Dirichlet problem (4)-(5) via a trace theorem of J. Necas (established for $\gamma(x) = d_M(x)^\epsilon$, $M = \partial\Omega$, $\epsilon \geq 0$, cf. [14], p. 309) generalized versions of which are discussed in Section 7 (cf. **Theorems 4** and **5**).

The authors are grateful to the referee who's remarks led to an improvement of **Theorem 2.**

## 2. ADMISSIBLE WEIGHTS

Let $f(x)$ be a function on $\Omega$ and $1 \leq s \leq N$, $h \in \mathbf{R}$. Set $\tau_{s,h}f(x) = \frac{1}{h}\,(f(x+he_s) - f(x))$, for any $x \in \Omega_{s,h}$. Let also $\Omega_h = \{x \in \Omega : dist(x, \partial\Omega) > |h|\}$. Then $\Omega_h \subset \Omega_{s,h}$ (because, for any $x \in \Omega_h$, $\overline{B}(x,|h|) \subset \Omega$ and $x + he_s \in \partial B\,(x, |h|)$). Next, if $\Omega' \subset\subset \Omega$ and $0 < \delta < \frac{1}{2} dist(\Omega', \partial\Omega)$ then $\Omega_h \neq \emptyset$ for any $|h| \leq \delta$ (indeed $\Omega' \subset \Omega_h$). Throughout $B(x,r)$ denotes the ball of center $x$ and radius $r > 0$.

Let $\gamma(x)$ be an admissible weight on $\Omega$. Then:

$$|\tau_{s,h}\gamma| \leq \frac{C(h)-1}{|h|}\gamma \tag{17}$$

a.e. in $\Omega$.

**Example 1.** Let $\gamma(x)$ be given by (14) for $s(t) = \exp(\epsilon t), t > 0, \epsilon \in \mathbf{R}$, (cf. [11], p. 187). If $\epsilon > 0$ then $\gamma$ is admissible. Indeed, we may take $C(h) = \exp(\epsilon|h|) \geq 1$ (and then $\lim_{h\to 0}|h|^{-1}(C(h)-1) = \epsilon$, $\sup_{|h|\leq\delta} C(h) = \exp(\epsilon h) < \infty$, $\sup_{|h|\leq\delta}|h|^{-1}\,(C(h)-1) = \delta^{-1}(\exp(\epsilon\delta)-1) < \infty$).

We end Section 2 by listing some properties of the operator $\tau_{s,h}$ i.e. 1) if $f \in W^{1,2}(\Omega,\gamma)$ then $\tau_{s,h} f \in W^{1,2}(\Omega_h, \gamma)$, 2) if $supp(f) \subset \Omega_h$ (or $supp(g) \subset \Omega_h$) then $\int_\Omega f\tau_{s,h}g dx = -\int_\Omega g\tau_{s,h} f dx$, 3) $\tau_{s,h}(fg) = f(x+he_s)\,\tau_{s,h}\,g + g(x)\tau_{s,h}f$. Only 1) needs to be justified (cf. [7], p. 46, for 2)-3)). If $\xi \in \mathbf{R}^N$ let $T_\xi : \mathbf{R}^N \to \mathbf{R}^N$ be given by $T_\xi(x) = x + \xi$. We may perform the following estimate:

$$\|\tau_{s,h}f\|^2_{L^2(\Omega_{h,\gamma})} \leq 2|h|^{-2}\|f\|^2_{L^2(\Omega,\gamma)} + I_{s,h}$$

where:

$$I_{s,h} = 2|h|^{-2} \int_{T_{he_s}(\Omega_h)} |f(x)|^2 \gamma(x - he_s)dx$$

Note that $I_{s,h}$ may be estimated by using (10) in the hypothesis of admissibility. Indeed:

$$I_{s,h} \leq 2|h|^{-2} C(h) \|f\|^2_{L^2(\Omega,\gamma)} < \infty$$

so that $\tau_{s,h} f \in L^2(\Omega_h, \gamma)$. The estimate:

$$\|\tau_{s,h} f\|_{L^2(\Omega_h,\gamma)} \leq \sqrt{2}|h|^{-1}(1 + C(h))^{1/2} \|f\|_{L^2(\Omega,\gamma)}$$

together with the identity $\tau_{s,h} D^\alpha f = D^\alpha \tau_{s,h} f$ complete the proof.

## 3. TWO LEMMAS

We shall need the following:

**Lemma 1.** *Let $\gamma(x)$ be an admissible weight. Then for any $\Omega' \subset\subset \Omega$ there is a constant $K = K(\Omega, \Omega', \gamma) > 0$ so that for any $v \in W^{1,2}(\Omega, \gamma)$ and any $|h| < \frac{1}{2} dist(\Omega', \partial\,\Omega)$ we have:*

$$\|\tau_{s,h} v\|_{L^2(\Omega',\gamma)} \leq K \|\frac{\partial\, v}{\partial\, x_s}\|_{L^2(\Omega,\gamma)} \tag{18}$$

**Proof.** Let $\Omega' = C \times (a, b) \subset \Omega$, where $C \subset \mathbf{R}^{N-1}$ is a cube. We may assume w.l.o.g. that $s = N$. Set $x' = (x_1, \ldots, x_{N-1})$. Then:

$$\int_{\Omega'} |\tau_{N,h} v(x)|^2 \gamma(x)dx = \int_C \int_b^a |\tau_{N,h} v(x', x_N)|^2 \gamma(x', x_N)dx_N dx' =$$

$$= \int_C \int_a^b \left| \frac{1}{h} \int_{x_N}^{x_N+h} \frac{\partial\, v}{\partial\, x_N}(x', t)dt \right|^2 \gamma(x', x_N)dx_N dx'$$

On the other hand:

$$\left| \frac{1}{h} \int_{x_N}^{x_N+h} \frac{\partial\, v}{\partial\, x_N}(x', t)dt \right|^2 = \left| \frac{1}{h} \int_0^h \frac{\partial\, v}{\partial\, x_N}(x', t + x_N)dt \right|^2 \leq$$

$$\leq \frac{1}{h} \int_0^h \left| \frac{\partial\, v}{\partial\, x_N}(x', t + x_N) \right|^2 dt$$

(one applies the Hölder inequality at the last step). Therefore:

$$\int_a^b \left| \frac{1}{h} \int_{x_N}^{x_N+h} \frac{\partial\, v}{\partial\, x_N}(x', t) \right|^2 \gamma(x', x_N)dx_N \leq \frac{1}{h} F(h)$$

where:

$$F(h) = \int_0^h dt \int_a^b \left| \frac{\partial v}{\partial x_N}(x', t + x_N) \right|^2 \gamma(x', x_N) dx_N$$

Set $h_0 = \frac{1}{2} dist(\Omega', \partial \Omega)$. By the mean value theorem (applied to $F$) there is $\xi$ between 0 and $h$ so that:

$$\frac{1}{h} F(h) = \int_a^b \left| \frac{\partial v}{\partial x_N}(x', \xi + x_N) \right|^2 \gamma(x', x_N) dx_N =$$

$$= \int_{a+\xi}^{b+\xi} \left| \frac{\partial v}{\partial x_N}(x', x_N) \right|^2 \gamma(x', x_N - \xi) dx_N \leq$$

$$\leq C(\xi) \int_{a+\xi}^{b+\xi} \left| \frac{\partial v}{\partial x_N}(x) \right|^2 \gamma(x) dx_N \leq$$

$$\leq K \int_{a-h_0}^{b+h_0} \left| \frac{\partial v}{\partial x_N}(x) \right|^2 \gamma(x) dx_N$$

(noting that $(x', x_N - \xi) = x - \xi e_N$ one has applied (10) followed by iii) in the definition of admissibility) where $K = \sup_{|\xi| \leq h_0} C(\xi)$. Finally:

$$\int_{\Omega'} |\tau_{N,h} v(x)|^2 \gamma(x) dx \leq K \int_{C \times (a-h_0, b+h_0)} \left| \frac{\partial v}{\partial x_N}(x) \right|^2 \gamma(x) dx \leq$$

$$\leq K \int_\Omega \left| \frac{\partial v}{\partial x_N} \right|^2 \gamma dx = K \left\| \frac{\partial v}{\partial x_N} \right\|^2_{L^2(\Omega, \gamma)}$$

So (18) is proved when $\Omega'$ is a cube. In general, it is sufficient to cover $\Omega'$ with a finite number of cubes, Q.E.D..

**Lemma 2.** *Let $v \in L^2(\Omega, \gamma)$ and assume that there is $K > 0$ and $h_0 > 0$ so that:*

$$||\tau_{s,h} v||_{L^2(\Omega_h, \gamma)} \leq K \tag{19}$$

*for any $|h| < h_0$. Then $\partial v / \partial x_s \in L^2(\Omega, \gamma)$ and:*

$$\left\| \frac{\partial v}{\partial x_s} \right\|_{L^2(\Omega, \gamma)} \leq K \tag{20}$$

**Proof.** Let $(h_j)_{j \geq 1}$ be a sequence of real numbers so that $h_j \to 0$ as $j \to \infty$. Set $g_j = \tau_{s,h_j} v, j \geq 1$. Let $\Omega' \subset\subset \Omega$. As $h_j \to 0$ there is $j_0 = j(\Omega', h_0) \geq 1$ so that $|h_j| < \min (h_0, \frac{1}{2} dist\,(\Omega', \partial \Omega))$ for any $j \geq j_0$. So on one hand:

$$||\tau_{s,h_j} v||_{L^2(\Omega_{h_j}, \gamma)} \leq K \tag{21}$$

for any $j \geq j_0$, and on the other:

$$\Omega' \subset \Omega_{h_j} \tag{22}$$

for any $j \geq j_0$. Using (21)-(22) we have:

$$\|g_j\|^2_{L^2(\Omega',\gamma)} = \int_{\Omega'} |g_j|^2 \gamma dx \leq \int_{\Omega_{h_j}} |g_j|^2 \gamma dx \leq K^2$$

so that:

$$(K^{-1} g_j)_{j \geq j_0}$$

is a sequence in the closed unit ball:

$$\overline{B}(0,1) \subset L^2(\Omega',\gamma)$$

But $L^2(\Omega',\gamma)$ is reflexive (any Hilbert space is reflexive) so that $\overline{B}(0,1)$ is weakly sequentially compact. Consequently, there is a subsequence $(h_{m_j})$ of $(h_j)$ so that:

$$g_{m_j} \longrightarrow g$$

as $j \longrightarrow \infty$ for some $g \in L^2(\Omega',\gamma)$ (weak convergence). Note that:

$$g = \partial v / \partial x_s$$

Indeed:

$$\int_{\Omega'} g\phi dx = \lim_{j\to\infty} \int_{\Omega'} g_{m_j} \phi dx = \lim_{j\to\infty} \int_{\Omega'} (\tau_{s,h_{m_j}} v)\phi dx =$$

$$= -\lim_{j\to\infty} \int_{\Omega'} v \tau_{s,-h_{m_j}} \phi dx = -\int_{\Omega'} v\, \partial \phi / \partial x_s dx$$

for any $\phi \in C_0^\infty(\Omega')$. Thus:

$$\partial v / \partial x_s \in L^2_{loc}(\Omega,\gamma)$$

and:

$$\lim_{h\to 0} \tau_{s,h} v = \partial v / \partial x_s$$

weakly in $L^2(\Omega',\gamma)$. Finally:

$$\left\| \frac{\partial v}{\partial x_s} \right\|_{L^2(\Omega',\gamma)} \leq \liminf_{h\to 0} \|\tau_{s,h} v\|_{L^2(\Omega',\gamma)} \leq$$

$$\leq \liminf_{h\to 0} \|\tau_{s,h} v\|_{L^2(\Omega_h \gamma)} \leq K$$

and thus (20) holds, Q.E.D.

If $a_{\alpha,\beta}, f_\alpha$ have locally integrable derivatives then a $\gamma$-weak solution $u \in C^2(\Omega)$ is also a classical solution of the equation $\gamma \sum_{|\alpha|=|\beta|=1} D^\alpha(a_{\alpha\beta} D^\beta u) = \sum_{|\alpha|=1} D^\alpha f_\alpha + f$ in $\Omega$, where $f, f_\alpha \in L^2(\Omega, \gamma^{-1})$.

We end Section 3 with a formal argument justifying our assumption $u_0 \equiv 0$ in the Dirichlet problem (4)-(5). Indeed, if $w = u - u_0$ then $\gamma \sum_{|\alpha|=|\beta|=1} D^\alpha(a_{\alpha\beta} D^\beta w) = \sum_{|\alpha|=1} D^\alpha F_\alpha + F$

where $F_\alpha = f_\alpha - \sum_{|\beta|=1} a_{\alpha\beta} \cdot D^\beta u_0 \cdot \gamma$ and $F = f + \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u_0 \cdot D^\alpha \gamma$. All we need to show is that $F, F_\alpha \in L^2(\Omega, \gamma^{-1})$. Indeed:

$$\|F\|^2_{L^2(\Omega,\gamma^{-1})} = \int_\Omega |F|^2 \gamma^{-1} dx \leq$$

$$\leq 2 \int_\Omega \left( |f|^2 + \left| \sum_{|=\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u_0 \cdot D^\alpha \gamma \right|^2 \right) \gamma^{-1} dx$$

Set $M = \max_{|\alpha|=|\beta|=1} \|a_{\alpha\beta}\|_\infty$. Using (7), i.e. $|D^\alpha \gamma| \leq C^* \gamma$ a.e. in $\Omega$, we may perform the estimate:

$$\left| \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u_0 \cdot D^\alpha \gamma \right|^2 \leq 2(MNC^*)^2 \gamma^2 \sum_{|\beta|=1} |D^\beta u_0|^2$$

Therefore:

$$\|F\|^2_{L^2(\Omega,\gamma^{-1})} \leq$$

$$\leq 2 \int_\Omega |f|^2 \gamma^{-1} dx + 4(MNC^*)^2 \sum_{|\beta|=1} \int_\Omega |D^\beta u_0|^2 \gamma dx \leq$$

$$\leq 2\|f\|^2_{L^2(\Omega,\gamma^{-1})} + 4(MNC^*)^2 \|u_0\|^2_{W^{1,2}(\Omega,\gamma)} < \infty$$

Finally, we leave it to the reader to verify that $\|F_\alpha\|^2_{L^2(\Omega,\gamma^{-1})} \leq 2\|f\|^2_{L^2(\Omega,\gamma^{-1})} + 4M^2 \|u_0\|^2_{W^{1,2}(\Omega,\gamma)} < \infty$.

## 4. THE MAIN ESTIMATE

At this point we may prove our **Theorem 1**. To this end, let $\eta \in C_0^\infty(\Omega), 0 \leq \eta \leq 1$, and $|h| < \frac{1}{4}$ dist $(supp(\eta), \partial\, \Omega)$ be fixed data. Set $\phi = \tau_{s,-h}(\eta^2, \tau_{s,h} u)$. Note that $\phi \in W^{1,2}(\Omega_h, \gamma)$ (and actually $\phi \in W_0^{1,2}(\Omega, \gamma)$ because $supp(\phi) \subseteq supp(\eta) \subset \Omega_h$). Then (5) may be written:

$$\sum_{|\alpha|=1} \int_\Omega \left( \sum_{|\beta|=1} a_{\alpha\beta} \cdot D^\beta u \cdot \gamma - f_\alpha \right) \cdot$$

$$\cdot \tau_{s,-h}(\eta^2 \tau_{s,h} D^\alpha u + 2\eta \cdot D^\alpha \eta \cdot \tau_{s,h} u) dx =$$

$$= - \int_\Omega (f + \sum_{\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u \cdot D^\alpha \gamma) \tau_{s,-h}(\eta^2 \tau_{s,h} u) dx$$

Let us use the properties 2)-3) of the operator $\tau_{s,h}$ (cf. Section 2) so that to obtain:

$$\sum_{|\alpha|=1} \int_\Omega \{ \sum_{|\beta|=1} [a_{\alpha\beta}(x + he_s)(\gamma(x + he_s)\tau_{s,h} D^\beta u +$$

$$+D^{\beta}u\cdot\tau_{s,h}\gamma)+D^{\beta}u\cdot\gamma\cdot\tau_{s,h}a_{\alpha\beta}]-$$

$$-\tau_{s,h}f_{\alpha}\}\{\eta^2\tau_{s,h}D^{\alpha}u+2\eta\cdot D^{\alpha}\eta\cdot\tau_{s,h}u\}dx=$$

$$=\int_{\Omega}(f+\sum_{|\alpha|=|\beta|=1}a_{\alpha\beta}\cdot D^{\beta}u\cdot D^{\alpha}\gamma)\tau_{s,-h}(\eta^2\tau_{s,h}u)dx \tag{23}$$

Consider the following integrals:

$$I_1=-\sum_{|\alpha|=|\beta|=1}2\int_{\Omega}\eta\cdot\tau_{s,h}D^{\beta}u\cdot a_{\alpha\beta}(x+he_s)\cdot D^{\alpha}\eta\cdot\tau_{s,h}u\cdot\gamma(x+he_s)dx$$

$$I_2=-\sum_{|\alpha|=1}\int_{\Omega}\eta^2\cdot\tau_{s,h}D^{\alpha}u\cdot\left(\sum_{|\beta|=1}D^{\beta}u\cdot\tau_{s,h}a_{\alpha\beta}\cdot\gamma-\tau_{s,h}f_{\alpha}\right)dx$$

$$I_3=-\sum_{|\alpha|=1}2\int_{\Omega}\eta\cdot D^{\alpha}\eta\cdot\tau_{s,h}u\cdot\left(\sum_{|\beta|=1}u\cdot\tau_{s,h}a_{\alpha\beta}\cdot\gamma-\tau_{s,h}f_{\alpha}\right)dx$$

$$I_4=\int_{\Omega}\left(f+\sum_{|\alpha|=|\beta|=1}a_{\alpha\beta}\cdot D^{\beta}u\cdot D^{\alpha}\gamma\right)\tau_{s,-h}(\eta^2\tau_{s,h}u)dx$$

$$I_5=-\sum_{|\alpha|=1}\int_{\Omega}\eta^2\cdot\tau_{s,h}D^{\alpha}u\cdot\sum_{|\beta|=1}a_{\alpha\beta}(x+he_s)\cdot D^{\beta}u\cdot\tau_{s,h}\gamma dx$$

$$I_6=-\sum_{|\alpha|=1}2\int_{\Omega}\eta\cdot D^{\alpha}\eta\cdot\tau_{s,h}u\cdot\sum_{|\beta|=1}a_{\alpha\beta}(x+he_s)\cdot D^{\beta}u\cdot\tau_{s,h}\gamma dx$$

With these notations the identity (23) becomes:

$$\sum_{|\alpha|=|\beta|=1}\int_{\Omega}a_{\alpha\beta}(x+he_s)\cdot\tau_{s,h}D^{\alpha}u\cdot\tau_{s,h}D^{\beta}u\cdot$$

$$\cdot\gamma(x+he_s)\cdot\eta^2\cdot\gamma(x+he_s)dx=\sum_{j=1}^{6}I_j \tag{24}$$

By the ellipticity assumption:

$$\sum_{|\alpha|=|\beta|=1}\int_{\Omega}a_{\alpha\beta}(x+he_s)\cdot\tau_{s,h}D^{\alpha}u\cdot\tau_{s,h}D^{\beta}u\cdot\gamma(x+he_s)\cdot\eta^2dx\geq$$

$$\geq C_0\int_{\Omega}\eta^2\gamma(x+he_s)|\tau_{s,h}\nabla u|^2dx$$

so that (24) yields:

$$C_0 \int_\Omega \eta^2(x+he_s)|\tau_{s,h}\nabla u|^2 dx \le \sum_{j=1}^{6} |I_j| \tag{25}$$

It remains to estimate the integrals $I_j$, $1 \le j \le 6$. Set:

$$L = \max_{|\alpha|=|\beta|=1} \| \partial a_{\alpha\beta} / \partial x_s \|_\infty$$

We obtain:

$$|I_1| \le 2MN \int_\Omega \eta |\tau_{s,h}\nabla u||\nabla\eta||\tau_{s,h}u|\gamma(x+he_s)dx \tag{26}$$

$$|I_2| \le N^{1/2} \int_\Omega \eta^2 |\tau_{s,h}\nabla u|(NL|\nabla u|\gamma + |\tau_{s,h}F|)dx \tag{27}$$

$$|I_3| \le 2N^{1/2} \int_\Omega \eta |\nabla\eta||\tau_{s,h}u|(NL|\nabla u|\gamma + |\tau_{s,h}F|)dx \tag{28}$$

Let us use $2ab \le \epsilon a^2 + \epsilon^{-1}b^2$, $\epsilon > 0$, to further estimate the integrals $I_j, j \in \{1,2,3\}$. For instance, set $a = \eta|\tau_{s,h}\nabla u|\gamma(x+he_s)^{1/2}$ and $b = M|\tau_{s,h}u|\ |\nabla\eta|\gamma\ (x+he_s)^{1/2}$. Then (26) furnishes:

$$|I_1| \le N\epsilon \int_\Omega \eta^2 |\tau_{s,h}\nabla u|^2 \gamma(x+he_s)dx +$$

$$+ \frac{M^2 N}{\epsilon} \int_\Omega |\tau_{s,h}u|^2 |\nabla\eta|^2 \gamma(x+he_s)dx \tag{29}$$

Similarly, to estimate $I_2$ (respectively $I_3$) let us choose:

$$a = |\tau_{s,h}\nabla u|\gamma(x+he_s)^{1/2}$$

and:

$$b = (NL|\nabla u|\gamma + |\tau_{s,h}F|)\gamma(x+he_s)^{-1/2}$$

(respectively:

$$a = |\nabla u||\tau_{s,h}u|\gamma(x+he_s)^{1/2}$$

$$b = \eta(NL|\nabla u|\gamma + |\tau_{s,h}F|)\gamma(x+he_s)^{-1/2}$$

and $\epsilon = 1$). We obtain:

$$|I_2| \le N^{1/2}\frac{\epsilon}{2} \int_\Omega \eta^2 |\tau_{s,h}\nabla u|^2 \gamma(x+he_s)dx +$$

$$+ \frac{N^{1/2}}{2\epsilon} \int_\Omega \eta^2 (NL|\nabla u|\gamma + |_{s,h}F|)^2 \gamma(x+he_s)^{-1} dx \tag{30}$$

$$|I_3| \le N^{1/2} \int_\Omega |\nabla\eta|^2 |\tau_{s,h}u|^2 \gamma(x+he_s)dx +$$

$$+N^{1/2}\int_{\Omega}\eta^2(NL|\nabla u|\gamma+|\tau_{s,h}F|)^2\gamma(x+he_s)^{-1}dx \tag{31}$$

The integral $I_4$ is somewhat trickier. Since:

$$\tau_{s,-h}(\eta^2\tau_{s,h}u)=\eta(x-he_s)\tau_{s,-h}(\eta\tau_{s,h}u)+\eta\cdot\tau_{s,h}u\cdot\tau_{s,-h}\eta$$

it follows that:

$$|I_4|\leq J_1+J_2 \tag{32}$$

where:

$$J_1=\int_{\Omega}|\tau_{s,-h}(\eta\cdot\tau_{s,h}u)|\cdot\left|f+\sum_{|\alpha|=|\beta|=1}a_{\alpha\beta}\cdot D^{\beta}u\cdot D^{\alpha}\gamma\right|dx$$

$$J_2=\int_{\Omega}\eta\cdot|\tau_{s,h}u|\cdot|\tau_{s,-h}\eta|\cdot|f+$$

$$+\sum_{|\alpha|=|\beta|=1}a_{\alpha\beta}\cdot D^{\beta}u\cdot D^{\alpha}\gamma|dx$$

To estimate $J_1$ (respectively $J_2$) choose:

$$a=|\tau_{s,-h}(\eta\cdot\tau_{s,h}u)|\gamma(x)^{1/2}$$

$$b=\left|f+\sum_{|\alpha|=|\beta|=1}a_{\alpha\beta}\cdot D^{\beta}u\cdot D^{\alpha}\gamma\right|\gamma(x)^{-1/2}$$

(respectively:

$$a=\eta|\tau_{s,h}u|\cdot|\tau_{s,-h}\eta|\gamma(x)^{1/2}$$

and $\epsilon=1$) and use again $2ab\leq\epsilon a^2+b^2/\epsilon$ so that to obtain:

$$J_1\leq\frac{\epsilon}{2}\int_{\Omega}|\tau_{s,-h}(\eta\ \tau_{s,h}u|^2\gamma dx+$$

$$+\frac{1}{2\epsilon}\int_{\Omega}\left|f+\sum_{|\alpha|=|\beta|=1}a_{\alpha\beta}\cdot D^{\beta}u\cdot D^{\alpha}\gamma\right|^2\gamma^{-1}dx \tag{33}$$

$$J_2\leq\frac{1}{2}\int_{\Omega}\eta^2|\tau_{s,-h}\eta|^2\gamma dx+$$

$$+\frac{1}{2}\int_{\Omega}\left|f+\sum_{|\alpha|=|\beta|=1}a_{\alpha\beta}D^{\beta}u\cdot D^{\alpha}\gamma\right|^2\gamma^{-1}dx \tag{34}$$

Next we use (32)-(33) and our **Lemma 1** with $v = \eta\tau_{s,h}u$ so that to perform the following estimates:

$$|I_4| \leq \frac{\epsilon}{2}\int_\Omega |\tau_{s,-h}(\eta\tau_{s,h}u)|^2\gamma dx+$$

$$+\frac{1}{2}(1+\frac{1}{\epsilon})\int_\Omega |f+\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}D^\beta u\cdot D^\alpha\gamma|^2\gamma^{-1}dx+$$

$$+\frac{1}{2}\int_\Omega |\tau_{s,-h}\eta|^2\eta^2|\tau_{s,h}u|^2\gamma dx \leq \frac{K\epsilon}{2}\int_\Omega |\nabla(\eta\tau_{s,h}u)|^2\gamma dx+$$

$$+(1+\frac{1}{\epsilon})\int_\Omega (|f|^2+\left|\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}D^\beta u\cdot D^\alpha\gamma\right|^2)\gamma^{-1}dx+$$

$$+\frac{a_0^2}{2}\int_\Omega |\tau_{s,h}u|^2\gamma dx$$

where $a_0 = \sup|\nabla\eta|$ (for the last integral one has used $\eta^2 \leq 1$ and:

$$|\tau_{s,-h}\eta(x)| = \frac{1}{|h|}|\eta(x-he_s)-\eta(x)| \leq$$

$$\leq \frac{1}{|h|}\sup_{y\in[x,x-he_s]}|\nabla\eta(y)||x-he_s-x| \leq \sup|\nabla\eta|$$

where $[x,y] = \{tx+(1-t)y : 0\leq t\leq 1\}$). On the other hand (by (7)) we have:

$$\left|\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}D^\beta u\cdot D^\alpha\gamma\right| \leq MN^2C^*|\nabla u|\gamma \tag{35}$$

a.e. on $\Omega$. Using again **Lemma 1** for $v = u$ and (35) it follows:

$$|I_4| \leq (\frac{Ka_0^2}{2}+M^2N^4(C^*)^2(1+\frac{1}{\epsilon}))\int_\Omega |\nabla u|^2\gamma dx+$$

$$+(1+\frac{1}{\epsilon})\int_\Omega |f|^2\gamma^{-1}dx+\frac{K\epsilon}{2}\int_\Omega |\nabla(\eta\tau_{s,h}u)|^2\gamma dx \tag{36}$$

for some $K = K(\Omega, supp(\eta), \gamma) > 0$ (furnished by **Lemma 1**). We need to estimate the last integral in (36). To this end, note that:

$$|\nabla(\eta\tau_{s,h}u)|^2 = \sum_{|\alpha|=1}|D^\alpha(\eta\tau_{s,h}u|^2 \leq$$

$$\leq \sum_{|\alpha|=1}(|D^\alpha\eta||\tau_{s,h}u|+\eta|D^\alpha\tau_{s,h}u|)^2 \leq$$

$$\leq 2 \sum_{|\alpha|=1} (|D^\alpha \eta|^2 |\tau_{s,h} u|^2 + \eta^2 |D^\alpha \tau_{s,h} u|^2) =$$

$$= 2(|\nabla \eta|^2 |\tau_{s,h} u|^2 + \eta^2 |\nabla \tau_{s,h} u|^2)$$

Consequently:

$$\int_\Omega |\nabla(\eta \tau_{s,h} u)|^2 \gamma dx \leq$$

$$\leq 2a_0^2 \int_\Omega |\tau_{s,h} u|^2 \gamma dx + 2 \int_\Omega \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx \leq$$

$$\leq 2a_0^2 K \int_\Omega |\nabla u|^2 \gamma dx + 2 \int_\Omega \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx$$

and our estimate (36) becomes:

$$|I_4| \leq K\epsilon \int_\Omega \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx + (1 + \frac{1}{\epsilon}) \int_\Omega |f|^2 \gamma^{-1} dx +$$

$$+ (a_0^2 \epsilon K^2 + \frac{a_0^2}{2} K + M^2 N^4 (C^*)^2 (1 + \frac{1}{\epsilon}) \int_\Omega |\nabla u|^2 \gamma dx \tag{37}$$

Finally, one has to estimate the integrals $I_5, I_6$. Note that $I_5, I_6$ have no counterpart in the classical (unweighted) theory (cf. [7], p. 50-51) i.e. $I_5 = I_6 = 0$ for $\gamma \equiv 1$. Using (17) we obtain:

$$|I_5| \leq MN \frac{C(h) - 1}{|h|} \int_\Omega \eta^2 |\tau_{s,h} \nabla u| |\nabla u| \gamma dx$$

$$|I_6| \leq 2MN \frac{C(h) - 1}{|h|} \int_\Omega \eta |\nabla \eta| |\tau_{s,h} u| |\nabla u| \gamma dx$$

To further estimate $I_5$ (respectively $I_6$) choose:

$$a = |\tau_{s,h} \nabla u| \gamma^{1/2}, b = |\nabla u| \gamma^{1/2}$$

(respectively:

$$a = \eta |\nabla \eta| |\nabla u| \gamma^{1/2}, b = |\tau_{s,h} u| \gamma^{1/2}$$

and $\epsilon = 1$). We obtain:

$$|I_5| \leq MNA \frac{\epsilon}{2} \int_\Omega \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx +$$

$$+ MNA \frac{1}{2\epsilon} \int_\Omega |\nabla u|^2 \gamma dx \tag{38}$$

$$|I_6| \leq MNA(K + a_0^2) \int_\Omega |\nabla u|^2 \gamma dx \tag{39}$$

where:

$$A = \sup_{|h| \leq h_0} \frac{C(h) - 1}{|h|}$$

and:

$$h_0 = \frac{1}{4} dist(supp(\eta), \partial\,\Omega)$$

(in (39) one has made use of **Lemma 1** for $v = u$). Set:

$$B = \sup_{|h| \leq h_0} C(h)$$

Then (29)-(31) yield:

$$|I_1| \leq NB\epsilon \int_\Omega \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx +$$
$$+ M^2 N a_0^2 K B \frac{1}{\epsilon} \int_\Omega |\nabla u|^2 \gamma dx \tag{40}$$

$$|I_2| \leq N^{1/2} B \frac{\epsilon}{2} \int_\Omega \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx +$$
$$+ N^{1/2} B \frac{1}{2\epsilon} \int_\Omega \eta^2 (NL |\nabla u| \gamma + |\tau_{s,h} F|)^2 \gamma^{-1} dx \tag{41}$$

$$|I_3| \leq N^{1/2} B K a_0^a \int_\Omega |\nabla u|^2 \gamma dx +$$
$$+ N^{1/2} B \int_\Omega \eta^2 (NL |\nabla u| \gamma + |\tau_{s,h} F|)^2 \gamma^{-1} dx \tag{42}$$

On the other hand, by **Lemma 1** for the weight $\gamma^{-1}$ (indeed, it is easily seen (by (10)) that $\gamma$ admissible yields $\gamma_{-1}$ admissible, as well):

$$\int_\Omega \eta^2 (NL |\nabla u| \gamma + |\tau_{s,h} F|)^2 \gamma^{-1} dx \leq$$
$$\leq 2 \int_\Omega \eta^2 (N^2 L^2 |\nabla u|^2 \gamma^2 + |\tau_{s,h} F|^2) \gamma^{-1} dx \leq$$
$$\leq 2N^2 L^2 \int_\Omega |\nabla u|^2 \gamma dx + 2K \int_\Omega |\nabla F|^2 \gamma^{-1} dx$$

(as $\eta^2 \leq 1$ and $(a+b)^2 \leq 2(a^2 + b^2)$) with the corresponding modification of (41)-(42). Recollecting the information in (37)-(42) our main estimate (25) becomes:

$$(C_0 B^{-1} - \epsilon C) \int_\Omega \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx \leq$$
$$\leq C_1(\epsilon) \int_\Omega |\nabla u|^2 \gamma dx + C_2(\epsilon) \int_\Omega |\nabla F|^2 \gamma^{-1} dx + C_3(\epsilon) \int_\Omega |f|^2 \gamma^{-1} dx \tag{43}$$

for some $C_i(\epsilon) > 0, i = 1, 2, 3$, (e.g. $C_3(\epsilon) = 1 + 1/\epsilon$). Let now $\Omega' \subset\subset \Omega$ and $\eta = 1$ on $\Omega'$. Take $\epsilon = \epsilon_0$ where:

$$0 < \epsilon_0 < \frac{C_0}{BC}$$

and set:

$$Q = \max_{i=1}^{3} \frac{C_i(\epsilon_0)}{C_0 B^{-1} - \epsilon_0 C}$$

Finally, we may use **Lemma 2** for $v = D^\alpha u$ so that to obtain (12), Q.E.D..

## 5. A WEIGHTED ESTIMATE

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain and $\gamma \in C^0(\overline{\Omega})$, $\gamma < 0$ in $\Omega$, a weight. Assume that $\gamma$ is admissible, in the sense of M. Troisi. Then (13) in [17], p. 50, holds with $\rho = \gamma^{1/2}$. Precisely, there are two positive constants $A_1, A_2$ so that:

$$A_1 \rho(x)^N \leq \int_\Omega \varphi(x, x_0) dx_0 \leq A_2 \rho(x)^N \tag{44}$$

for any $x \in \Omega$. Here $\varphi(\cdot, x_0)$ denotes the characteristic function of $I(x_0)$. The constant $A_2$ depends only on $N$, while $A_1$ depends on $N$ and on the characteristic cone $\mathcal{C}$ of $J(x_0)$, where:

$$J(x_0) = T_{\gamma,x_0}(I(x_0))$$

and the transformation $\xi = T_{\gamma,x_0}(x)$ is defined by:

$$x - x_0 = \rho(x_0)(\xi - x_0) \tag{45}$$

We recall (cf. [17], p. 50) that $\mathcal{C}$ does not depend upon $x_0$. The aim of the present section is to establish the following:

**Theorem 3.** *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain and $\gamma \in C^0(\overline{\Omega}), \gamma > 0$ in $\Omega$, an admissible (in the sense of M. Troisi) weight. Let $j, m \in \mathbf{Z}$ such that $0 \leq j < m$. Let $q, r \in [1, +\infty]$. Assume that:*

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{1-a}{1}$$

$$a \in \left[\frac{j}{m}, 1\right]$$

$$m - j - \frac{N}{r} \notin \{\dots, -2, -1\}$$

*(i.e. $m - j - N/r$ is not a negative integer). Then:*

$$|D^j \phi|_{p, s+N/p} \leq C |D^m \phi|^a_{r,\mu} |\phi|^{1-a}_{q,\mu_0} \tag{46}$$

*for any $\phi \in C_0^\infty(\Omega)$ and any $s \in \mathbf{R}$. The constant $C$ in (46) is of the form:*

$$C = C_0 (A_1^{-1} A_2)^{1/p} A_0^{|s| + |\mu - N/p| a + |\mu_0 - N/p|(1-a)}$$

*where $C_0$ is a positive constant depending only on the characteristic cone $\mathcal{C}$ and on $m, j, r, p$. Also:*

$$\mu_0 = s - \left(j + \frac{N}{q}\right) + \frac{2N}{p}$$

$$\mu = s + \frac{1-a}{a}\left(j + \frac{N}{q} - \frac{N}{p}\right) + \frac{N}{p}$$

As to the weighted norms in (46) we adopt the following notation. We set:

$$|u|_{p,\alpha} = \|\rho^{\alpha} u\|_{L^p(\Omega)}$$

for any $0 < p \leq +\infty$, $\alpha \in \mathbf{R}$. Also $D^j\phi$ stands for $D^\sigma\phi$ for any fixed multiindex $\sigma$ with $|\sigma| = j$. To prove **Theorem 3**, we use Theorem 9.3 in [5] so that tp get:

$$\|D^j\phi\|_{L^p(J(x_0))} \leq C_0 \|D^m\phi\|^a_{L^r(J(x_0))} \|\phi\|^{1-a}_{L^q(J(x_0))} \tag{47}$$

where $\phi$ is thought of as a function of $\xi = T_{\gamma,x_0}(x)$. Under the change of variable (45) the inequality (47) becomes:

$$\rho(x_0)^{j-N/p} \|D^j\phi\|_{L^p(I(x_0))} \leq$$
$$\leq C_0 \rho(x_0)^{a(m-N/r)-(1-a)N/q} \|D^m\phi\|^a_{L^r(I(x_0))} \|\phi\|^{1-a}_{L^q(I(x_0))}$$

or (because of $-j + N/p + a(m - N/r) - (1-a)N/q = 0$):

$$\|D^j\phi\|_{L^p(I(x_0))} \leq C_0 \|D^m\phi\|^a_{L^r(I(x_0))} \|\phi\|^{1-a}_{L^q(I(x_0))}$$

hence (for any $v \geq 1, s \in \mathbf{R}$):

$$\int_\Omega \rho(x_0)^{sv} \|D^j\phi\|^v_{L^p(I(x_0))} dx_0 \leq$$

$$\leq C_0^v \int_\Omega \rho(x_0)^{sv} \|D^m\phi\|^{av}_{L^r(I(x_0))} \|\phi\|^{(1-a)v}_{L^q(I(x_0))} dx_0$$

Then (by Hölder's inequality):

$$\int_\Omega \rho(x_0)^{sv} \|D^j\phi\|^v_{L^p(I(x_0))} dx_0 \leq$$

$$\leq C_0^v \left( \int_\Omega \rho(x_0)^{[s+(1-a)(j+N/q-N/p)/a]v} \|D^m\phi\|^v_{L^r(I(x_0))} dx_0 \right)^a \cdot$$
$$\cdot \left( \int_\Omega \rho(x_0)^{[s-j-N/q+N/p]v} \|\phi\|^v_{L^q(I(x_0))} dx_0 \right)^{1-a} \tag{48}$$

We shall need the Lemmas 1.1 and 1.2 in [17], p. 52. That is:

**Lemma 3.** *Let $p > 0$ and $s \in \mathbf{R}$. For any function $v$ so that $(x_0 \mapsto \rho(x_0)^s \, \|v\|_{L^p(I(x_0))}) \in L^p(\Omega)$ we have:*

$$\int_\Omega \left(\rho(x_0)^s \|v\|_{L^p(I(x_0))}\right)^p dx_0 \geq A_0^{-|s|p} A_1 |v|^p_{p,s+N/p}$$

**Lemma 4.** *Let $\nu \geq p > 0$ and $s \in \mathbf{R}$. For any function $v$ so that $\rho^{s+N/\nu} v \in L^p(\Omega)$ we have:*

$$\int_\Omega \left(\rho(x_0)^s \|v\|_{L^p(I(x_0))}\right)^\nu dx_0 \leq A_0^{|s|\nu} A_2 |v|^\nu_{p,s+N/\nu}$$

$\nu = p$ in (48) and use **Lemmas 3** and **4** so that to obtain:

$$A_0^{-|s|p} A_1 |D^j \phi|^p_{p,s+N/p} \leq$$

$$\leq C_0^p A_0^{|s+(1-a)(j+N/q-N/p)/a|ap} A_2^a |D^m \phi|^{ap}_{r,s+(1-a)(j+N/q-N/p)/a+N/p} \cdot$$

$$\cdot A_0^{|s-j-N/q+N/p|(1-a)p} A_2^{1-a} |\phi|^{(1-a)p}_{q,s-j-N/q+2N/p}$$

which is clearly equivalent to (46). Taking $a = 1, j = 0, m = 1$ and $r = 2$ we obtain the following:

**Corollary 1.** *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain and $\gamma \in C^0(\overline{\Omega}), \gamma > 0$ in $\Omega$, an admissible (in the sense of M. Troisi) weight. If N is an odd integer $\geq 3$ then:*

$$\|\phi\|_{L^q(\Omega,\gamma^{q/2})} \leq C(A_1^{-1}A_2)^{1/q} A_0^{|4-N|} \|D^1\phi\|_{L^2(\Omega,\gamma)} \tag{49}$$

*for any $\phi \in C_0^\infty(\Omega)$, where $q = 2N/(N-2)$ and C is a positive constant depending on N and C.*

## 6. A MAXIMUM PRINCIPLE

We adopt the Einstein summation convention (as to upper and lower indices). Let $\gamma \in C^0(\overline{\Omega}) \cap C^1(\Omega)$, $\gamma > 0$ in $\Omega$, be an admissible weight and rewrite (6) as:

$$a_\gamma(u,\phi) = \int_\Omega \{(a^{ij}D_j u + b^i u)D_i(\gamma\phi) - (c^i D_i u + du)\gamma\phi\}dx \tag{50}$$

where $a^{ij}, b^i, c^i, d, (i,j = 1, \ldots, N)$ are measurable functions on $\Omega$. Then $u \in W^1(\Omega)$ satisfies $Lu \geq 0$ (by definition) if $a^{ij}D_j u + b^i u$, $c^i D_i u + du$ are locally integrable and $a_\gamma(u,\phi) \leq 0$ for any $\phi \in W_0^{1,2}(\Omega,\gamma)$, $\phi \geq 0$. We assume (8), i.e. in the new notations:

$$\sum_{i \leq i,j \leq N} a^{ij}(x)\xi_i\xi_j \geq C_0|\xi|^2 \tag{51}$$

for any $x \in \Omega$, $\xi \in \mathbf{R}^N$ and:

$$C_0^{-2}\sum_{i=1}^N (|b^i(x)|^2 + |c^i(x)|^2 + C_0^{-1}|d(x)| \leq \nu^2 \tag{52}$$

for some $\nu > 0$. Finally, we request that:

$$\int_\Omega (d\gamma\phi - b^i D_i(\gamma\phi))dx \leq 0 \tag{53}$$

for any $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$.

Let $u \in W^{1,2}(\Omega,\gamma)$. Then $u \le 0$ on $\partial\,\Omega$ (by definition) if:

$$u^+ = \max\{u, 0\} \in W_0^{1,2}(\Omega,\gamma)$$

Next, we set:

$$\sup_{\partial\Omega} u = \inf\{k \in \mathbf{R} : u - k \le 0 \ on \ \partial\,\Omega\}$$

Note that (53) continues to hold for any nonnegative $\phi \in W_0^{1,1}(\Omega,\gamma)$, assuming that $b^i$ and $d$ are bounded. Indeed, let $\phi \in W_0^{1,1}(\Omega,\gamma)$ and $\{\phi_k\}_{k\ge 1} \subset C_0^\infty(\Omega)$ so that $\|\phi_k - \phi\|_{W^{1,1}(\Omega,\gamma)} \to 0$ as $k \to \infty$ and $\phi_k \ge 0$ for any $k \ge 1$. Then:

$$\int_\Omega \{d\phi_k\gamma - b^i D_i(\gamma\phi_k)\}dx \le 0$$

and we may perform the following estimates:

$$\int_\Omega (d\phi\gamma - b^i D_i(\gamma\phi))dx \le$$

$$\le \left| \int_\Omega \{d(\phi - \phi_k)\gamma - b^i D_i((\phi - \phi_k)\gamma)\}dx \right| \le$$

$$\le M \int_\Omega \{|\phi - \phi_k|\gamma + \sum_{i=1}^N |D_i(\phi - \phi_k)|\gamma + \sum_{i=1}^N |\phi - \phi_k||D_i\gamma|\}dx \le$$

$$\le M\|\phi - \phi_k\|_{W^{1,1}(\Omega,\gamma)} + M\int_\Omega \sum_{i=1}^N |\phi - \phi_k| C^*\gamma dx \le$$

$$\le M(1 + NC^*)\|\phi - \phi_k\|_{W^{1,1}(\Omega,\gamma)}$$

for some $M > 0$ with $|b^i| \le M$, $|d| \le M$, $1 \le i \le N$. Note that we made use of (7) as well.

At this point we may prove **Theorem 2**. Let $u \in W^{1,2}(\Omega,\gamma)$ and $\phi \in W_0^{1,2}(\Omega,\gamma)$. Then $u\phi \in W_0^{1,1}(\Omega,\gamma)$ and $\nabla(u\phi) = \phi\nabla u + u\nabla\phi$. Indeed, as $u \in W^{1,2}(\Omega,\gamma)$ it follows that $u \in L^1_{loc}(\Omega)$ (by definition). Let $\rho \in C^\infty(\mathbf{R}^N)$, $\rho \ge 0$, $\int \rho dx = 1$, and $\rho = 0$ outside $B(0,1)$, be some mollifier and:

$$u_h(x) = h^{-N}\int_\Omega \rho(\frac{x-y}{h})u(y)dy, \quad h > 0, \quad x \in \Omega_h$$

the regularization of $u$. Also, consider $\{\phi_k\}_{k\ge 1} \subset C_0^\infty(\Omega)$ so that $\phi_k \to \phi$ in the norm (1) as $k \to \infty$. Now, on one hand $u\phi \in W^{1,1}(\Omega,\gamma)$ as a consequence of:

$$\|u\phi\|_{W^{1,1}(\Omega,\gamma)} \le (2N+1)\|u\|_{W^{1,2}(\Omega,\gamma)}\|\phi\|_{W^{1,2}(\Omega,\gamma)}$$

and on the other hand:

$$\|u_h\phi_k - u\phi\|_{W^{1,1}(\Omega,\gamma)} \to 0 \tag{54}$$

as $h \to 0$, $k \to \infty$, as a consequence of the following considerations:

$$\|u_h\phi_k - u\phi\|_{W^{1,1}(\Omega,\gamma)} =$$

$$= \int_\Omega |u_h\phi_k - u\phi|\gamma dx + \int_\Omega \sum_{i=1}^N |D_i(u_h\phi_k - u\phi)|\gamma dx$$

These integrals are estimated separately. Firstly:

$$\int_\Omega |u_h\phi_k - u\phi|\gamma dx \le$$

$$\le M_k \int_{\Gamma_k} |u_h - u|\gamma dx + \|u\|_{W^{1,2}(\Omega,\gamma)}\|\phi_k - \phi\|_{W^{1,2}(\Omega,\gamma)} \tag{55}$$

where $0 < M_k = \sup_{\Gamma_k} |\phi_k|$ and $\Gamma_k = supp(\phi_k)$, $k \ge 1$. We need:

**Lemma 5.** *If $u \in L^p_{loc}(\Omega,\gamma)$ then $u_h \to u$ in $L^p_{loc}(\Omega,\gamma)$, i.e. $\|u_h - u\|_{L^p(\Omega',\gamma)} \to 0$ as $h \to 0$, for any $\Omega' \subset\subset \Omega$.*

**Proof.** Note that:

$$|u_h(x)|^p \le \int_{|z|\le 1} \rho(z)|u(x - hz)|^p dz \tag{56}$$

**Step 1.** *For any $\Omega' \subset\subset \Omega$, $0 < \delta < \frac{1}{2}$ $dist(\Omega', \partial\,\Omega)$ set:*

$$M = M(\gamma, \Omega', \delta) = \inf\{\gamma(y) : y \in B_\delta(\Omega')\}$$

*where $B_\delta(\Omega') = \{x \in \Omega : dist(x, \Omega') < \delta\}$. Then:*

*i) $M > 0$*

*ii) For any $0 < h < \min(\delta, M)$ we have $u_h \in L^p(\Omega', \gamma)$ and:*

$$\|u_h\|_{L^p(\Omega',\gamma)} \le A_0^{2/p}\|u\|_{L^p(B_h(\Omega'),\gamma)} \tag{57}$$

**Proof.** Assume $M = 0$. There is a sequence $\{y_n\}$ in $B_\gamma(\Omega')$ with $lim_{n\to\infty}\gamma(y_n) = 0$. As $\{y_n\}$ is bounded, we may extract a subsequence $\{y'_n\}$ so that $y'_n \to y_0$ as $n \to \infty$, for some $y_0 \in \overline{B_\delta(\Omega')} \subset \Omega$. Thus $\gamma(y_0) = 0$ for some $y_0 \in \Omega$, a contradiction.

To prove ii) in **Step 1**, let $\Omega'_z = T_{-hz}(\Omega') \subset \Omega$, $|z| \le 1$. As

$$|y + hz - y| = h|z| \le h < M$$

$$\Omega'_z \subset B_h(\Omega') \subset B_\delta(\Omega')$$

it follows that $y + hz \in I(y)$ for any $y \in \Omega'_z$. Thus (by (15)):

$$\gamma(y + hz) \le A_0^2\gamma(y)$$

for any $y \in \Omega'_z$. Thus (by (56)) we have the following estimates:

$$\int_{\Omega'} |u_h(x)|^p \gamma(x) dx \leq$$

$$\leq \int_{\Omega'} \gamma(x) [\int_{|z|\leq 1} \rho(z) |u(x-hz)|^p dz] dx =$$

$$= \int_{|z|\leq 1} \rho(z) [\int_{\Omega'} |u(x-hz)|^p \gamma(x) dx] dz$$

and:

$$\int_{\Omega'} |u(x-hz)|^p \gamma(x) dx = \int_{\Omega'_z} |u(y)|^p \gamma(y+hz) dy \leq$$

$$\leq A_0^2 \int_{\Omega'_z} |u(y)|^p \gamma(y) dy \leq A_0^2 \int_{B_h(\Omega')} |u(y)|^p \gamma(y) dy$$

because $\Omega'_z \subset B_h(\Omega')$. Indeed, if $x \in \Omega'_z$ then $x = y - hz$ for some $y \in \Omega'$. Thus $dist(x, \Omega') = \inf_{\xi \in \Omega'} |x - \xi| \leq |x - y| = h|z| < h$. Finally:

$$\int_{\Omega'} |u_h(x)|^p \gamma(x) dx \leq$$

$$\leq \int_{|z|\leq 1} \rho(z) [A_0^2 \int_{B_h(\Omega')} |u(y)|^p \gamma(y) dy] dz =$$

$$= A_0^2 \int_{B_h(\Omega')} |u(y)|^p \gamma(y) dy = A_0^2 \left( \|u\|_{L^p(B_h(\Omega'),\gamma)} \right)^p$$

and (57) is completely proved.

**Step 2.** *For any $\Omega' \subset\subset \Omega$, $u_h \to u$ in $L^p(\Omega', \gamma)$ as $h \to 0$.*
Note that $C^0(\Omega) \subset L^p_{loc}(\Omega, \gamma)$ by (2). Indeed:

$$\int_{\Omega'} |w|^p \gamma dx \leq \sup_{\overline{\Omega'}} |w|^p \int_{\Omega'} \gamma dx < \infty$$

Let $w \in C^0(\Omega)$. Then we may apply (57) for the function $u - w \in L^p_{loc}(\Omega, \gamma)$, that is:

$$\|u_h - w_h\|_{L^p(\Omega',\gamma)} \leq A_0^{2/p} \|u - w\|_{L^p(B_h(\Omega'),\gamma)} \tag{58}$$

for any $w \in C^0(\Omega)$. Set $\Omega" = B_h(\Omega')$. Let $\epsilon > 0$ arbitrary. By a classical result in measure theory (cf. e.g. [16], p. 213) there is $w \in C^0(\Omega)$ so that:

$$\|u - w\|_{L^p(\Omega",\gamma)} \leq \frac{1}{n} \tag{59}$$

Using (58)-(59) we have:

$$\|u-u_h\|_{L^p(\Omega',\gamma)} \le$$
$$\le \|u-w\|_{L^p(\Omega',\gamma)} + \|w-w_h\|_{L^p(\Omega',\gamma)} + \|w_h-u_h\|_{L^p(\Omega',\gamma)} \le$$
$$\le \epsilon(1+A_0^{1/p}) + \|w-w_h\|_{L^p(\Omega',\gamma)}$$

But $w_h$ converges uniformly to $w$ (as $w \in C^0(\Omega)$, cf. [8], p. 147) on any relatively compact subset of $\Omega$. Thus for any $\epsilon > 0$, there is $\delta = \delta(\epsilon, \Omega')$ so that $|w_h(x) - w(x)| < \epsilon^p \left(\|\gamma\|_{L^1(\Omega')}\right)^{-1}$, for any $|h| < \delta$, $x \in \Omega'$. Then $\|w - w_h\|_{L^p(\Omega',\gamma)} < \epsilon$, $|h| < \delta$. The proof of **Lemma 5** is complete.

Let us go back to the proof of (54). By (55) and **Lemma 5** it follows that $\int_\Omega |u_h\phi_k - u\phi|\gamma dx \to 0$. The integral:

$$\int_\Omega \sum_{i=1}^N |D_i(u_h\phi_k - u\phi)|\gamma dx$$

may be dealt with in a similar way (by using $D_i u \in L^2(\Omega,\gamma)$ and $D_i u_h = (D_i u)_h$, as in particular $u \in L^1_{loc}(\Omega)$).

Let $\phi \in W_0^{1,2}(\Omega,\gamma)$ so that $\phi \ge 0$ and $u\phi \ge 0$ in $\Omega$. Then $Lu \ge 0$ and (53) yield:

$$\int_\Omega \{a^{ij}D_j u D_i(\gamma\phi) - (b^i + c^i)\phi\gamma D_i u\}dx \le$$
$$\le \int_\Omega \{du\gamma\phi\; b^i D_i(u\phi\gamma)\}dx \le 0$$

Next we use (52) in the form:

$$\sum_{i=1}^N \left(|b^i|^2 + |c^i|^2\right) \le C_0^2 \nu^2$$

so that to perform the estimates:

$$\int_\Omega a^{ij}D_j u D_i(\gamma\phi)dx \le \int_\Omega (b^i + c^i)\phi \cdot D_i u \cdot \gamma dx \le$$
$$\le \int_\Omega |b^i + c^i||\phi||D_i u|\gamma dx \le 2C_0\nu \int_\Omega |\nabla u|\phi\gamma dx$$

Let $\ell = \sup_{\partial\Omega} u^+$. The rest of the proof of (16) is by contradiction. Assume that $\sup_\Omega u > \ell$. Then there is $k \in \mathbf{R}$ so that $\ell < k < \sup_\Omega u$. As previously established:

$$\int_\Omega a^{ij}D_j u \cdot D_i(\phi\gamma)dx \le 2C_0\nu \int_\Omega |\nabla u|\phi\gamma dx \qquad (60)$$

for any $\phi \in W_0^{1,2}(\Omega, \gamma)$ with $\phi \geq 0$, $\phi u \geq 0$. Then:

$$\int_\Omega a^{ij} D_j u \cdot D_i \phi \cdot \gamma dx \leq$$

$$\leq 2C_0 \nu \int_\Omega |\nabla u| \phi \gamma dx - \int_\Omega a^{ij} D_j u \cdot \phi \cdot D_i \gamma dx \leq$$

$$\leq 2C_0 \nu \int_\Omega |\nabla u| \phi \gamma dx + \int_\Omega |a^{ij}||D_j u||\phi||D_i \gamma| dx \leq$$

$$\leq (2C_0 \nu + NMC^*) \int_\Omega |\nabla u| \phi \gamma dx$$

(where $M = \max_{1 \leq i,j \leq N} \|a^{ij}\|_\infty$). Note that we used (7). At this point we may use:

$$\int_\Omega a^{ij} D_j u \cdot D_i \phi \cdot \gamma dx \leq (2C_0 \nu + N^2 MC^*) \int_\Omega |\nabla u| \phi \gamma dx \tag{61}$$

for $\phi = (u - k)^+$. As $k > 0$, $\phi$ vanishes at the points where $u < 0$ (thus $\phi u \geq 0$). Set $C_1 = 2C_0 \nu + N^2 MC^* > 0$. Then (61) furnishes:

$$\int_\Omega a^{ij} D_j \phi \cdot D_i \phi \cdot \gamma dx \leq C_1 \int_\Gamma \phi |\nabla \phi| \gamma dx$$

where $\Gamma = supp(\nabla \phi) \subset supp(\phi)$. Hence (by (51)):

$$C_0 \int_\Omega |\nabla \phi|^2 \gamma dx \leq C_1 \int_\Gamma \phi |\nabla \phi| \gamma dx \leq$$

$$\leq C_1 \|\phi\|_{L^2(\Gamma,\gamma)} \|\nabla \phi\|_{L^2(\Gamma,\gamma)} \|\nabla \phi\|_{L^2(\Omega,\gamma)}$$

so that:

$$\|\nabla \phi\|_{L^2(\Omega,\gamma)} \leq C \|\phi\|_{L^2(\Gamma,\gamma)} \tag{62}$$

where $C = C_1 / C_0 > 0$. Let us now apply the weighted inequality (49). Indeed:

$$\|\phi\|_{L^q(\Omega,\gamma^{q/2})} \leq$$

$$\leq B \max_{|\alpha|=1} \|d^\alpha \phi\|_{L^2(\Omega,\gamma)} \leq B \|\nabla \phi\|_{L^2(\Omega,\gamma)}$$

where $q = 2N/(N-2)$ and $B = B(N, A_0, A_1, A_2, C)$ is some positive constant. Set $\lambda = BC$. Then:

$$\|\phi\|_{L^q(\Omega,\gamma^{q/2})} \leq \lambda \|\phi\|_{L^2(\Gamma,\gamma)} \leq$$

$$\leq \lambda |\Gamma|^{1/n} \left( \int_\Gamma |\phi|^q \gamma^{q/2} dx \right)^{1/q} \leq \lambda |\Gamma|^{1/n} \|\phi\|_{L^q(\Omega,\gamma^{q/2})}$$

so that:

$$|supp(\nabla\phi)| \geq \lambda^{-n} \tag{63}$$

This inequality is independent of $k$ so it must hold as $k$ tends to $\sup_\Omega u$. The contradiction is attained by Lemma 7.7 in [8], p. 152.

## 7. ON A TRACE THEOREM BY J. NECAS

We recall the notion of domain of class $C^0$ (respectively $C^{0,1}$). This is best understood in the context of domains in $C^\infty$ manifolds. Let $X$ be a real $N$-dimensional $C^\infty$ differentiable manifolds. Set $\Delta = \{y = (y_1, \ldots, y_{N-1}) \in \mathbf{R}^{N-1} : |y_1| < \delta, 1 \leq i \leq N-1\}$. Consider a domain $\Omega \subseteq X$. We say that $\Omega \in C^0$ iff there exist $m$ local charts $(U_\alpha, \varphi_\alpha)$ of $X$, $\varphi_\alpha(U_\alpha) = \mathbf{R}^N$, and there exist $m$ functions $a_\alpha : \overline{\Delta} \to \mathbf{R}$, $1 \leq \alpha \leq m$, so that:

i) For any $x \in \partial\Omega$, there is $\alpha \in \{1, \ldots, m\}$ so that:

$$x'(x) = (x^1(x), \ldots, x^{N-1}(x)) \in \Delta$$

and:

$$x^N(x) = a_\alpha(x'(x))$$

where $\varphi_\alpha = (x^1, \ldots, x^N)$.

ii) Each $a_\alpha : \overline{\Delta} \to \mathbf{R}$ is continuous, $1 \leq \alpha \leq m$.

iii) There is $0 < \beta < 1$ so that the sets $\Omega_\alpha = B_\alpha \cap \Omega$, $\Gamma_\alpha = B_\alpha \cap \partial\Omega$ satisfy:

$$\Omega_\alpha = \{x \in U_\alpha : x'(x) \in \Delta, a_\alpha(x'(x)) - \beta < x^N(x) < a_\alpha(x'(x))\}$$

and:

$$\Gamma_\alpha = \{x \in U_\alpha : x'(x) \in \Delta, x^N(x) = a_\alpha(x'(x))\}$$

for any $1 \leq \alpha \leq m$, where $B_\alpha$ are given by:

$$B_\alpha = \{x \in U_\alpha : x'(x) \in \Delta,$$

$$a_\alpha(x'(x)) - \beta < x^N(x) < a_\alpha(x'(x)) + \beta\}$$

Next:

$$\Omega \in C^{0,1}$$

iff $\Omega \in C^0$ and the functions $a_\alpha$ in ii) are Lipschitz on $\overline{\Delta}$:

$$|a_\alpha(y) - a_\alpha(z)| \leq A|y - z|$$

for some $A > 0$ and any $y, z \in \overline{\Delta}$.

Let $\Omega \subset \mathbf{R}^N$ be a domain and let $\gamma(x)$ be given by (14) for $s(t) = t^\epsilon, t > 0$, $\epsilon \in \mathbf{R}$. If $\Omega$ is bounded and $\epsilon \geq 0$ then $C^\infty(\overline{\Omega}) \subset W^{1,p}(\Omega, \gamma)$, by Theor. 6.3 in [10], p. 48. If additionally $\Omega \in C^{0,1}$, $0 \leq \epsilon < p - 1$ and $M = \partial\Omega$ then (by Theor. 1.2 in [14], p. 309) $W^{1,p}(\Omega, \gamma) \to L^p(\partial\Omega)$, i.e. there is a continuous linear map $Z : W^{1,p}(\Omega, \gamma) \hookrightarrow L^p(\partial\Omega)$ so that $Z(\phi) = \phi|_{\partial\Omega}$ for any $\phi \in C^\infty(\overline{\Omega})$.

Let us recall that $L^p(\partial\Omega,\gamma)$ consists of all $u(x)$ so that:

$$\|u\|_{L^p(\partial\Omega,\gamma)} \equiv \left(\sum_{\alpha=1}^{m}\int_{\Delta}|u_\alpha(y,a_\alpha(y))|^p\gamma_\alpha(y,a_\alpha(y))dy\right)^{1/p} < \infty$$

where $\Omega \in C^{0,1}$ and $u_\alpha = u\circ\varphi_\alpha^{-1}$. The space $L^p(\partial\Omega,\gamma)$ is denoted by $L^p(\partial\Omega,d_M,\omega)$ (cf. [10], p. 100) when $\gamma(x) = d_M(x)^\omega$. If $M = \partial\Omega$ then $d_M(y,a_\alpha(y)) \equiv 0$ and the space $L^p(\partial\Omega,d_M,\omega)$ makes sense only for $\omega = 0$ (and then $L^p(\partial\Omega;\, d_{\partial\Omega},0) = L^p(\partial\Omega)$). We obtain:

**Theorem 4.** *Let $1<p<\infty$. Let $\Omega \subset \mathbf{R}^N$ be a bounded domain and $\gamma \in C^1(\Omega)\cap C^0(\overline{\Omega})$ a weight possessing the property $(P_1)$. If $\Omega \in C^{0,1}$ and $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega,\gamma)$ then there is a continuous linear map $Z : W^{1,p}(\Omega,\gamma) \to L^p(\partial\Omega,\gamma)$ so that $Z(\phi) = \phi_{|\partial\Omega}$ for any $\phi \in C^\infty(\overline{\Omega})$.*

**Proof.** Let $y \in \Delta$ and $0<\eta<\beta$. Let $\phi \in C^\infty(\overline{\Omega})$. Then:

$$\phi_\alpha(y,a_\alpha(y))\gamma_\alpha(y,a_\alpha(y))^{1/p} =$$

$$= \phi_\alpha(y,a_\alpha(y)-\eta)\gamma_\alpha(y,a_\alpha(y)-\eta)^{1/p} + \int_{a_\alpha(y)-\eta}^{a_\alpha(y)}\frac{\partial}{\partial t}(\phi_\alpha\gamma_\alpha^{1/p})(y,t)dt$$

where $\phi_\alpha = \phi\cdot\varphi_\alpha^{-1}, \gamma_\alpha = \gamma\circ\varphi_\alpha^{-1}$. Consequently:

$$|\phi_\alpha(y,a_\alpha(y))|\gamma_\alpha(y,a_\alpha(y))^{1/p} \le$$

$$\le |\phi_\alpha(y,a_\alpha(y)-\eta)|\gamma_\alpha(y,a_\alpha(y)-\eta)^{1/p} + |J_1| + \frac{1}{p}|J_2|$$

where:

$$J_1 = \int_{a_\alpha(y)-\eta}^{a_\alpha(y)}\frac{\partial\phi_\alpha}{\partial t}(y,t)\gamma_\alpha(y,t)^{1/p}dt$$

$$J_2 = \int_{a_\alpha(y)-\eta}^{a_\alpha(y)}\phi_\alpha(y,t)\gamma_\alpha(y,t)^{(1-p)/p}\frac{\partial\gamma_\alpha}{\partial t}(y,t)dt$$

We estimate the integrals $J_i, i = 1,2$, as follows (cf. Hölder's inequality):

$$|J_1| \le \beta^{1/q}\left(\int_{a_\alpha(y)-\beta}^{a_\alpha(y)}\left|\frac{\partial\phi_\alpha}{\partial t}(y,t)\right|^p\gamma_\alpha(y,t)dt\right)^{1/p}$$

where $q = p/(p-1)$. Next, using (7) we have:

$$|J_2| \le \int_{a_\alpha(y)-\eta}^{a_\alpha(y)}|\phi_\alpha(y,t)|\gamma_\alpha(y,t)^{(1-p)/p}C^*\gamma_\alpha(y,t)dt \le$$

$$\leq \beta^{1/q} C^* \left( \int_{a_\alpha(y)-\beta}^{a_\alpha(y)} |\phi_\alpha(y,t)|^p \gamma_\alpha(y,t) dt \right)^{1/p}$$

Using these estimates and the inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, for $a>0, b>0$, we obtain:

$$|\phi_\alpha(y, a_\alpha(y))|^p \gamma_\alpha(y, a_\alpha(y)) \leq$$

$$\leq 2^{p-1}(|\phi_\alpha(y, a_\alpha(y)-\eta)|^p \gamma_\alpha(y, a_\alpha(y)-\eta)+$$

$$+2^{p-1}\beta^{p/q}[\int_{a_\alpha(y)-\beta}^{a_\alpha(y)} \left|\frac{\partial \phi_\alpha}{\partial t}(y,t)\right|^p \gamma_\alpha(y,t)dt+$$

$$+(\frac{C^*}{p})^p \int_{a_\alpha(y)-\beta}^{a_\alpha(y)} |\phi_\alpha(y,t)|^p \gamma_\alpha(y,t)dt])$$

Furthermore, we integrate with respect to $\eta$ on the interval $[\beta/2, \beta]$ so that to yield:

$$\frac{\beta}{2}|\phi_\alpha(y, a_\alpha(y))|^p \gamma_\alpha(y, a_\alpha(y)) \leq$$

$$\leq 2^{p-1}\{\int_{a_\alpha(y)-\beta}^{a_\alpha(y)} |\phi_\alpha(y,t)|^p \gamma_\alpha(y,t)dt+$$

$$+2^{p-2}\beta^{1+p/q}[(\frac{C^*}{p})^p \int_{a_\alpha(y)-\beta}^{a_\alpha(y)} |\phi_\alpha(y,t)|^p \gamma_\alpha(y,t)dt+$$

$$+\int_{a_\alpha(y)-\beta}^{a_\alpha(y)} \left|\frac{\partial \phi_\alpha}{\partial t}(y,t)\right|^p \gamma_\alpha(y,t)dt]\} \leq$$

$$\leq C_1 \int_{a_\alpha(y)-\beta}^{a_\alpha(y)} |\phi_\alpha(y,t)|^p \gamma_\alpha(y,t)dt+$$

$$+C_2 \int_{a_\alpha(y)-\beta}^{a_\alpha(y)} \left|\frac{\partial \phi_\alpha}{\partial t}(y,t)\right|^p \gamma_\alpha(y,t)dt$$

where $C_1 = 2^{p-1}(1 + 2^{p-2}p^{-p}(C^*)^p\beta^{1+p/q})$ and $C_2 = 2^{2p-3}\beta^{1+p/q}$. Finally, we integrate with respect to $y$ on $\Delta$. We also make use of the identity:

$$\int_{\Omega_\alpha} f(x)dx = \int_\Delta dy \int_{a_\alpha(y)-\beta}^{a_\alpha(y)} f_\alpha(y,t)dt$$

where $f_\alpha = f \circ \varphi_\alpha^{-1}$. We obtain:

$$\frac{\beta}{2}\int_\Delta |\phi_\alpha(y, a_\alpha(y))|^p \gamma_\alpha(y, a_\alpha(y))dy \leq$$

$$\leq C_1 \int_{\Omega_\alpha} |\phi(x)|^p \gamma(x)dx + C_2 \int_{\Omega_\alpha} |D_N\phi(x)|^p \gamma(x)dx \leq$$

$$\leq Q\|\phi Vert^p_{W^{1,p}(\Omega,\gamma)}$$

where $Q = \max\{C_1, C_2\}$. We now take the sum over $1 \leq \alpha \leq m$ and obtain:

$$\|\phi_{|\partial\Omega}\|_{L^p(\partial\Omega,\gamma)} \leq \left(\frac{2mQ}{\beta}\right)^{1/p} \|\phi\|_{W^{1,p}(\Omega,\gamma)} \tag{64}$$

Thus $C^\infty(\overline{\Omega}) \hookrightarrow L^p(\partial\Omega,\gamma)$ and due to the hypothesis:

$$\overline{C^\infty(\overline{\Omega})} = W^{1,p}(\Omega,\gamma) \tag{65}$$

we obtain $Z$ by continuous extension.

Recall that, given $x \in \partial\Omega$, we say that $\Omega \in \mathcal{C}(x)$ iff $\Omega \in C^0$ and there is an open bounded cone $\mathcal{C}_x$ with vertex at $x$ and such that $\mathcal{C}_x \cap \overline{\Omega} = \emptyset$ (cf. [10], p. 29).

**Remark 1.** Assume that $\Omega \in \mathcal{C}(\overline{M})$, i.e. for any $x \in \overline{M}$, $\Omega \in \mathcal{C}(x)$ and the cones $\mathcal{C}_x$ are mutually congruent. Let $\gamma$ be given by (14) with $s(t) = t^\epsilon$, $t > 0$, $\epsilon \geq 0$. Then (65) holds (cf. Prop. 7.6 in [10], p. 62) and our **Theorem 4** applies.

**Theorem 5.** *Let $1 < p < \infty$. Let $\Omega \subset \mathbf{R}^N$ be a bounded domain of class $C^{0,1}$ and $\gamma(x)$ a weight on $\Omega$. Assume that i) (65) holds and ii) there is a constant $0 < C < \infty$ so that:*

$$\int_{a_\alpha(y)-\beta}^{a_\alpha(y)} \gamma_\alpha(y,t)^{1/(1-p)} dt \leq C \tag{66}$$

*for any $y \in \Delta$ and any $1 \leq \alpha \leq m$. Then $W^{1,p}(\Omega,\gamma) \to L^p(\partial\Omega)$. Moreover, if additionally $\Omega \in \mathcal{C}(\overline{M})$ and $\gamma(x)$ is given by (14) with $s(t) = t^\epsilon$, $t > 0$, $0 \leq \epsilon < p-1$, then the hypothesis i) - ii) are satisfied (with a constant $C > 0$ depending only on $\epsilon, p, \beta$ and $A$).*

**Proof.** Let $y \in \Delta, 0 < \eta < \beta$ and $\phi \in C^\infty(\overline{\Omega})$. We wish to derive an estimate similar to (64). The proof is analogous to that of **Theorem 4**, so that we allow ourselves to be somewhat sketchy. Using (66) and Hölder's inequality we obtain:

$$|\phi_\alpha(y, a_\alpha(y))| \leq |\phi_\alpha(y, a_\alpha(y) - \eta)| +$$

$$+ C^{(p-1)/p} [\int_{a_\alpha(y)-\beta}^{a_\alpha(y)} \left|\frac{\partial\phi_\alpha}{\partial t}(y,t)\right|^p \gamma_\alpha(y,t) dt]^{1/p}$$

for any $y \in \Delta$, $1 \leq \alpha \leq m$. Moreover, we integrate with respect to $\eta$ from $\beta/2$ to $\beta$ and use the estimate:

$$\int_{a_\alpha(y)-\beta}^{a_\alpha(y)-\beta/2} |\phi_\alpha(y,t)| dt \leq C^{(p-1)/p} [\int_{a_\alpha(y)-\beta}^{a_\alpha(y)} |\phi_\alpha(y,t)|^p \gamma_\alpha(y,t) dt]^{1/p}$$

so that to yield (as $\beta/2 < 1$):

$$\left(\frac{\beta}{2}\right)^p |\phi_\alpha(y, a_\alpha(y))|^p \leq$$

$$\leq (2C)^{p-1} \int_{a_\alpha(y)-\beta}^{a_\alpha(y)} \{|\phi_\alpha(y,t)|^p + \left|\frac{\partial \phi_\alpha}{\partial t}(y,t)\right|^p\}\gamma_\alpha(y,t)dt$$

Finally, let us integrate with respect to $y$ on $\Delta$ and sum over $1 \leq \alpha \leq m$. This procedure furnishes:

$$\frac{\beta}{2}\|\phi_{|\partial\Omega}\|_{L^p(\partial\Omega)} \leq m^{1/p}(2C)^{(p-1)/p}\|\phi\|_{W^{1,p}(\Omega,\gamma)}$$

and we are done. To check the second statement in **Theorem 5** it suffices to show that:

$$\int_{a_\alpha(y)-\beta}^{a_\alpha(y)} d_M(\varphi_\alpha^{-1}(y,t))^{\epsilon/(1-p)}dt < \infty$$

for any $y \in \Delta$, provided that $0 \leq \epsilon < p-1$. Set $x = \varphi_\alpha^{-1}(y,t), y \in \Delta$, $a_\alpha(y) - \beta < t < a_\alpha(y)$. By Lemma 4.6 in [10], p. 25.

$$|a_\alpha(y) - t| \leq (A+1)d_{\partial\Omega}(x) \leq (A+1)d_M(x)$$

Set $q = \epsilon/(1-p)$. As $0 \leq q < 1$, if $a \geq b > 0$ then $a^{-q} \leq b^{-q}$. Thus:

$$|a_\alpha(y) - t|^{-q} \geq (A+1)^{-q}\gamma_\alpha(y,t)^{1/(1-p)}$$

and we obtain:

$$\int_{a_\alpha(y)-\beta}^{a_\alpha(y)} \gamma_\alpha(y,t)^{1/(1-p)}dt \leq (A+1)^q\frac{\beta^{1-q}}{1-q}$$

and **Theorem 5** is completely proved.

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