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FOUR-DIMENSIONAL CONFORMALLY FLAT RIEMANNIAN MANIFOLDS

G. CALVARUSO[1]

Abstract. *In this paper we study some aspects of the classification of the four-dimensional, compact, conformally flat Riemannian manifolds which have constant scalar curvature* τ, *in particular for* $\tau = 0$.

0. INTRODUCTION

Compact, conformally flat Riemannian manifolds have been studied intensively by several people. In view of the positive solution of the Yamabe problem [10] it is fundamental to consider those which have constant scalar curvature. Such manifolds have harmonic curvature.

A. Derdzinski [4] produced several examples which are not Ricci-parallel and hence not locally symmetric. M.H. Noranha [8] treated compact, conformally flat Riemannian manifolds which have non-negative scalar curvature τ. For vanishing τ, the manifolds covered by $\mathbb{R}^4$ or by the product $\mathbb{S}^2(c) \times \mathbb{H}^2(-c)$ provide the locally symmetric examples but there also exist examples which are not locally symmetric. In fact, P. Braam [3] constructed a compactification of $(\mathbb{H}^3 / \Gamma) \times \mathbb{S}^1$, where Γ is a Kleinian group, which is conformally flat and, if the Hausdorff dimension of the limit set of Γ is equal to 1, has scalar curvature $\tau = 0$. In particular, the connected sum $(\mathbb{S}^1 \times \mathbb{S}^3)^{\#n}$, for $n \geq 2$, admits a conformally flat metric with scalar curvature $\tau = 0$, as a consequence of Remark 1 in [3].

Notwithstanding the existence of examples which are not locally symmetric inside the class of four-dimensional compact conformally flat manifolds with zero scalar curvature, it is still interesting to find sufficient conditions to ensure local symmetry for manifolds in this class. In Section 2 of this paper, we shall correct and extend Proposition 6.9 of [8], using the following

Theorem. *Let* (M, g) *be a four-dimensional, compact, orientable, conformally flat Riemannian manifold with* τ *constant. Then we have:*

$$(28 / 3)\pi^2\tau\mathcal{X}(M) \ \leq \ (\tau^3 / 9) \text{ Vol } (M, g) - \int_M \check{\rho} dM, \tag{1}$$

where $\mathcal{X}(M)$ *and* ρ *denote, respectively, the Euler-Poincaré characteristic of* M *and the Ricci tensor of* (M, g) *and* $\check{\rho} = tr(\rho^3)$. *Moreover, the equality in (1) holds if and only if* (M, g) *has constant sectional curvature or is locally isometric to* $\mathbb{R} \times \mathbb{S}^3(c)$, $\mathbb{R} \times \mathbb{H}^3(-c)$, $\mathbb{S}^2(c) \times \mathbb{H}^2(-c)$.

Moreover, we shall also apply the above Theorem to the case of four-dimensional, compact, conformally flat manifolds with $\mathcal{X}(M) = 0$.

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Section 1 is devoted to the proof of this Theorem and to an application about the sign of the Euler-Poincaré characteristic. In Section 3 we develop some considerations about four-dimensional, conformally flat Riemannian manifolds which have $\tau \geq 0$ and vanishing Betti number $b_2(M)$ (for $b_2(M) > 0$, these manifolds have been completely classified by Noronha, see [8], Theorem 2, p. 256).

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1. PROOF OF THE THEOREM

The following formula holds for any compact, orientable Riemannian manifold (M, g) (see [9], formula 2.14, p. 592):

$$\int_M [|\nabla R|^2 - 4|\nabla p|^2 + |\nabla\tau|^2]dM = \int_M [4\check{\overline{R}} + 4\check{\rho} - 4 < \rho\otimes\rho, \overline{R} > -2 < \rho, \dot{R} > + \check{R}]dM, \quad (1.1)$$

where $< \rho\otimes\rho, \overline{R} > = \sum \rho_{ab}\rho_{cb}R_{acbd}$, $< \rho, \dot{R} > = \sum \rho_{uv}R_{uabc}R_{vabc}$, $\check{R} = \sum R_{abcd}R_{abuv}R_{cduv}$, $\check{\overline{R}} = \sum R_{abcd}R_{aucv}R_{budv}$, with respect to any orthonormal basis. Then, since $\dim(M) = 4$ and (M, g) is conformally flat, the components of R are given by

$$R_{ijkh} = (1/2)(g_{ik}\rho_{jh} + g_{jh}\rho_{ik} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih}) - (\tau/6)(g_{ik}g_{jh} - g_{ih}g_{jk}). \quad (1.2)$$

This yelds:

$$\begin{aligned} |\nabla R|^2 &= 2|\nabla\rho|^2 - (1/3)|\nabla\tau|^2, \\ < \rho\otimes\rho, \overline{R} > &= (7/6)\tau|\rho|^2 - \tau^3/6 - \check{\rho}, \\ < \rho, \hat{R} > &= (5/6)\tau|\rho|^2 - \tau^3/6, \\ \check{R} &= \tau|\rho|^2 - 2\tau^3/9, \\ \check{\overline{R}} &= (3/4)\tau|\rho|^2 - \tau^3/9 - \check{\rho}. \end{aligned} \quad (1.3)$$

Consequently, (1.1) becomes

$$\int_M [(2/3)|\nabla\tau|^2 - 2|\nabla\rho|^2]dM = \int_M [4\check{\rho} - (7/3)\tau|\rho|^2 + \tau^3/3]dM$$

and, since τ is constant, we have

$$\int_M |\nabla\rho|^2 dM = \int_M [-2\check{\rho} + (7/6)\tau|\rho|^2 - \tau^3/6]dM. \quad (1.4)$$

Further, the Euler-Poincaré characteristic of a four-dimensional, compact, orientable manifold (M, g) is given by (see [2], p. 82)

$$\mathcal{X}(M) = (1/32\pi^2)\int_M [|R|^2 - 4|\rho|^2 + \tau^2]dM.$$

Since (M, g) is conformally flat, it follows from (2.2) that $|R|^2 = 2|\rho|^2 - \tau^2/3$, and so

$$\mathcal{X}(M) = (1/16\pi^2)\int_M [\tau^2/3 - |\rho|^2]dM. \tag{1.5}$$

This and (1.4) then yield

$$(56/3)\pi^2\tau\mathcal{X}(M) - \int_M [2\tau^3/9 - 2\check{\rho}]dM = -\int_M |\nabla\rho|^2 dM \leq 0, \tag{1.6}$$

and hence, (1) holds.

Moreover, from (1.6) it follows that we have the equality in (1) if and only if $\nabla\rho = 0$. But a four-dimensional, compact, orientable, conformally flat Riemannian manifold is Ricci-parallel (or equivalently, is locally symmetric) if and only if it is locally isometric to $\mathbb{R}^4$, $\mathbb{S}^4(c)$, $\mathbb{H}^4(-c)$, $\mathbb{S}^2(c)\times\mathbb{H}^2(-c)$, $\mathbb{R}\times\mathbb{S}^3(c)$ or $\mathbb{R}\times\mathbb{H}^3(-c)$ (see [7], Proposition 3.3, or [8], Proposition 4.2). Hence, this classification completes the proof of the theorem.

As an immediate consequence of the Theorem we have:

Corollary. *Let (M, g) be a four-dimensional, compact, orientable, conformally flat Riemannian manifold with constant scalar curvature τ. If $\check{\rho} \geq \tau^3/9$ and $\tau > 0$ (respectively < 0), then $\mathcal{X}(M) \leq 0$ (respectively ≥ 0). Moreover, $\mathcal{X}(M) = 0$ if and only if (M, g) is locally isometric to $\mathbb{R}\times\mathbb{S}^3(c)$ (resp. to $\mathbb{R}\times\mathbb{H}^3(-c)$).*

Proof. We can rewrite (1) in this way:

$$(28/3)\pi^2\tau\mathcal{X}(M) \leq \int_M [\tau^3/9 - \check{\rho}]dM.$$

Since $\check{\rho} \geq \tau^3/9$, we obtain $\tau\mathcal{X}(M) \leq 0$, that is, if $\tau > 0$, $\mathcal{X}(M) \leq 0$, and if $\tau < 0$, then $\mathcal{X}(M) \geq 0$.

Now suppose $\mathcal{X}(M) = 0$. Then we have the equality in (1) and so (M, g) is locally isometric to $\mathbb{R}\times\mathbb{S}^3(c)$ if $\tau > 0$, or to $\mathbb{R}\times\mathbb{H}^3(-c)$ if $\tau < 0$.

2. APPLICATIONS OF THE THEOREM

Now we give some applications of our main theorem and we start with a result which corrects and extends Proposition 6.9 of [8].

Proposition 2.1. *Let (M, g) be a four-dimensional, compact, conformally flat Riemannian manifold with $\tau = 0$ and let $\rho_1 \geq \rho_2 \geq \rho_3 \geq \rho_4$ be the eigenvalues of the Ricci operator ρ of M. If*

a) *ρ has less than three distinct eigenvalues in any point of M; or*

b) *$\rho_1 \geq |\rho_4|$; or*

c) *$\rho_3 = \rho_4$,*

then (M, g) is flat or locally isometric to $\mathbb{S}^2(c)\times\mathbb{H}^2(-c)$.

Proof. a) Let (M, g) be a four-dimensional, compact, conformally flat Riemannian manifold with $\tau = 0$ and which has less than three distinct eigenvalues for the Ricci operator at any

point. Sicne (M, g) has harmonic curvature it is analytic in suitable coordinates. If (M, g) is not Ricci-parallel it follows from a classification theorem of Derdzinski (see [4], Theorem 3, p. 280) that (M, g) is covered isometrically by some compact manifold that does not have zero scalar curvature, which is an evident contradiction. So, (M, g) is locally symmetric, that is, it is flat or locally isometric to $\mathbb{S}^2(c) \times \mathbb{H}^2(-c)$.
b) We first suppose that M is orientable. Since $\tau = 0$, (1.3) yields

$$\check{\rho} = -<\rho \otimes \rho, \overline{R}> = -\sum \rho_{ij}\rho_{kh}R_{ikjh}. \tag{2.1}$$

Let $\{e_i\}$ be an orthonormal basis of eigenvectors of the Ricci operator. With respect to such a basis, the only non-vanishing components of R are $R_{ijkj} = K_{ij} = (1/2)(\rho_i + \rho_j)$ for $i \neq j$ and where K_{ij} denotes the sectional curvature for the plane spanned by e_i and e_j. Then, with respect to this basis, (2.1) becomes

$$\check{\rho} = -\sum_{i \neq j} \rho_i \rho_j K_{ij} = -2\sum_{i<j} \rho_i \rho_j K_{ij}. \tag{2.2}$$

On the other hand, a result of Kulkarni (see [6]) states that a n-dimensional Riemannian manifold of dimension ≥ 4 is conformally flat if and only if

$$K_{12} + K_{34} = K_{13} + K_{24} = K_{14} + K_{23}.$$

Therefore, in our case we have

$$\tau = 6(K_{12} + K_{34}) = 6(K_{13} + K_{24}) = 6(K_{23} + K_{14}).$$

Since $\tau = 0, K_{34} = -K_{12}, K_{24} = -K_{13}, K_{23} = -K_{14}$ and so (2.2) becomes

$$\check{\rho} = -2[K_{12}(\rho_1\rho_2 - \rho_3\rho_4) + K_{13}(\rho_1\rho_3 - \rho_2\rho_4) + K_{14}(\rho_1\rho_4 - \rho_2\rho_3)]. \tag{2.3}$$

But $\rho_i = \sum_h K_{ih}$, and hence $\rho_1\rho_2 - \rho_3\rho_4 = -4K_{13}K_{14}, \rho_1\rho_3 - \rho_2\rho_4 = -4K_{12}K_{14}$, $\rho_1\rho_4 - \rho_2\rho_3 = -4K_{12}K_{13}$. This gives

$$\check{\rho} = 24K_{12}K_{13}K_{14}.$$

Consequently, by (1) it follows $\int_M K_{12}K_{13}K_{14}dM \leq 0$.

Since $\tau = 0$ and $\rho_1 \geq \rho_2 \geq \rho_3 \geq \rho_4$, it is easy to see that $K_{12} \geq 0$ and $K_{13} \geq 0$ (see also [8], pp. 265-266). Finally, from the hypothesis $\rho_1 \geq |\rho_4|$ it follows $\int_M \check{\rho}dM = 0$, and hence our theorem completes the proof for the orientable case.

If M is not orientable, one has only to consider the orientable double covering of (M, g) to finish the proof.
c) Since $\tau = 0, K_{12} \geq 0$. Further, we have $K_{13}K_{14} = (K_{13})^2$. So, $\check{\rho} = 24K_{12}K_{13} \cdot K_{14} \geq 0$, and hence, as in the case b), we may conclude that (M, g) is locally isometric to $\mathbb{R}^4$ or to $\mathbb{S}^2(c) \times \mathbb{H}^2(-c)$.

Next, let (M, g) be a four-dimensional, compact, orientable, conformally flat Riemannian manifold with $\tau = 0$ and $\mathcal{X}(M) = 0$. Then, by (1.5) we have $\rho = 0$ and so, by (1.2), (M, g) is flat. If $\tau \neq 0$, the following result holds:

Proposition 2.2. *Let (M, g) be a four-dimensional, conformally flat manifold with $\tau \neq 0$ constant and $\mathcal{X}(M) = 0$. If the distinct eigenvalues of the Ricci operator are $\lambda \neq 0$ and 0, then (M, g) is locally isometric to $\mathbb{R}^3(c)$ or $\mathbb{R} \times \mathbb{H}^3(-c)$.*

Proof. First, let M be orientable. Then, since $\mathcal{X}(M) = 0$, by (1) we obtain

$$\int_M \check{\rho} dM \leq (\tau^3 / 9) \cdot \operatorname{Vol}(M, g). \tag{2.4}$$

The scalar curvature and the norm of the Ricci tensor are expressed in function of the eigenvalues of the Ricci operator by $\tau = m(\lambda) \cdot \lambda$ (constant) and $|\rho|^2 = m(\lambda) \cdot \lambda^2$, respectively, where $m(\lambda)$ denotes the multiplicity of the eigenvalue λ. So, $m(\lambda) = \tau / \lambda$ is a continuous, integer-valued function, that is, $m(\lambda)$ is constant. Next, since $\mathcal{X}(M) = 0$, from (1.5) we obtain easily

$$\int_M m(\lambda)\lambda^2 \cdot [3 - m(\lambda)] dM = 0. \tag{2.5}$$

We observe that $1 \leq m(\lambda) \leq 3$. In fact, if $m(\lambda) = 0$, $\tau = 0$ and if $m(\lambda) = 4$, (M, g) has constant sectional curvature $k = \lambda / 3 \neq 0$, and so $\mathcal{X}(M) \neq 0$, which contradicts our hypothesis. Since $m(\lambda) \leq 3$ and $m(\lambda) \neq 0$, from (2.5) it follows that $m(\lambda) = 3$ and so, since $\check{\rho} = m(\lambda) \cdot \lambda^3$, we have the equality in (2.4). Therefore, (M, g) is locally isometric to $\mathbb{R} \times \mathbb{S}^3(c)$ or $\mathbb{R} \times \mathbb{H}^3(-c)$.

If M is non-orientable, it is enough to consider the orientable double covering of (M, g).

Remark 2.1. Relating to the results of Proposition 2.2, it is interesting to remark that A. Derdzinski [4] gave some examples of four-dimensional, compact, conformally flat Riemannian manifolds with $\mathcal{X}(M) = 0$ and τ constant, which have less than three distinct eigenvalues for the Ricci operator, but are not locally symmetric. So, these manifolds do not enter in the classification given in Proposition 2.2.

3. FOUR-DIMENSIONAL, CONFORMALLY FLAT MANIFOLDS WITH NON-NEGATIVE SCALAR CURVATURE

We first recall the following result of M.H. Noronha:

Theorem (cf. [8], p. 256). *Let M be a $2k$-dimensional, $k > 1$, orientable, compact conformally flat manifold with $\tau \geq 0$. Then either the kth Betti number is 0 or M is covered by $\mathbb{R}^{2k}$ or $\mathbb{S}^k(c) \times \mathbb{H}^k(-c)$.*

This result can also be obtained from two known theorems. Indeed, if (M, g) verifies the hypotheses of the Theorem of Noronha, the possibilities are the following:

a) $\tau \geq 0$, but τ does not vanish identically. Then, by a theorem of Avez and Heslot (see [1], p. 771), $b_k(M) = 0$.

b) $\tau = 0$. By a theorem of Lafontaine (see [7], p. 316), either $b_k(M) = 0$, or (M, g) is locally isometric to $\mathbb{R}^{2k}$ or to $\mathbb{S}^k(c) \times \mathbb{H}^k(-c)$.

For dim $(M) = 4$, we now prove the following result:

Proposition 3.1. *Let (M, g) be a four-dimensional, compact, conformally flat Riemannian manifold with scalar curvature $\tau \geq 0$. For $\mathcal{X}(M) > 0$ and putting $c := a_0 / 12$ where*

$a_0^2 = \max_M(\tau^2)$, *we then have*

$$Vol(M,g) \geq \ Vol(\mathbb{S}^4(c)), \qquad (respectively \ \geq \ Vol(\mathbb{P}^4\mathbb{R}(c))),$$

if M is orientable (resp. non-orientable). Moreover, the equality holds if and only if (M,g) *is isometric to* $\mathbb{S}^4(c)$ *(resp.* $\mathbb{P}^4\mathbb{R}(c)$*).*

Proof. If M is orientable, (1) holds. Consequently, since $|\rho|^2 \geq \tau^2 / 4$, we obtain

$$\mathcal{X}(M) \leq (1 / 192\pi^2) \int_M \tau^2 dM. \tag{3.1}$$

Hence, we have

$$\mathcal{X}(M) \leq (a_0^2 / 192\pi^2) \cdot Vol(M,g). \tag{3.2}$$

Since $\mathcal{X}(M) > 0$, (3.1) assures that τ does not vanish identically, and the theorem of Avez and Heslot [1] implies $b_2(M) = 0$. Next, since $2 - 2b_1 = \mathcal{X}(M) > 0$, $b_1 = 0$. Hence, from (3.2) we have

$$Vol(M,g) \geq 384\pi^2 / a_0^2 = Vol(\mathbb{S}^4(c)) \tag{3.3}$$

where $c := a_0 / 12$. Moreover, if in (3.3) the equality holds, the equality also holds in (3.1). Thus $|\rho|^2 = \tau^2 / 4$, that is, (M,g) is an Einstein manifold. Then, (M,g) is isometric to $\mathbb{S}^4(c)$ (see for example Propositions E.I.3 and E.III.5 in [2]).

If M is not orientable, we derive the same result for the double covering of (M,g) and so (M,g) is isometric to $\mathbb{P}^4\mathbb{R}(c)$.

Remark 3.1. It is well-known that if M is a four-dimensional compact manifold with finite fundamental group, and if there exist on M a conformally flat metric g, then M is diffeomorphic to $\mathbb{S}^4$ if M is orientable and to $\mathbb{P}^4\mathbb{R}$ if M is not orientable (see [5]).

By proceeding in the same way as in the proof of Proposition 3.1, we may prove that for such manifold

$$Vol(M,g) \geq \ Vol(\mathbb{S}^4(c)) \qquad (resp.\ Vol(M,g) \geq \ vol(\mathbb{P}^4\mathbb{R}(c))),$$

and the equality holds if and only if (M,g) is isometric to $\mathbb{S}^4(c)$ (*resp.* $\mathbb{P}^4\mathbb{R}(c)$).

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G. CALVARUSO
Dipartimento di Matematica
Università degli Studi di Lecce
provinciale Lecce-Arnesano
73100 Lecce
ITALIA