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EXAMPLES OF LOCALLY CONFORMAL KÄHLER STRUCTURES

K. MATSUO

Dedicated to Professor Haruo Kitahara in occasion of his sixtieth birthday

Abstract. *We shall prove that there are locally conformal symplectic structures on products of two even-dimensional (resp. odd-dimensional) manifolds endowed with certain 2-forms of maximal rank. In particular, we shall show that the Riemannian product of a Sasakian manifold and a Kenmotsu manifold carrys a locally conformal Kähler structure.*

1. INTRODUCTION

Let V be an even-dimensional differentiable manifold endowed with a non-degenerate 2-form Ω, i.e., (V,Ω) is an *almost symplectic manifold*. An almost symplectic manifold (V,Ω) is called a *locally conformal symplectic manifold* by Vaisman in [9] if there is a global 1-form ω, called the *Lee form*, on V such that

$$d\Omega = \omega \wedge \Omega, \quad d\omega = 0. \tag{1}$$

If $\dim V \geq 6$, the first equation of (1) implies the second one. In particular, if the Lee form $\omega = 0$ on V, then (V,Ω) is a *symplectic manifold*. Hermitian manifolds are typical examples of almost symplectic manifolds. If, for the Kähler form Ω of a Hermitian manifold V, there is a 1-form ω satisfying the equation (1), then V is called a *locally conformal Kähler manifold* in [9]. Vaisman in [9] gave some examples of locally conformal symplectic manifolds and locally conformal Kähler manifolds.

In this paper, we shall prove the following theorem.

Theorem A. *There are locally conformal symplectic structures on the products of the following differentiable manifolds M and N.*

(a) *M is a $2m$-dimensional symplectic manifold with the exact symplectic form $d\theta$, and N is a $2n$-dimensional locally conformal symplectic manifold with a 2-form Φ of maximal rank and the Lee form φ, i.e., $d\Phi = \varphi \wedge \Phi$.*

(b) *M is a $(2m+1)$-dimensional contact manifold with the contact form θ, i.e., $\theta \wedge (d\theta)^m \neq 0$, and N is a $(2n+1)$-dimensional differentiable manifold with a 2-form Φ and a 1-form φ such that $\varphi \wedge \Phi^n \neq 0$ (i.e., almost contact), $d\Phi = \varphi \wedge \Phi$ and $d\varphi = 0$.*

For example, the cotangent bundles of m-dimensional manifolds and the tangent bundles of m-dimensional Riemannian manifolds are $2m$-dimensional symplectic manifolds with the exact symplectic form (see, for instance, [2],[9]). On the other hand, there are some examples of locally conformal symplectic manifolds in [9] as mentioned above. Hence we obtain new examples of locally conformal symplectic manifolds by (a) of Theorem A.

In Section 3, we apply (b) of Theorem A to the products of certain Riemannian manifolds, and give some examples of locally conformal Kähler structures.

Throughout this paper, we shall be in C^∞-category and deal with paracompact, connected differentiable manifolds without boundary. We assume that the dimensions of manifolds are at least two.

2. PROOF OF THEOREM A

We define a 2-form Ω on $M \times N$ as follows:

$$\Omega = d\theta + \Phi + \theta \wedge \varphi. \tag{2}$$

First we prove that Ω is non-degenerate on $M \times N$. Let X be a vector field on $M \times N$, and assume that $\Omega(X, Y) = 0$ for any vector field Y on $M \times N$. Then it is sufficient to show that $X = 0$. By (2),

$$0 = \Omega(X, Y) = d\theta(X_1, Y_1) + \Phi(X_2, Y_2) + \frac{1}{2}\{\theta(X_1)\varphi(Y_2) - \theta(Y_1)\varphi(X_2)\} \tag{3}$$

for any Y, where X_1 and Y_1 (resp. X_2 and Y_2) are M (resp. N)-components of X and Y, respectively. Since Y is arbitrary, by setting $Y = Y_1$ or $Y = Y_2$ in (2), we obtain

$$d\theta(X_1, Y_1) = \frac{1}{2}\,\theta(Y_1)\varphi(X_2), \tag{4}$$

$$\Phi(X_2, Y_2) = -\frac{1}{2}\,\theta(X_1)\varphi(Y_2) \tag{5}$$

for any Y_1 and Y_2. Moreover, setting $Y_1 = X_1$ in (4), we obtain $\theta(X_1)\varphi(X_2) = 0$, i.e., $\theta(X_1) = 0$ or $\varphi(X_2) = 0$.

In the case (a) of Theorem A, if $\theta(X_1) = 0$, then, by (5), we have

$$\Phi(X_2, Y_2) = 0 \qquad \text{for any } Y_2.$$

Since $\Phi^n \neq 0$, we then obtain $X_2 = 0$. Thus, by (4), we also have

$$d\theta(X_1, Y_1) = 0 \qquad \text{for any } Y_1.$$

Since $(d\theta)^m \neq 0$, we obtain $X_1 = 0$, and hence $X = 0$. Similarly, if $\varphi(X_2) = 0$, then we conclude $X = 0$.

In the case (b) of Theorem A, since $\theta \wedge (d\theta)^m \neq 0$ (resp. $\varphi \wedge \Phi^n \neq 0$), there is a nonvanishing vector field ζ on M (resp. ξ on N) such that $\theta(\zeta) = 1$ and $d\theta(\zeta, Y_1) = 0$ for any Y_1 (resp. $\varphi(\xi) = 1$ and $\Phi(\xi, Y_2) = 0$ for any Y_2). If $\theta(X_1) = 0$, then, by (5), we have

$$\Phi(X_2, Y_2) = 0 \qquad \text{for any } Y_2.$$

Thus $X_2 = b\xi$, where b is a function on N. So, by (4), we have

$$d\theta(X_1, Y_1) = \frac{b}{2}\,\theta(Y_1) \qquad \text{for any } Y_1.$$

Setting $Y_1 = \zeta$ in this equation, we get $b = 0$, i.e., $X_2 = 0$. Thus, by (4), we also have

$$d\theta(X_1, Y_1) = 0 \qquad \text{for any } Y_1.$$

Thus, $X_1 = a\zeta$, where a is a function on M. Since $0 = \theta(X_1) = a$, we obtain $X_1 = 0$, and hence $X = 0$. Similarly if $\varphi(X_2) = 0$, then we conclude $X = 0$.

Next we prove $d\Omega = \varphi \wedge \Omega$. In fact, by (2),

$$\begin{aligned} d\Omega &= d^2\theta + d\Phi + d\theta \wedge \varphi - \theta \wedge d\varphi \\ &= \varphi \wedge \Phi + d\theta \wedge \varphi \\ &= \varphi \wedge (d\theta + \Phi + \theta \wedge \varphi) \\ &= \varphi \wedge \Omega. \end{aligned}$$

Hence Ω and φ defines a locally conformal symplectic structure on $M \times N$.

3. LOCALLY CONFORMAL KÄHLER STRUCTURES

A Hermitian manifold V is called a locally conformal Kähler manifold if, for the Kähler form Ω of V, there is a 1-form ω satisfying (1). It is known that there are some examples of such manifolds, for instance, the Hopf manifolds $S^{2m-1} \times S^1$, the Inoue surfaces, locally conformal Kähler nilmanifolds, locally conformal Kähler solvmanifolds and so forth (cf. [5],[9]). In this section, we shall give new examples of locally conformal Kähler structures.

Let M be a $(2m+1)$-dimensional almost contact manifold with the structure tensor (ϕ, ξ, η), i.e.,

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

where I denotes the identity transformation of the tangent spaces. An almost contact structure (ϕ, ξ, η) is said to be *normal* (cf. [3]) if $[\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ. A Riemannian metric g on M is said to be *compatible* if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on M. An almost contact manifold M with a compatible Riemannian metric g is said to have an *almost contact metric structure* (ϕ, ξ, η, g). It is known that there always exists an almost contact metric structure on an almost contact manifold. The *fundamental 2-form* Φ on an almost contact metric manifold M is defined by

$$\Phi(X, Y) = g(X, \phi Y)$$

for all vector fields X, Y on M. Then we have $\eta \wedge \Phi^m \neq 0$. If $\Phi = d\eta$, then M is, by definition, a *contact manifold*. Such an almost contact metric structure is called a *contact metric structure.* Moreover if a contact metric structure is normal, it is called a *Sasakian structure* (cf. [3]). On the other hand, if there is a closed 1-form φ on an almost contact metric manifold M such that $[\phi, \phi] = 0$, $d\Phi = \varphi \wedge \Phi$ and $d\eta = \frac{1}{2}\varphi \wedge \eta$, then M is called a *locally conformal cosymplectic manifold* (cf. [4],[5],[8]). If a locally conformal cosymplectic manifold M is normal, then we obtain $\varphi = f\eta$, where f is a function on M such that $df \wedge \eta = 0$. In particular, if $f = 2$, then M is, by definition (cf. [6],[8]), a *Kenmotsu manifold*, i.e., $d\Phi = 2\eta \wedge \Phi$ and $d\eta = 0$.

We now recall the result of Morimoto [7] that the product of two normal almost contact manifolds is a complex manifold. Let M_i be an almost contact manifolds with the structure tensor (ϕ_i, ξ_i, η_i) for each $i = 1, 2$. On the product manifold $V = M_1 \times M_2$, there is an almost complex structure J defined by

$$J = \phi_1 + \eta_2 \otimes \xi_1 + \phi_2 - \eta_1 \otimes \xi_2. \tag{6}$$

Morimoto in [7] proved that this almost complex structure J is integrable if and only if both almost contact structures of M_1 and M_2 are normal.

Moreover, if g_1 and g_2 are the compatible Riemannian metrics on M_1 and M_2 respectively, the Riemannian product metric $g = g_1 + g_2$ on V is compatible with J, that is, g is a Hermitian metric on V. Thus, if M_i is normal for each $i = 1, 2$, the product manifold V is a Hermitian manifold. Then its Kähler form Ω is given by

$$\Omega = \Phi_1 + \Phi_2 + 2\eta_1 \wedge \eta_2, \tag{7}$$

where Φ_i denotes the fundamental 2-form on M_i for each $i = 1, 2$. Especially if M_1 and M_2 are a $(2m+1)$-dimensional Sasakian manifold and a $(2n+1)$-dimensional Kenmotsu manifold respectively, then we have

$$\eta_1 \wedge \Phi_1^m \neq 0, \qquad \Phi_1 = d\eta_1, \tag{8}$$

$$\eta_2 \wedge \Phi_2^n \neq 0, \qquad d\Phi_2 = 2\eta_2 \wedge \Phi_2, \qquad d\eta_2 = 0. \tag{9}$$

Thus, by Theorem A, the Kähler form Ω of $V = M_1 \times M_2$ with the Hermitian structure mentioned above satisfies $d\Omega = 2\eta_2 \wedge \Omega$. Hence we have

Theorem B. *The Riemannian product of a Sasakian manifold and a Kenmotsu manifold endowed with the complex structure* (6) *is locally conformal Kähler.*

Moreover, the following fact is well-known (for instance, see 1.167 in [1]).

Proposition. *A Riemannian product $(M_1 \times M_2, g_1 + g_2)$ is conformally flat if and only if either*

(*a*) *(M_i, g_i) is one-dimensional, and (M_j, g_j) $(i \neq j)$ is of constant curvature, or*

(*b*) *(M_1, g_1) and (M_2, g_2) are of dimension at least two, one with constant curvature* 1, *and other with constant curvature* -1.

The Hopf manifolds are Riemannian products of type (a). A conformally flat, locally conformal Kähler manifold is said to be *locally conformally Kähler-flat* in Vaisman [10]. The following theorem implies the existence of locally conformally Kähler-flat structures which are Riemannian products of type (b).

Theorem C. *The Riemannian product of a Sasakian manifold M_1 and a Kenmotsu manifold M_2 endowed with the complex structure* (6) *is locally conformally Kähler-flat if and only if M_1 and M_2 have constant curvature* 1 *and* -1, *respectively.*

Remark. It is well-known (cf. [3]) that there is the standard Sasakian structure on the unit sphere S^{2m+1} in $\mathbf{C}^{m+1}$, and its compatible Riemannian metric has constant curvature 1. On the other hand, $\mathbf{R}\times_f\mathbf{C}^n$, where $f(t) = ce^t$ (c: positive constant, $t \in\mathbf{R}$), is an example of Kenmotsu manifolds with constant curvature -1 (see [6]). Thus, by Theorem C, $S^{2m+1}\times(\mathbf{R}\times_f\mathbf{C}^n)$ carrys a locally conformally Kähler-flat structure. But its Lee form ω is exact, i.e., $\omega = 2\eta_2 = 2dt$. Hence its locally conformally Kähler-flat structure is globally conformal Kähler (cf. [9]).

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Received November 26, 1996
Department of Mathematics
Ichinoseki National College of Technology
JAPAN
E-mail: matsuo@ichinoseki.ac.jp