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LIPSCHITZ STABILITY OF LINEAR IMPULSIVE DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. *Sufficient conditions are found for the Lipschitz stability of linear impulsive differential-difference equations. The impulses are realized at fixed moments.*

1. INTRODUCTION

The impulsive differential-difference equations are natural generalization of the impulsive ordinary differential equations without delay and the differential-difference equations without impulses. They are an adequate apparatus in the mathematical modelling of numerous real processes and phenomena. The fact that impulse effect and delay are present simultaneously leads to necessity to modify the standard methods, as well as to introduce new methods in investigations. That is why the theory of these equations is developing rather slowly.

At the present paper Lipschitz stability is studied for a linear system of impulsive differential-difference equations. Let us note that the notion of Lipschitz stability is introduced by Dannan and Elaydi [2] for nonlinear differential equations without impulses.

By virtue of suitable auxiliary equations and differential inequalities for piecewise continuous functions, we prove sufficient conditions for Lipschitz stability of linear system of impulsive differential-difference equations with fixed moments of impulse effect.

2. PRELIMINARY NOTES AND DEFINITIONS

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with elements $x = \text{col}\,(x_1, \dots, x_n)$ and the norm $|\cdot|$; $h > 0$ and $t_0 \in \mathbb{R}$.

We consider the system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h), \quad t \neq \tau_k, t > t_0,$$

$$\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = C_k x(\tau_k), \quad \tau_k > t_0, k = 1, 2, \dots, \tag{1}$$

where $x \in \mathbb{R}^n$; $A(t)$ and $B(t)$ are $n \times n$-matrix valued functions; $t \in (t_0, \infty)$; $C_k, k = 1, 2, \dots,$ are constant $n \times n$-matrices; $t_0 < \tau_1 < \tau_2 < \dots$.

Let $\varphi_0 \in C[[t_0 - h, t_0], \mathbb{R}^n] \equiv C_0$.

We denote by $x(t) = x(t;\, t_0, \varphi_0)$ the solution of (1), that satisfies the initial condition

$$x(t) = \varphi_0(t), \quad t \in [t_0 - h, t_0]. \tag{2}$$

The solution $x(t) = x(t;\, t_0, \varphi_0)$ of the initial problem (1), (2) is a piecewise continuous function on (t_0, ∞), that admits points of discontinuity of the first kind (i.e., the left and right limits exist there, and they are bounded) τ_k, $k = 1, 2, \ldots$, at which it is continuous from the left, i.e., the relations

$$x(\tau_k - 0) = x(\tau_k),$$
$$x(\tau_k + 0) = x(\tau_k) + C_k x(\tau_k), \quad k = 1, 2, \ldots$$

are valid.

In the sequel we shall use the following notations:

$\|\varphi\| = \max_{t \in [t_0 - h, t_0]} |\varphi(t)|$ is the norm of the function $\varphi \in C_0$;

$|A| = \sup_{|x|=1} |Ax|$ is the norm of the $n \times n$-matrix A;

E is the unit $n \times n$-matrix;

$\mu(A) = \limsup_{\sigma \to 0^+} \frac{1}{\sigma} \left(|E + \sigma A| - 1\right)$ is Losinski's "logarithmic norm" of the matrix A.

We introduce the following definition of Lipschitz stability of the system (1):

Definition 1. *The system (1) is said to be* globally uniformly Lipschitz stable *if*

$$(\exists M > 0)(\forall \varphi_0 \in C_0)(\forall t > t_0):$$

$$|x(t; t_0, \varphi_0)| \leq M \|\varphi_0\|.$$

Together with the system (1) we will consider the system

$$\begin{aligned} \dot{u} &= \mu\left(A(t) + B(t)\right) u, \quad t \neq \tau_k, t > t_0, \\ \Delta u(\tau_k) &= (d_k - 1) u(\tau_k), \quad \tau_k > t_0, \end{aligned} \tag{3}$$

$u \in R_+$; $d_k = \text{const} > 0, k = 1, 2, \ldots$.

Definition 2. *The system (3) is said to be* globally uniformly Lipschitz stable *if*

$$(\exists M > 0)(\forall u_0 \in \mathbb{R}_+)(\forall t > t_0):$$

$$u^+(t; t_0, u_0) \leq M u_0,$$

where $u^+(t; t_0, u_0)$ is the maximal solution of the system (3), that satisfies $u^+(t_0; t_0, u_0) = u_0$.

We introduce the following assumptions:

H1. $t_0 \equiv \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \tau_{k+1} < \ldots$.

H2. $\lim_{k \to \infty} \tau_k = \infty$.

H3. The matrices $A(t)$ and $B(t)$ are piecewise continuous on (t_0, ∞), and they admit points of discontinuity of the first kind $\tau_k, k = 1, 2, \ldots$, where they are continuous from the left.

The following lemma will be used later in the proofs of the main results:

Lemma 1. *Let the following conditions hold:*

1. Assumptions H1-H2 *are fulfilled.*

2. The function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ is continuous in $\cup_{k=1}^{\infty} (\tau_{k-1}, \tau_k) \times \mathbb{R}^n$, it satisfies the local Lipschitz condition on each of the sets $(\tau_{k-1}, \tau_k) \times \mathbb{R}^n$, and for each $k = 1, 2, \ldots$ there exist the finite limits

$$V(\tau_k - 0, x) = \lim_{\substack{t \to \tau_k \\ t < \tau_k}} V(t, x), \quad V(\tau_k + 0, x) = \lim_{\substack{t \to \tau_k \\ t > \tau_k}} V(t, x),$$

the equality $V(\tau_k - 0, x) = V(\tau_k, x)$ *holds true, and* $V(t_0, \varphi_0(t_0)) \leq u_0$.

3.

$$V(\tau_k + 0, x(\tau_k) + C_k x(\tau_k)) \leq d_k V(\tau_k, x(\tau_k)), \quad k = 1, 2, \ldots, \tag{4}$$

and

$$D_+ V(t, x(t)) = \limsup_{\sigma \to 0^+} \frac{1}{\sigma} \Bigg[V\Bigg(t + \sigma, x(t) +$$

$$+ \sigma(A(t)x(t) + B(t)x(t-h)) \Bigg) - V(t, x(t)) \Bigg] \leq$$

$$\leq \mu\,(A(t) + B(t))\, V(t, x(t)), t \geq t_0, t \neq \tau_k, x \in \Omega,$$

where $\Omega = \{x \in PC[[t_0, \infty), \mathbb{R}^n] : V(s, x(s)) \leq V(t, x(t)), t - h \leq s \leq t, t \geq t_0\}$, $d_k = const.$

Then

$$V(t, x(t; t_0, \varphi_0)) \leq u^+(t; t_0, u_0) \tag{5}$$

for $t > t_0$.

Proof. The maximal solution $u^+(t; t_0, u_0)$ of the system (3) with initial condition $u^+(t_0 + 0; t_0, u_0) = u_0$ is defined for $t > t_0$ by the equality

$$u^+(t; t_0, u_0) = \begin{cases} u_0^+(t; t_0, u_0), & t_0 < t \leq \tau_1, \\ u_1^+(t; \tau_1, u_1), & \tau_1 < t \leq \tau_2, \\ \ldots\ldots\ldots & \ldots\ldots\ldots, \\ u_k^+(t; \tau_k, u_k), & \tau_k < t \leq \tau_{k+1}, \\ \ldots\ldots\ldots & \ldots\ldots\ldots, \end{cases}$$

where $u_k^+(t; \tau_k, u_k)$ is the maximal solution of the equation $\dot{u} = \mu(A(t) + B(t))u$ in the interval $(\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \ldots$, for which $u_k = d_k u_{k-1}^+(\tau_k; \tau_{k-1}, u_{k-1})$, $k = 1, 2, \ldots$.

Assume that $t \in (t_0, \tau_1]$. Then, by means of the comparison lemma for the continuous case [4], we obtain

$$V(t, x(t; t_0, \varphi_0)) \leq u^+(t; t_0, u_0),$$

i.e., the inequality (5) is valid for $t \in (t_0, \tau_1]$.

Let us suppose now that (5) is fulfilled for $t \in (\tau_{k-1}, \tau_k]$, $k > 1$. Then, it follows from (4) that

$$V(\tau_k + 0, x(\tau_k + 0; t_0, \varphi_0)) \leq d_k V(\tau_k, x(\tau_k; t_0, \varphi_0)) \leq$$

$$\leq d_k u^+(\tau_k; t_0, u_0) = d_k u_{k-1}^+(\tau_k; \tau_{k-1}, u_{k-1}) = u_k.$$

We apply again the comparison lemma of [4] for $t \in (\tau_k, \tau_{k+1}]$. Hence

$$V(t, x(t; t_0, \varphi_0)) \leq u_k^+(t; \tau_k, u_k) = u^+(t; t_0, u_0),$$

i.e., the inequality (5) is fulfilled for $t \in (\tau_k, \tau_{k+1}]$.

The proof is completed by running induction. ∎

3. MAIN RESULTS

Theorem 1. *Let the following conditions hold:*

1. *Assumptions* H1-H3 *are fulfilled.*
2. $\limsup_{t\to\infty} \int_{t_0}^{t} \mu(A(s)+B(s))ds < \infty$.
3. $|E+C_k| \le d_k, k=1,2,\dots$.
4. $\prod_{k=1}^{\infty} d_k < \infty$.

Then the system (1) *is globally uniformly Lipschitz stable.*

Proof. It follows from conditions 1,2 and 4 of Theorem 1 that the system (3) is globally uniformly Lipschitz stable [1], [3]. Therefore, there exists a constant $M>0$ such that for $u_0 \ge 0$ and $t > t_0$ we have

$$u^+(t;t_0,u_0) \le Mu_0. \tag{6}$$

Let us introduce the notations $V(t,x(t)) = |x(t)|$ and $\|\varphi_0\| = u_0$. Then the set Ω is defined by the equality

$$\Omega \doteq \{x \in PC[[t_0,\infty),\mathbb{R}^n] : |x(s)| \le |x(t)|, \quad t-h \le s \le t, t \ge t_0\}.$$

On the other hand, for $t>t_0, x \in \Omega$ and sufficiently small $\sigma > 0$, we have

$$|x(t)+\sigma(A(t)x(t)+B(t)x(t-h))| \le |x(t)+\sigma(A(t)+B(t))x(t)| \le$$
$$\le |x(t)| + \sigma(A(t)+B(t))|x(t)| + \varepsilon(\sigma), \tag{7}$$

where $\frac{\varepsilon(\sigma)}{\sigma} \to 0^+$ as $\sigma \to 0$.

It follows from the inequality (7) for $t \ne \tau_k$ that

$$D_+V(t,x(t)) = \limsup_{\sigma\to 0^+} \frac{1}{\sigma}\Big[|x(t+\sigma)| - |x(t)|\Big] \le$$
$$\le \lim_{\sigma\to 0^+} \frac{1}{\sigma}\Big[|x(t+\sigma)| + \sigma\mu(A(t)+B(t))|x(t)| + \varepsilon(\sigma) -$$
$$-|x(t)+\sigma(A(t)x(t)+B(t)x(t-h))|\Big] \le$$
$$\le \mu(A(t)+B(t))|x(t)| + \lim_{\sigma\to 0^+}\frac{\varepsilon(\sigma)}{\sigma} +$$
$$+\lim_{\sigma\to 0^+}\left|\frac{1}{\sigma}[x(t+\sigma)-x(t)] - (A(t)x(t)+B(t)x(t-h))\right| =$$
$$= \mu(A(t)+B(t))|x(t)| = \mu(A(t)+B(t))V(t,x(t)). \tag{8}$$

Now, condition 3 of Theorem 1 implies

$$V(\tau_k+0, x(\tau_k)+C_kx(\tau_k)) = |x(\tau_k)+C_kx(\tau_k)| \le$$
$$\le |E+C_k||x(\tau_k)| \le d_k|x(\tau_k)| = d_kV(\tau_k,x(\tau_k)) \tag{9}$$

$k = 1, 2, \ldots$.

Moreover,

$$V(t_0, \varphi_0(t_0)) = |\varphi_0(t_0)| \leq \|\varphi_0\| = u_0. \tag{10}$$

It follows from (8), (9) and (10) that all conditions of Lemma 1 are satisfied and therefore

$$V(t, x(t;t_0, \varphi_0)) \leq u^+(t;t_0, u_0), \quad t > t_0.$$

The last inequality and (6) imply the inequalities

$$|x(t;t_0, \varphi_0)| \leq u^+(t;t_0, u_0) \leq Mu_0 = M\|\varphi_0\|, \quad t > t_0,$$

i.e., $|x(t;t_0, \varphi_0)| \leq M\|\varphi_0\|, t > t_0$.

Therefore, the system (1) is globally uniformly Lipschitz stable. ■

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