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A NOTE ON SETS HAVING THE STEINHAUS PROPERTY

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Abstract. *A planar set is said to have the Steinhaus property if however it is placed on $\mathbb{R}^2$, it contains exactly one integer lattice point. The main result proved in this note is that if a set S satisfying the Steinhaus property contains a circle of positive radius, then it contains the disk enclosed by the circle. A result of Mihai Ciucu [2] would then imply that any set having the Steinhaus property cannot contain a circle of positive radius in it*

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1. INTRODUCTION

A subset of $\mathbb{R}^2$ is said to satisfy the Steinhaus property if under any rigid motion of $\mathbb{R}^2$, it contains exactly one integer lattice point. The major open problem regarding such sets is whether such a set exists at all or not.

The earliest result known about Steinhaus sets was obtained by Sierpiński [4] who proved that any Steinhuaus set can be neither compact nor bounded open.

Recently, Mihai Ciucu [2] has proved the following result.

Theorem 1.1. *Any set S having the Steinhaus property has empty interior.*

As a corollary to the above theorem, it was derived in [2] that closed sets do not have the Steinhaus property, thus removing the boundedness assumption in Sierpiński's result.

Regarding other results related to sets having the Steinhaus property J. Beck [1] has proved that any Steinhaus set cannot the bounded and Lebesgue measurable at the same time.

The purpose of this note is to show how the ideas in [2] can be pushed to yield finer results. More precisely, we prove

Theorem 1.2. *If a set S satisfying the Steinhaus property contains a circle of positive radius, then it contains the disk enclosed by the circle.*

Now, in light of Theorem 1.1 mentioned above, as a corollary to Theorem 1.2 we get the following strenghtening of Theorem 1.1.

Corollary 1.3. *Any set having the Steinhaus property cannot contain a circle of positive radius in it.*

In section 2 we give a proof a Theorem 1.2.

Conjecture. We conjecture that a set satisfying the Steinhaus property can not contain any homeomorphic image of the unit circle in $\mathbb{R}^2$.

2. PROOF OF THE MAIN THEOREM

Let us first fix some notations. We denote a circle centered at $\mathbf{a}$ and of radius $r>0$ by $C(\mathbf{a}, r)$. Also, for $m, r > 0$, $A(\mathbf{0};\ m, m+r)$ will denote the closed annulus centered at the origin $\mathbf{0} = (0,0)$ and of inner radius m and outer radius $m+r$.

Assuming the coordinates to increase downward and to the right, for $m, n \in \mathbb{Z}$, let $A_{m,n}$ denote the unit square of $\mathbb{R}^2$ having its upper left corner at (m,n) and containing the north and west sides.

For any subset $M \subset A_{m,n}$, we denote by M (mod 1) the set $M + (-m, -n)$.

For $\theta \in [0, 2\pi)$, $S(\theta)$ will be the set obtained from S by a rotation through an angle θ around the origin in the anticlockwise direction.

Throughtout our discussion, S will be a subset of $\mathbb{R}^2$ having the Steinhaus property containing a circle of positive radius r.

We need two lemmas.

Lemma 2.1. *(M. Ciucu) A set S satisfies the Steinhaus property if and only if $\{S(\theta) \cap A_{m,n}$ mod 1)$\}_{m,n}$ is a partition of $A_{O,O}$, for all $\theta \in [0, 2\pi)$.*

Proof. See [2] for the proof.

Lemma 2.2. *If S contains a circle $C(\mathbf{0}, r)$ of radius $r>0$ and $D = \cup_{n=x^2+y^2} A(\mathbf{0}, \sqrt{n} - r, \sqrt{n} + r)$, where in the union x, y vary over integers, then $S \cap D = \phi$.*

Proof. Suppose S contains a circle $C(\mathbf{0}, r)$. Then for any integer n representable in the forme $x^2 + y^2$ where x and y are integers, the annulus $A(\mathbf{0}\colon \sqrt{n} - r, \sqrt{n} + r)$ cannot contain points of S. For if $\mathbf{u} \in S \cap A(\mathbf{0};\ \sqrt{n} - r, \sqrt{n} + r)$, then there exists $\theta \in [0, 2\pi)$ such that $\mathbf{u} \in S(\theta) \cap C(\mathbf{n}, r)$, where $\mathbf{n} = (x, y)$. Now, we can find a point $\mathbf{v} \in C(0, r)$ such that the vectors $\vec{\mathbf{un}}$ and $\vec{\mathbf{v0}}$ have the same direction. Hence if we translate $S(\theta)$ through a distance r in the direction $\vec{\mathbf{un}}$ the translated set will contain two lattice points $\mathbf{0}$ and $\mathbf{n}$, which is contradictory to the fact that S has the Steinhaus property. Hence $S \cap D = \phi$.

Remark 2.3. *Regarding the estimate of the number of integer lattice points on the disc bounded by the circle of radius $\sqrt{N}$ centered at the origin, Sierpiński [3] proved*

$$\sum_{n=x^2+y^2, n \le \sqrt{N}} 1 = \pi N + O(N^{1/3} \log N).$$

In fact, one knows better estimates for the error term and the conjecture that the error term satisfies $O(N^{1/4+\epsilon})$ for any $\epsilon > 0$ is the famous 'Circle problem'. However, even from the estimate stated above, or from a still weaker result of the type

$$\sum_{n=x^2+y^2, n \le \sqrt{N}} 1 = \pi N + O(N^{1/2-\epsilon}), \quad \text{form some } \epsilon > 0,$$

one can easily obtain that if $a_1, a_2, \ldots$ is the sequence of numbers, in increasing order, which are of the form $x^2 + y^2$, then

$$\lim_{n \to \infty} (\sqrt{a_{n+1}} - \sqrt{a_n}) = 0.$$

Hence there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ any two successive annuli will overlap and $S \cap D = \phi$ will then imply that S is bounded.

Proof of Theorem 1.2. Without loss of generality, we may assume that S contains a circle of positive radius having the centre at the origin $\mathbf{0}$. Let the connected component of $A_{m,n} \cap D$ (mod 1) containing the origin be denoted by $C_{m,n}$. Consider the set

$$I(r) \stackrel{\text{def}}{=} \cap_{(m,n)\neq(0,0)} C_{m,n}.$$

We claim thta $I(r) \subset S$. To prove che claim, let $x \in I(r) \subseteq A_{O,O}$. By Lemma 2.1 there exist m, n such that $x \in S \cap A_{m,n}$ (mod 1). That is, $x + (m, n) \in S$. We shall show that $(m, n) = (0, 0)$, which will prove our claim. If $(m, n) \neq (0, 0)$, then, since $x \in I(r)$, we have $x \in C_{m,n} \subseteq A_{m,n} \cap D$ (mod 1) and so $x + (m, n) \in D$. This gives a contradiction to Lemma 2.2 which says that $S \cap D$ is empty. Hence $(m, n) = (0, 0)$.

We observe that for each $(m, n) \neq (0, 0)$, $C_{m,n}$ contains $Q \cap A_{0,0}$ where Q is the closed disk of radius r centered at the origin. Therefore, $Q \cap A_{0,0}$ is contained in $I(r)$ and hence in S, by the claim established above. This proves that S contains a closed disk.

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