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A REMARK ON CYLINDRICAL CURVES AND INDUCED CURVES IN RP^2

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1. INTRODUCTION

Given an immersion of a manifold into euclidean space R^n there are a few maps which can be defined using tangent vector subspaces (affine tangent subspaces) or normal vector subspaces (affine normal subspaces). In what follows we shall take a closed curve $f : S^1 \to R^3$ and a fixed point p not lying on any tangent (binormal, principal normal) line. We then associate to each $x \in S^1$ the 2-plane determined by the tangent (binormal, principal normal) line of f at x and p. This way we obtain maps from S^1 into the pencil of 2-planes which pass through p. Such a pencil can obviously be identified with the real projective space RP^2.

We shall assume that p is the origin in R^3 and that our curves lie in the cylinder C given by $x^2 + y^2 = 1$. We show that for two of the maps defined above we can always get rid of the singularities in the sense that a suitable translation will allow us to have a curve which will induce an immersion in RP^2. For the other map, the one defined using the binormal lines, that is not true.

This short note is a by-product of some previous joint work [1] and the work reported here was done during a visit of the first named author to the Faculty of Mathematical Studies - Southampton University - U. K.

2. SINGULAR POINTS

Throughout this note we use the term curve to mean a smooth ($= C^\infty$) immersion $f : R \to R^3$ of period 1. The curvature function k of f is assumed to vanish nowhere. Then, for every $s \in R$, there is a well defined Serret-Frenet frame $(T(s), N(s), B(s))$. We can always choose a point p not lying on any tangent (binormal, principal normal) line as the next proposition shows.

Proposition 1. *The family of tangent (binormal, principal normal) lines of any curve does not fill R^3.*

Proof. Let S be a sphere enclosing $f(R)$. For every $s \in R$, the tangent half-line $\{x \in R^3 \mid x = f(s) + \lambda f'(s), \lambda > 0\}$ meets S in a unique point $g(s)$, say. Then g is smooth. Likewise, there is a smooth map $g_1 : R \to S$ obtained by considering the other tangent half-lines, that is, with $\lambda < 0$. The set $X = g(R) \cup g_1(R)$ has measure zero in S, by Sard's theorem. Then, for any $p \in S \setminus X$, no line through p is tangent to f.

From now on we assume that $p = (0, 0, 0)$ and that no tangent line passes through the origin in R^3. We define a map $F_t : R \to G_2(3)$, where $G_2(3)$ is the Grassmannian of 2-dimensional vector subspaces in R^3, by associating to every $s \in R$ the 2-plane determined by p and the tangent line at s. Since we are going to be interested in the singularities we observe that

$F_t = \lambda \circ f_t$, where $\lambda : S^2 \to G_2(3) \equiv RP^2$ is such that $\lambda(x)$ is the orthogonal complement of the space x generates and $f_t(s) = \frac{f(s)\wedge T(s)}{\|f(s)\wedge T(s)\|}$. Consequently F_t and f_t have the same singularities. Similar considerations can be made using $B(s)$ or $N(s)$.

We are therefore going to be concerned with the maps f_t, f_n, f_b. While f_t and f_b are constant maps if and only if $f(R)$ is contained in a plane passing through the origin it happens that f_b is never a constant map. If f_b were constant then f would be a plane curve and $B(s)$ would belong to the direction of the plane. This cannot happen obviously.

Proposition 2. *f_t is singular at s if and only if $(0,0,0) \in \Pi_s$, where Π_s is the osculating plane at s.*

f_n is singular at s if and only if $(0,0,0) \in \Pi_s$ and the torsion $\tau(s)$ is zero.

f_b is singular at s if and only if $(0,0,0) \in \alpha_s$, where α_s is the rectifying plane at s, and $\tau(s) = 0$.

Proof. We prove, for instance, the only if part for f_n. Assume that s is a singular point of f_n. Then there is a real number α such that $(f \wedge N)'(s) = \alpha f(s) \wedge N(s)$ which implies that

$$v(s)B(s) - k(s)v(s)f(s) \wedge T(s) + \tau(s)v(s)f(s) \wedge B(s) = \alpha f(s) \wedge N(s),$$

with v standing for velocity. Write $f(s) = a_1T(s) + a_2N(s) + a_3B(s)$ and obtain the equalities

$$\begin{aligned} k(s)v(s)a_3 + \tau(s)v(s)a_1 &= 0 \\ v(s)(1 + k(s)a_2) &= \alpha a_1 \\ \tau(s)v(s)a_2 &= -\alpha a_3. \end{aligned}$$

It follows that $\tau(s) = 0$ and from

$$v(s)B(s) - k(s)v(s)f(s) \wedge T(s) = \alpha f(s) \wedge N(s)$$

we can obtain $[f(s), T(s), N(s)] = 0$, where $[\ldots]$ stands for the triple scalar product, scalar multiplying both members by $N(s)$. Therefore the osculating plane Π_s passes through $(0,0,0)$.

3. CURVES ON A CYLINDER

Suppose now that $f(R) \subset C$, where C is the right cylinder given by $x^2 + y^2 = 1$.

Proposition 1. *Let $T(s) = (0,0,\pm 1)$. Then f_t and f_n are not singular at s but f_b is singular at s.*

Proof. If $f(R) \subset C$ then

$$[f(s), T(s), N(s)] = k_g(s)k(s)^{-1} + f_3(s)(T_1(s)N_2(s) - N_1(s)T_2(s)),$$

where $k_g(s)$ is the geodesic curvature of f at s. Since $T(s) = (0,0,\pm 1)$ the normal curvature $k_n(s)$ is zero and $k(s)^2 = k_g(s)^2 \neq 0$. It follows that $[f(s), T(s), N(s)] \neq 0$. On the other hand

$$[f(s), T(s), B(s)] = -k(s)^{-1}k_n(s) - N_3(s)f_3(s).$$

Since $T(s) = (0,0,\pm 1)$ the normal curvature and $N_3(s)$ are 0. Therefore $[f(s), T(s), B(s)] = 0$. Also $\tau(s) = 0$ as a direct calculation shows if $f(t)$ is written as $(cos\theta(t), sin\theta(t), F(t))$.

If $s \in R$ is such that $T(s) \neq (0,0,\pm 1)$ then

$$| T_1(s)N_2(s) - N_1(s)T_2(s) | = k(s)^{-1}A(s),$$

where $A(s) = \| (T_1(s), T_2(s), 0) \|^3$. Therefore if $[f(s),\ T(s),\ N(s)] = 0$ then $f_3(s) = \pm k_g(s)A(s)^{-1}$ and we have

Proposition 2. *Let $A \in R$ be such that, for $s \in R$, $| k_g(s)A(s)^{-1} | \leq A$ and $| f_3(s) | \leq A$. Then, for $| B | > 2A$, a translation of f through $(0,0,B)$ yields a curve h such that h_t and h_n, if defined, are immersions.*

Proof. Under the above conditions $| h_3(s) | > A, s \in R$.

We have no analogue of Proposition 2 when we consider $B(s)$ instead of $T(s)$ or $N(s)$. The curve given by $(cos\ s, sin\ s, cos\ s + c)$, $|c| > 2$, is a counterexample, for instance.

In the particular case of f being the graph of some smooth map $h : S^1 \to R$ the conditions of Proposition 2 become $\frac{1}{4\pi^2} \mid H''(s) \mid \leq A$, $| H(s) | \leq A$, with $H(s) = h(e^{2\pi i s})$.

REFERENCES

[1] F.J. CRAVEIRO DE CARVALHO, S.A. ROBERTSON, *Space curves and their duals*, Portugaliae Math. 54 (1997) 95-100.

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