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OPERATORS OF SOLUTION FOR CONVOLUTION EQUATIONS[1]

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Abstract. *We prove that the existence of a solution operator for a convolution operator from the space of ultradifferentiable functions to the corresponding space of ultradistributions is equivalent to the existence of a continuous solution operator in the space of functions. Our results are in the spirit of a classical characterization of the surjectivity of convolution operators due to Hörmander. The behaviour of a fixed convolution operator in different classes of ultradifferentiable functions of Beurling type concerning the existence of a continuous linear right inverse is also considered.*

1 Introduction

In the early fifties L. Schwartz posed the problem to characterize when a partial differential operator $P(D)$ admits a continuous linear right inverse, that is, under which conditions a continuous linear map $R : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega)$ or $R : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ exists such that $P(D) \circ R$ equals the identity. After several partial results, the problem was completely solved in 1990 by Meise, Taylor and Vogt [16]. They characterized this property by giving many equivalent conditions. In particular, they proved that the existence of a continuous linear right inverse for partial linear differential operators with constant coefficients is equivalent to the existence of fundamental solutions with prescribed large lacunas in their support. In [17] the same problem was completely solved for non-quasianalytic classes of Beurling and Roumieu type. The splitting of differential complexes was considered in [10, 18]. Though the surjectivity of convolution operators between spaces of C^∞-functions or distributions on open sets of $\mathbb{R}^N$ was characterized by Ehrenpreis [11] and Hörmander [13], the existence of a continuous linear right inverse for such operators was attacked for the first time only in 1987 by Meise and Vogt [15] in dimension 1.

The purpose of this article is twofold. First we extend the results of [15, 16, 17] for convolution operators on spaces of ultradifferentiable functions on arbitrary open sets. Second we show in Theorem 2 that if $T_\mu : \mathcal{E}_{(\omega)}(X) \to \mathcal{E}_{(\omega)}(Y)$ is a convolution operator between spaces of non-quasianalytic functions of Beurling type defined on open sets of $\mathbb{R}^N$ such that $X = Y - \operatorname{supp}\mu$, which admits a continuous linear solution operator $R : \mathcal{E}_{(\omega)}(Y) \to \mathcal{D}'_{(\omega)}(X)$, then T_μ itself admits a continuous linear right inverse. The case of functions of Roumieu type is also considered in Theorem 4. These results are inspired by the classical result of Hörmander [13], which states that a convolution equation $\mu * f = g$ which can be solved for

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every $g \in C^\infty$ with a distribution solution $f \in \mathcal{D}'$, can be also solved in the class of C^∞-functions. Extensions of this last statement for non-quasianalytic classes have been recently given in Bonet, Galbis and Meise [6]. Our last section includes some examples.

This paper complements the articles of Meise, Taylor and Vogt [16, 17]. The methods of these two articles and [6] had a strong influence in ours. On the other hand, the results of [17], which are valid for linear partial differential operators with constant coefficients, are extended here for arbitrary convolution operators. Sohr in [20] was the first one who observed that the techniques of [16, 17] could be used for convolution operators $T_\mu : C^\infty(X) \to C^\infty(Y)$ if $X = Y - \mathrm{supp}\mu$. Condition (e) in Theorem 2 and in Theorem 4 appears here for the first time.

2 Preliminaries

In this preliminary section we recall the definition of the standard spaces of ultradifferentiable functions of Beurling type, convolution operators on these spaces and most of the notation which will be used in the sequel. The definitions are taken from Braun, Meise and Taylor [8]. Compare also with Beurling [2] and Björck [3].

Definition. Let $\omega : \mathbb{R} \to [0,\infty[$ be a continuous even function which is increasing on $[0,\infty[$ and satisfies $\omega(0) = 0$ and $\omega(1) > 0$. It is called a weight function if it satisfies the following conditions:
(α) $\omega(2t) \le K(1+\omega(t))$ for all $t \in \mathbb{R}$
(β) $\int_{-\infty}^{\infty} \frac{\omega(t)}{1+t^2} dt < \infty$
(γ) $\log(1+t^2) = o(\omega(t))$ as t tends to ∞
(δ) $\varphi : t \to \omega(e^t)$ is convex on $\mathbb{R}$.

For a weight function ω we define $\tilde{\omega} : \mathbb{C}^N \to [0,\infty[$ by $\tilde{\omega}(z) = \omega(|z|)$ and again call this function ω, by abuse of notation. The Young conjugate of φ is defined by $\varphi^*(x) = \sup_{y>0}\{xy - \varphi(y)\}$.

Examples. It is easy to check that the following functions ω (possibly after a suitable change on $[-A,A]$ for some $A > 0$) are weight functions:
(1) $\omega(t) = t^\beta$, $0 < \beta < 1$,
(2) $\omega(t) = (\log(1+t))^\beta$, $\beta > 1$,
(3) $\omega(t) = t(\log(e+t))^{-\beta}$, $\beta > 1$.
In case (1) we get the classical Gevrey-classes of exponent $d := \frac{1}{\beta}$.

Definition. Let ω be a weight function.
(a) For a compact set $K \subset \mathbb{R}^N$ and $\ell > 0$ we set

$$\mathcal{E}_\omega^\ell(K) = \{f \in C^\infty(K) : \| f \|_{K,\ell} < \infty \},$$

where

$$\| f \|_{K,\ell} := \sup_{x\in K} \sup_\alpha | f^{(\alpha)}(x) | \exp(-\ell\varphi^*(\frac{|\alpha|}{\ell})).$$

(b) For an open set $\Omega \subset \mathbb{R}^N$ we define

$$\mathcal{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega) : \| f \|_{K,\ell} < \infty \forall K \subset\subset \Omega, \forall \ell \in \mathbb{N}\}$$

and

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \forall K \subset\subset \Omega, \exists \ell \in \mathbb{N} \ \| f \|_{K,\ell} < \infty\}.$$

The elements of $\mathcal{E}_{(\omega)}(\Omega)$ (resp. $\mathcal{E}_{\{\omega\}}(\Omega)$) are called ω-ultradifferentiable functions of Beurling (resp. Roumieu) type on Ω. We write $\mathcal{E}_*(\Omega)$, where $*$ can be either (ω) or $\{\omega\}$ at all occuring places.

(c) For a compact set K in $\mathbb{R}^N$ we set

$$\mathcal{D}_*(K) = \{f \in \mathcal{E}_*(\mathbb{R}^N) : \mathrm{supp}(f) \subset K\}$$

endowed with the induced topology. For an open set $\Omega \subset \mathbb{R}^N$ and a fundamental sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of Ω we define $\mathcal{D}_*(\Omega) := \mathrm{ind}_{j\rightarrow}\mathcal{D}_*(K_j)$.

The dual $\mathcal{D}'_*$ of $\mathcal{D}_*$ is endowed with its strong topology. The elements of $\mathcal{D}'_{(\omega)}(\Omega)$ (resp. $\mathcal{D}'_{\{\omega\}}(\Omega)$) are called ω-ultradistributions of Beurling (resp. Roumieu) type on Ω.

(d) For an open set $\Omega \subset \mathbb{R}^N$, an open subset U of Ω and $X(\Omega)$, being one of the spaces introduced above, we set $X(\Omega, U) := \{f \in X(\Omega) : f|_U \equiv 0\}$.

The classical case $\mathcal{E}_{(\omega)} = C^\infty$ is formally not a subcase of what we present here since we assume condition (γ). However, all our results also hold in this case after obvious modifications.

Convolution operators. Given $\mu \in \mathcal{E}'_*(\mathbb{R}^N)$ and two open sets X and Y in $\mathbb{R}^N$ with $Y - \mathrm{supp}\mu \subset X$ we define

$$T_\mu : \mathcal{E}_*(X) \to \mathcal{E}_*(Y) \text{ by } T_\mu(f) = \mu * f, \ \mu * f(x) = \langle \mu_y, f(x-y)\rangle.$$

T_μ is a continuous linear operator on the space $\mathcal{E}_*(X)$, and μ induces also a continuous linear operator $S_\mu : \mathcal{D}'_*(X) \to \mathcal{D}'_*(Y)$ where

$$S_\mu(\nu)(\varphi) = \langle \mu * \nu, \varphi\rangle = \langle \nu, \check{\mu} * \varphi\rangle \text{ and } \langle \check{\mu}, \varphi\rangle = \langle \mu, \check{\varphi}\rangle \ , \check{\varphi}(x) = \varphi(-x)$$

for $\nu \in \mathcal{D}'_*(X)$, and $\varphi \in \mathcal{D}_*(Y)$.

Note that for $\mu = \sum_{|\alpha|\le m} a_\alpha \frac{1}{i^{|\alpha|}} \delta_0^\alpha$ the corresponding convolution operator is

$$P(D) = \sum_{|\alpha|\le m} a_\alpha \frac{1}{i^{|\alpha|}} (\frac{\partial}{\partial x})^\alpha,$$

while for $\mu = \varphi \in \mathcal{D}_*(\mathbb{R}^N)$ we get

$$T_\mu(f) = \int f(y)\varphi(x-y)dy.$$

In the rest of this article we assume $X = Y - \mathrm{supp}\mu$. For an open subset G of Y, if we denote by H the set $G - \mathrm{supp}\mu$, we define

$$\mathcal{N}(H) := \{\nu \in \mathcal{D}'_*(H) \ : \ S_\mu(\nu) = 0 \text{ on } G\}.$$

We observe that $S_\mu(u)|_G = S_\mu(u|_H)$ if $u \in \mathcal{D}'_*(X)$.

Right inverses. For locally convex spaces E and F we let $L(E,F) = \{A : E \to F : A$ is continuous and linear$\}$. A map $A \in L(E,F)$ is said to admit a right inverse, if there exists $R \in L(F,E)$ so that $A \circ R = \mathrm{id}|_F$.

Definition. (a) An ultradistribution $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ is called *slowly decreasing for* (ω) if there exists $C > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| > C$ there is $\zeta \in \mathbb{C}^N$ with $|x - \zeta| \leq C\omega(x)$ and $|\hat{\mu}(\zeta)| \geq \exp(-C|\operatorname{Im}\zeta| - C\omega(\zeta))$.

(b) An ultradistribution $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^N)$ is called *slowly decreasing for* $\{\omega\}$ if for each $m \in \mathbb{N}$ there exists $R > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| > R$ there exists $\zeta \in \mathbb{C}^N$ satisfying $|x - \zeta| \leq \frac{1}{m}\omega(x)$ such that $|\hat{\mu}(\zeta)| \geq \exp(-\frac{1}{m}\omega(\zeta))$.

Remark 1 *If $\mu \in \mathcal{E}'_*(\mathbb{R}^N)$ is slowly decreasing for $*$, it follows from [6, 2.9, 3.4] that the convolution operator $S_{\check{\mu}}$ and consequently S_μ is surjective on $\mathcal{D}'_*(\mathbb{R}^N)$.*

3 Results

In this section we show that the existence of a right inverse for a convolution operator is equivalent to a (a priori) weaker condition. Throughout this section, ω denotes a weight function and $\mu \neq 0$ an element of $\mathcal{E}'_*$. For a non-empty open set Y in $\mathbb{R}^N$ and $\varepsilon > 0$ we put

$$Y_\varepsilon := \{x \in Y : |x| < \frac{1}{\varepsilon} \text{ and } \operatorname{dist}(x, \partial Y) > \varepsilon\}.$$

We define $X := Y - \operatorname{supp}\mu$. For $\varepsilon > 0$ we set $X(\varepsilon) := Y_\varepsilon - \operatorname{supp}\mu$. We assume in the rest of this section without mentioning explicitely that $\varepsilon < \varepsilon_0$ and Y_{ε_0} is non-empty.

Theorem 2 *Let $\mu \neq 0$ be an element of $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$. Let Y be a non-empty open subset of $\mathbb{R}^N$. The following conditions are equivalent:*
(a) $S_\mu : \mathcal{D}'_{(\omega)}(X) \to \mathcal{D}'_{(\omega)}(Y)$ admits a continuous linear right inverse,
(b) μ is (ω)-slowly decreasing and there exists an increasing sequence $(X_k)_{k\in\mathbb{N}}$ of relatively compact open sets of X, with $\bigcup_{k\in\mathbb{N}} \overline{X_k} = X$ such that for every $k \in \mathbb{N}$ and every $u \in \mathcal{N}(X_{k+1})$ there exists $v \in \mathcal{N}(X)$ with $v|_{X_k} = u|_{X_k}$,
(c) μ is (ω)-slowly decreasing and for every $\varepsilon > 0$ there exists δ_0 so that for all $0 < \sigma < \eta < \delta < \delta_0$ and each $\xi \in Y_\eta \setminus \overline{Y_\delta}$ there exists $E_\xi \in \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ such that

(i) $\operatorname{supp} E_\xi \subset (\mathbb{R}^N \setminus X(\varepsilon)) - \xi$

(ii) $S_\mu E_\xi = \delta + T_\xi$ *with* $\operatorname{supp} T_\xi \subset (\mathbb{R}^N \setminus \overline{Y_\sigma}) - \xi$,

(d) $T_\mu : \mathcal{E}_{(\omega)}(X) \to \mathcal{E}_{(\omega)}(Y)$ admits a continuous linear right inverse,
(e) there is a continuous linear operator $R : \mathcal{E}_{(\omega)}(Y) \to \mathcal{D}'_{(\omega)}(X)$ such that $S_\mu R f = f$ for every $f \in \mathcal{E}_{(\omega)}(Y)$.

Proof. $(a) \Rightarrow (b)$: Since S_μ is surjective we can apply [6, 2.9] to get that μ is (ω)-slowly decreasing. Let R denote a right inverse for S_μ which exists by hypothesis. Given $\varepsilon > 0$, let B be a total bounded set in $\mathcal{D}_{(\omega)}(\overline{X(\varepsilon)})$. The continuity of R gives $0 < \delta_0 < \varepsilon$ and a bounded set D in $\mathcal{D}_{(\omega)}(\overline{Y_{\delta_0}})$ such that

$$\sup_{\varphi\in B} |\langle Rf, \varphi\rangle| \leq C \sup_{\psi\in D} |\langle f, \psi\rangle|$$

for some constant $C > 0$. Consequently, if $0 < \delta < \delta_0$ and $f \in \mathcal{D}'_{(\omega)}(Y, Y_\delta)$, $Rf \in \mathcal{D}'_{(\omega)}(X, X(\varepsilon))$. Given $0 < \eta < \delta < \delta_0$ and $u \in \mathcal{N}(X(\eta))$. We take $\varphi \in \mathcal{D}_{(\omega)}(X(\eta))$, $\varphi \equiv 1$ on a neighbourhood of X_δ. Hence $\varphi u \in \mathcal{D}'_{(\omega)}(X)$, $S_\mu(\varphi u)|_{Y_\delta} = S_\mu(u|_{X(\delta)}) = 0$. Therefore $S_\mu(\varphi u) \in \mathcal{D}'_{(\omega)}(Y, Y_\delta)$ and $RS_\mu(\varphi u) \in \mathcal{D}'_{(\omega)}(X, X(\varepsilon))$. For $v = \varphi u - RS_\mu(\varphi u)$, we obtain $v|_{X(\varepsilon)} = u|_{X(\varepsilon)}$ and $S_\mu v = 0$. An induction argument gives the conclusion.

$(b) \Rightarrow (c)$ We may adapt the proof of [17, 2.6].

$(a) \Rightarrow (e)$ and $(d) \Rightarrow (e)$ are obvious.

$(e) \Rightarrow (c)$ The first step is to show that condition (e) implies

$$\begin{aligned}&\forall\varepsilon > 0 \exists 0 < \delta < \varepsilon \forall 0 < \eta < \delta \exists \ell \in \mathbb{N} \forall f \in \mathcal{E}^\ell_{(\omega)}(Y, Y_\delta)\\ &\exists g \in \mathcal{D}'_{(\omega)}(X(\eta), X(\varepsilon)) \text{ with } S_\mu g = f|_{Y_\eta} \text{ in } \mathcal{D}'_{(\omega)}(\mathrm{Y}_\eta).\ (\text{cf. } [17, 2.5\ (*)])\end{aligned} \tag{1}$$

Given $\varepsilon > 0$, we select a total bounded set D in $\mathcal{D}_{(\omega)}(\overline{X(\varepsilon)})$. Since $R : \mathcal{E}_{(\omega)}(Y) \to \mathcal{D}'_{(\omega)}(X)$ is continuous, there is $0 < \delta_0 < \varepsilon$ such that, for some $C > 0$, $m \in \mathbb{N}_0$,

$$\sup_{\varphi\in D} |\langle Rf, \varphi\rangle| \leq C \| f \|_{\overline{Y_{\delta_0}}, m}. \tag{2}$$

We take $0 < \eta < \delta < \delta_0$ and consider the bilinear map

$$B : \mathcal{E}_{(\omega)}(Y) \times \mathcal{D}_{(\omega)}(\overline{X(\eta)}) \longrightarrow \mathbb{K},$$

defined as $B(f, \varphi) = \langle Rf, \varphi\rangle$. The map B is separately continuous, hence continuous. Therefore we can find $0 < \sigma < \eta$, $\tilde{m}, \tilde{\ell} \in \mathbb{N}$, $C > 0$ such that

$$|\langle Rf, \varphi\rangle| \leq C \| \varphi \|_{\tilde{m}} \| f \|_{\overline{Y_\sigma}, \tilde{\ell}} \tag{3}$$

for every $f \in \mathcal{E}_{(\omega)}(Y)$ and $\varphi \in \mathcal{D}_{(\omega)}(\overline{X(\eta)})$.
Note that, as in the proof of [17, 2.5], there are $\ell \in \mathbb{N}$ and $D > 0$ such that, for all $p \in \mathbb{N}_0$,

$$\exp(-\tilde{\ell}\varphi^*(\frac{p}{\tilde{\ell}})) \leq D \exp(-\ell\varphi^*(\frac{p+1}{\ell})).$$

To show that $\ell \in \mathbb{N}$ just selected satisfies the statement, we fix $f \in \mathcal{E}^\ell_{(\omega)}(Y, Y_\delta)$ and choose $\psi \in \mathcal{D}_{(\omega)}(Y)$ with $\psi \equiv 1$ on a neighbourhood of $\overline{Y_\sigma}$. We take $\rho \in \mathcal{D}_{(\omega)}(B_1(0))$ with $\int \rho(x)dx = 1$, we let $\rho_t(x) = t^{-n}\rho(\frac{x}{t})$ for $t > 0$ and define $f_t := \psi(f * \rho_t)$. For $t > 0$ small enough $f_t \in \mathcal{E}_{(\omega)}(Y, Y_{\delta_0})$. Moreover, $\| f_t - f \|_{\overline{Y_\sigma}, \tilde{\ell}}$ converges to 0 as $t \downarrow 0$. Therefore $(f_t)_{t>0}$ is a Cauchy net with respect to $\| \cdot \|_{\overline{Y_\sigma}, \tilde{\ell}}$. By (3), we conclude that $(Rf_t)_{t>0}$ is a Cauchy net in $\mathcal{D}_{(\omega)}(X(\eta))'_b$, and there is $g \in \mathcal{D}_{(\omega)}(X(\eta))'$ with $g = \mathcal{D}_{(\omega)}(X(\eta))'_b - \lim_{t\downarrow 0} Rf_t$. Moreover, since $f_t|_{Y_{\delta_0}} \equiv 0$ and D is a total bounded subset of $\mathcal{D}_{(\omega)}(X(\varepsilon))$, we can apply (2) to conclude

$g|_{X(\varepsilon)} \equiv 0$.
Finally, since $Y_\sigma \supset Y_\eta$, we have, for $\varphi \in \mathcal{D}_{(\omega)}(Y_\eta)$,

$$\langle S_\mu g, \varphi\rangle = \langle g, \check{\mu} * \varphi\rangle = \lim_{t\downarrow 0}\langle Rf_t, \check{\mu} * \varphi\rangle = \langle S_\mu Rf_t, \varphi\rangle =$$

$$\lim_{t\downarrow 0}\langle f_t, \varphi\rangle = \langle f, \varphi\rangle.$$

This completes the proof of (1).

Now, the proof of [17, 2.6] can be adapted to show that condition (1) implies

$$\forall \varepsilon > 0\, \exists \delta_0 > 0\, \forall 0 < \sigma < \eta < \delta < \delta_0\, \forall \xi \in Y_\eta \setminus \overline{Y_\delta}\, \exists E_\xi \in \mathcal{D}'_{(\omega)}(\mathbb{R}^N) \qquad (4)$$

such that
(i) $\operatorname{supp} E_\xi \subset (\mathbb{R}^N \setminus X(\varepsilon)) - \xi$
(ii) $S_\mu E_\xi = \delta + T_\xi$ with $\operatorname{supp} T_\xi \subset (\mathbb{R}^N \setminus \overline{Y_\sigma}) - \xi$.

To see this, one only has to observe that convolution operators S_μ commute with ultradifferential operators. Finally, observe that (e) implies $S_\mu \mathcal{D}'_{(\omega)}(X) \supset \mathcal{E}_{(\omega)}(Y)$. By [6, 2.5], μ is (ω)-slowly decreasing.

$(c) \Rightarrow (d)$ and $(c) \Rightarrow (a)$. We take $0 < \varepsilon_1 < 2^{-1}$ with $Y_{\varepsilon_1} \neq \emptyset$. If $0 < \varepsilon_k < \ldots < \varepsilon_1$ are already selected, we apply the assumption for $\varepsilon = \varepsilon_k$ to find δ_0 and select $0 < \varepsilon_{k+1} < \min(\delta_0, \frac{1}{k+1})$. We let $Y_k := Y_{\varepsilon_{k+1}}$, $k \in \mathbb{N}$, $X_k := Y_k - \operatorname{supp}\mu$ and $X_0 = Y_0 = X_{-1} = Y_{-1} = X_{-2} = Y_{-3} = \emptyset$.
Since μ is (ω)-slowly decreasing we apply [6, 2.5] to find a fundamental solution $E \in \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ with $S_\mu E = \delta$. At this point, proceeding as in the proof of [17, 2.4], we can construct, for each $k \in \mathbb{N}_0$, a continuous linear map

$$A_k : \mathcal{D}'_{(\omega)}(Y, Y_k) \to \mathcal{D}'_{(\omega)}(X, X_{k-2})$$

such that $S_\mu A_k(f)|_{Y_{k+1}} = f|_{Y_{k+1}}$ for all $f \in \mathcal{D}'_{(\omega)}(Y, Y_k)$. It also follows from the construction of the maps A_k that $A_k(\mathcal{E}'_{(\omega)}(Y, Y_k)) \subset \mathcal{E}'_{(\omega)}(X, X_{k-2})$. We put $R_0 = A_0$ and $R_{k+1}(f) = R_k(f) + A_{k+1}(f - \mu * R_k(f))$. Therefore R_k is a continuous linear map from $\mathcal{D}'_{(\omega)}(Y)$ into $\mathcal{D}'_{(\omega)}(X)$ and $(\mu * R_k f)|_{Y_{k+1}} = f|_{Y_{k+1}}$ for all $k \in \mathbb{N}_0$. Consequently $A_{k+1}(f - \mu * R_k)$ is identically 0 on X_{k-1}, that is $R_{k+1}(f)$ coincides with $R_k(f)$ on X_{k-1}. Hence $R := \lim_{k\to\infty} R_k$ is well-defined and $S_\mu \circ R = id$. This shows $(c) \Rightarrow (a)$. The same arguments also give $(c) \Rightarrow (d)$. □

The proof of Theorem 2 should be compared with [17] and [20]. Sohr [20], in the frame of distributions, observed that the condition $X = Y - \operatorname{supp}\mu$ permitted to extend many arguments of [16, 17] to convolution operators.

Remark 3 *By the results of Meise and Vogt [15, 3.9], the conditions of Theorem 2 for $N = 1$ and $X = Y = \mathbb{R}$ are also equivalent to the following condition: μ is (ω)-slowly decreasing and*

$$\sup\{\frac{|\operatorname{Im} a|}{1 + \omega(a)}, a \in \mathbb{C}, \hat{\mu}(a) = 0\} < \infty.$$

Now we consider convolution operators on spaces of ultradifferentiable functions of Roumieu type.

Theorem 4 *Let $\mu \neq 0$ be an element of $\mathcal{E}'_{\{\omega\}}(\mathbb{R}^N)$. Let Y be a non-empty open subset of $\mathbb{R}^N$ and let $X := Y - \operatorname{supp}\mu$. The following conditions are equivalent:*
(a) $S_\mu : \mathcal{D}'_{\{\omega\}}(X) \to \mathcal{D}'_{\{\omega\}}(Y)$ admits a continuous linear right inverse.
(b) μ is $\{\omega\}$-slowly decreasing and there exists an increasing sequence $(X_k)_{k\in\mathbb{N}}$ of relatively compact open sets of X, with $\bigcup_{k\in\mathbb{N}} \overline{X_k} = X$ such that for every $k \in \mathbb{N}$ and every $u \in \mathcal{N}(X_{k+1})$ there exists $v \in \mathcal{N}(X)$ with $v|_{X_k} = u|_{X_k}$,
(c) μ is $\{\omega\}$-slowly decreasing and for every $\varepsilon > 0$ there exists δ_0 so that for all $0 < \sigma < \eta < \delta < \delta_0$ and each $\xi \in Y_\eta \setminus \overline{Y_\delta}$ there exists $E_\xi \in \mathcal{D}'_{\{\omega\}}(\mathbb{R}^N)$ such that

(*i*) $\operatorname{supp} E_\xi \subset (\mathbb{R}^N \setminus X(\varepsilon)) - \xi$

(*ii*) $S_\mu E_\xi = \delta + T_\xi$ *with* $\operatorname{supp} T_\xi \subset (\mathbb{R}^N \setminus \overline{Y_\sigma}) - \xi$,

(d) $T_\mu : \mathcal{E}_{\{\omega\}}(X) \to \mathcal{E}_{\{\omega\}}(Y)$ admits a continuous linear right inverse.
(e) There is a continuous linear operator $R : \mathcal{E}_{\{\omega\}}(Y) \to \mathcal{D}'_{\{\omega\}}(X)$ such that $S_\mu R f = f$ for every $f \in \mathcal{E}_{\{\omega\}}(Y)$.

Proof. $(a) \Rightarrow (b)$: Let $R : \mathcal{D}'_{\{\omega\}}(Y) \to \mathcal{D}'_{\{\omega\}}(X)$ be a continuous linear right inverse for S_μ. Given $\varepsilon > 0$ the set

$$B := \{\varphi \in \mathcal{D}_{\{\omega\}}(\overline{X(\varepsilon)}) : \| \varphi \|_{X(\varepsilon),1} \leq 1\}$$

is bounded in $\mathcal{D}_{\{\omega\}}(\overline{X(\varepsilon)})$, hence in $\mathcal{D}_{\{\omega\}}(X)$. Therefore

$$q_B : \mathcal{D}_{\{\omega\}}(X) \to \mathbb{R},\ q_B(\nu) := \sup_{\varphi\in B} | \langle \nu, \varphi \rangle |$$

is a continuous seminorm on $\mathcal{D}'_{\{\omega\}}(X)$. By the continuity of R there exists a bounded set C in $\mathcal{D}_{\{\omega\}}(Y)$ and $M > 0$ so that

$$q_B(R\nu) \leq M q_C(\nu)$$

for all $\nu \in \mathcal{D}'_{\{\omega\}}(Y)$. Without loss of generality, since $\mathcal{D}'_{\{\omega\}}(Y)$ is a (DFS)-space, we may assume that

$$C = \{\varphi \in \mathcal{D}_{\{\omega\}}(\overline{Y_{\delta_0}}) : \| \varphi \|_{\overline{Y_{\delta_0}},1/m} \leq 1\}$$

for some $m \in \mathbb{N}$ and some $0 < \delta_0 < \varepsilon$. Proceeding as in [17, 3.4,(1) $\Rightarrow$ (2)], for each $0 < \delta < \delta_0$, and each $f \in \mathcal{D}'_{\{\omega\}}(Y, Y_\delta)$ we have that $Rf \in \mathcal{D}'_{\{\omega\}}(X, X(\varepsilon))$. Now, the same argument as in Theorem 2, $(a) \Rightarrow (b)$ finishes the proof.

$(b) \Rightarrow (c)$, $(c) \Rightarrow (a)$, and $(c) \Rightarrow (d)$: This can be shown as in Theorem 2.

$(a) \Rightarrow (e)$ and $(d) \Rightarrow (e)$: are obvious.

$(e) \Rightarrow (c)$: The first step is to show that (e) implies

$$\begin{aligned}&\forall \varepsilon > 0 \exists 0 < \delta < \varepsilon \forall 0 < \eta < \delta\ \exists \sigma = o(\omega) \text{ as } t \to \infty \text{ such that} \\ &\forall f \in \mathcal{E}_{(\sigma)}(Y, Y_\delta)\ \exists g \in \mathcal{D}'_{\{\omega\}}(X(\eta), X(\varepsilon)) \text{ with } S_\mu g = f|_{Y_\eta} \text{ in } \mathcal{D}'_{\{\omega\}}(Y_\eta).\end{aligned} \quad (5)$$

Given $\varepsilon > 0$, the set $B := \{\varphi \in \mathcal{D}_{\{\omega\}}(\overline{X(\varepsilon)}) : \| \varphi \|_{X(\varepsilon),1} \leq 1\}$ is bounded in $\mathcal{D}_{\{\omega\}}(\overline{X_\varepsilon})$, hence bounded in $\mathcal{D}_{\{\omega\}}(X)$. Therefore

$$q_B : \mathcal{D}'_{\{\omega\}}(X) \to \mathbb{R},\ q_B(\upsilon) = \sup_{\varphi\in B} | \upsilon(\varphi) |$$

is a continuous seminorm on $\mathcal{D}'_{\{\omega\}}(X)$. By the continuity of R and the description of the continuous seminorms on $\mathcal{E}_{\{\omega\}}(Y)$ in [17, 3.3] we may find $0<\delta_0<\varepsilon$ and $\sigma=o(\omega)$ such that $q_B(Rh)\le M\,\|h\|_{Y_{\delta_0},\sigma}$ for every $h\in\mathcal{E}_{\{\omega\}}(Y)$. Moreover, proceeding as in [17, 3.4] we obtain that $Rh\in\mathcal{D}'_{\{\omega\}}(X,X(\varepsilon))$ whenever $h\in\mathcal{E}_{\{\omega\}}(Y,Y_{\delta_0})$. We take $0<\eta<\delta<\delta_0$ and we consider the bilinear map

$$B:\mathcal{E}_{\{\omega\}}(Y)\times\mathcal{D}_{\{\omega\}}(\overline{X(\eta)})\longrightarrow\mathbb{K},\ (h,\varphi)\to\langle Rh,\varphi\rangle.$$

The map B is separately continuous and, proceeding as in [6, 3.5], for every $m\in\mathbb{N}$ there are $\sigma_m=o(\omega)$, a compact subset K_m of Y and a positive constant C_m such that

$$|\langle Rh,\varphi\rangle|\le C_m\,\|h\|_{K_m,\sigma_m}|\varphi|_{\overline{X(\eta)},m}\,. \tag{6}$$

Now, using [8, 1.9], we find $\sigma=o(\omega)$ such that $\sigma_m=o(\sigma)$ for all m. Hence (6) implies the existence of D_m such that

$$|\langle Rh,\varphi\rangle|\le D_m\,\|h\|_{K_m,\sigma}|\varphi|_{\overline{X(\eta)},m}\,. \tag{7}$$

The inclusion $\mathcal{E}_{\{\omega\}}(Y)\hookrightarrow\mathcal{E}_{(\sigma)}(Y)$ is continuous and has dense range. Then, given $f\in\mathcal{E}_{(\sigma)}(Y,Y_\delta)$ we may find $(\psi_j)_{j\in\mathbb{N}}$ in $\mathcal{E}_{\{\omega\}}(Y)$ such that $f=\mathcal{E}_{(\sigma)}(Y)-\lim_{j\to\infty}\psi_j$. It is easy to see that $(\psi_j)_{j\in\mathbb{N}}$ can be taken in $\mathcal{E}_{\{\omega\}}(Y,Y_{\delta_0})$. Therefore, $(R\psi_j)_{j\in\mathbb{N}}\in\mathcal{D}'_{\{\omega\}}(X,X(\varepsilon))$ and, on account of (7), it is a Cauchy sequence in $\mathcal{D}'_{\{\omega\}}(X(\eta))$ endowed with the strong topology. Therefore, it converges to some $g\in\mathcal{D}'_{\{\omega\}}(X(\eta),X(\varepsilon))$ and $S_\mu g=f\,|_{Y_\eta}$. This proves (5). Now, the proof of [16, 2.1] can be adapted and we may proceed as in the proof $(e)\Rightarrow(c)$, in Theorem 2. □

The remark after Theorem 2 also holds in the Roumieu case by the results in [15, 4.4].

Corollary 5 *Let $\omega\le\sigma$ be two weights. Let $0\ne\mu\in\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ (resp. $0\ne\mu\in\mathcal{E}'_{\{\omega\}}(\mathbb{R}^N)$). Let Y be a non-empty set of $\mathbb{R}^N$ and let $X:=Y-\operatorname{supp}\mu$. If $T_\mu:\mathcal{E}_{(\omega)}(X)\to\mathcal{E}_{(\omega)}(Y)$ (resp. $T_\mu:\mathcal{E}_{\{\omega\}}(X)\to\mathcal{E}_{\{\omega\}}(Y)$) has a continuous linear right inverse, so does $T_\mu:\mathcal{E}_{(\sigma)}(X)\to\mathcal{E}_{(\sigma)}(Y)$ (resp. $T_\mu:\mathcal{E}_{\{\sigma\}}(X)\to\mathcal{E}_{\{\sigma\}}(Y)$).*

Observe that it can be proved as follows: Let $L:\mathcal{E}_{(\omega)}(Y)\to\mathcal{E}_{(\omega)}(X)$ be a continuous linear right inverse of $T_\mu:\mathcal{E}_{(\omega)}(X)\to\mathcal{E}_{(\omega)}(Y)$. Let $j:\mathcal{E}_{(\sigma)}(Y)\to\mathcal{E}_{(\omega)}(Y)$ and $J:\mathcal{D}'_{(\omega)}(X)\to\mathcal{D}'_{(\sigma)}(X)$ the canonical continuous injections. Then $R:=J\circ L\circ j:\mathcal{E}_{(\sigma)}(Y)\to\mathcal{D}'_{(\sigma)}(X)$ is continuous and $S_\mu Rf=f$ for all $f\in\mathcal{E}_{(\sigma)}(Y)$. The conclusion follows from Theorem 2 for the Beurling case. For the Roumieu case one proceeds in the same way and uses Theorem 4.

Corollary 5 follows also from the characterization using fundamental solutions. It served as a motivation to conjecture and prove Theorem 2. Corollary 5 must be compared with the examples provided below, in which condition (γ) of the weights is needed.

Our next result shows that the converse of Corollary 5 does not hold.

Proposition 6 *Let σ and ω be two weights such that $\omega = o(\sigma)$. There is $\mu \in \mathcal{E}_{(\omega)}(\mathbb{R})$ such that $T_\mu : \mathcal{E}_{(\sigma)}(\mathbb{R}) \to \mathcal{E}_{(\sigma)}(\mathbb{R})$ has a continuous linear right inverse, but $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$ does not.*

Proof. By [8, 1.7] there is $m : [0,\infty[\to [0,\infty[$ such that $\omega(t) = o(m(t))$ and $m(t) = o(\sigma(t))$ as $t \to \infty$. Proceeding by recurrence starting with $r_1 \geq 2$, we select a sequence $(r_j)_{j\in\mathbb{N}}$, $4r_j \leq r_{j+1}$, such that, for all $j \in \mathbb{N}$,
(i) $1 + \sigma(r_j) \geq m(r_j)$,
(ii) $j(\omega(r_j) + 1) \leq m(r_j)$
(iii) $(j+1)^2 \leq \inf_{t\geq r_j} \frac{\omega(t)}{\log t}$.
By (iii), if $n(t) = \mathrm{card}\{j \in \mathbb{N};\, r_j \leq t\}, t > 0$, we have $n(t)\log t = o(\omega(t))$ as $t \to \infty$. For each $j \in \mathbb{N}$ we select $z_j \in \mathbb{C}$ with $|\, z_j \,| = r_j$,$\mathrm{Im}\, z_j = m(r_j)$, and we set $f(z) := \prod_{j\in\mathbb{N}}(1 - \frac{z}{z_j})$, $z \in \mathbb{C}$. By [19, p. 325], f is an entire function and its zeros are the z_j's. Proceeding as in [9, 3.11],

$$\log |\, f(z) \,| \leq n(|\, z \,|) \log |\, z \,| + \log 2 + \frac{4}{9}, \quad z \in \mathbb{C}.$$

Therefore there is $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ with $\hat{\mu} = f$, $\mathrm{supp}\mu = \{0\}$, and T_μ is surjective on $\mathcal{E}_{(\omega)}(\mathbb{R})$, hence on $\mathcal{E}_{(\sigma)}(\mathbb{R})$ (cf [5]).

By our construction,

$$\sup\left\{ \frac{|\, \mathrm{Im}\, z_j \,|}{1 + \sigma(z_j)} : j \in \mathbb{N} \right\} \leq 1$$

and

$$\frac{|\, \mathrm{Im}\, z_j \,|}{1 + \omega(z_j)} > j \text{ for each } j \in \mathbb{N}.$$

According to the characterization given in [15, 3.9], T_μ has a continuous linear right inverse on $\mathcal{E}_{(\sigma)}(\mathbb{R})$ but not on $\mathcal{E}_{(\omega)}(\mathbb{R})$. □

Remark 7 *It would be interesting to know whether it is possible to characterize the existence of a right inverse for $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \to \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ in terms of a Phragmén-Lindelöf condition, similar to the one which appears in the work of Meise, Taylor and Vogt [16, 17].*

4 Examples

In this section it is proved that $*$-hypoelliptic convolution operators do not admit a right inverse, and that (ω)-hyperbolic operators have a global right inverse.

The following proposition is proved as in [16, 2.11]. We include the short proof for the sake of completeness.

Proposition 8 *Let $\mu \in \mathcal{E}'_*(\mathbb{R}^N)$, $N \geq 2$, X and Y be as in Theorem 2. If $\mathcal{N}(X) \subset \mathcal{E}_*(X)$, then $S_\mu : \mathcal{D}'_*(X) \to \mathcal{D}'_*(Y)$ does not admit a continuous linear right inverse.*

Proof. Let $(X_k)_{k\in\mathbb{N}}$ be as in Theorems 2, 4, (b), and consider the space $\mathcal{N}(X_k)\cap L^\infty(X_k)$, endowed with the sup-norm, which is a Banach space. By (b) in Theorem 2 for the Beurling case or Theorem 4 for the Roumieu case, the restriction map

$$\rho_k : \frac{\mathcal{N}(X)}{\mathcal{N}(X,X_{k-1})} \longrightarrow \frac{\mathcal{N}(X_k)\bigcap L^\infty(X_k)}{\mathcal{N}(X_k,X_{k-1})\bigcap L^\infty(X_k)}$$

is an isomorphism. Therefore, $\frac{\mathcal{N}(X)}{\mathcal{N}(X,X_{k-1})}$ being a nuclear and a Banach space, must be finite dimensional. However, the set

$$\{\exp(\langle\cdot,\xi\rangle) : \hat{\mu}(\xi)=0\}$$

is non-empty (see [14, 16.7]), not finite (since $N\geq 2$) and it is linearly independent in $\frac{\mathcal{N}(X)}{\mathcal{N}(X,X_{k-1})}$, a contradiction. □

Remark 9 *Linear partial differential operator satisfying $\mathcal{N}(X)\subset\mathcal{E}_{(\omega)}(X)$ are exactly (ω)-hypoelliptic operators, as it is proved in [3]. The same is true for surjective convolution operators on $C^\infty(\mathbb{R}^N)$ (see [14]) and for convolution operators on $\mathcal{E}_*(\mathbb{R}^N)$ admitting a fundamental solution [15, section 4], [4]. There are examples of non (ω)-hypoelliptic operators satisfying $\mathcal{N}(\mathbb{R}^N)\subset\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ (see [4]).*

To obtain positive examples, we recall that hyperbolic (resp. (ω)-hyperbolic) convolution operators were defined by Ehrenpreis in [12] and Abdullah in [1] in the following way. Let $0\neq\mu\in\mathcal{E}'(\mathbb{R}\times\mathbb{R}^N)$, $N\geq 0$ be given (resp. in $\mathcal{E}'_{(\omega)}(\mathbb{R}\times\mathbb{R}^N)$). The convolution operator T_μ is $((\omega)$-) hyperbolic in x if there exist two fundamental solutions E_+ and E_- in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^N)$ (resp. in $\mathcal{D}'_{(\omega)}(\mathbb{R}\times\mathbb{R}^N)$) such that, for some $a', a''>0$,

$$\operatorname{supp} E_+ \subset \{(x,y)\in\mathbb{R}\times\mathbb{R}^N : x\leq a' - \frac{\|y\|}{a''}\}$$

$$\operatorname{supp} E_- \subset \{(x,y)\in\mathbb{R}\times\mathbb{R}^N : x\geq -a' + \frac{\|y\|}{a''}\}.$$

Proceeding as in [16, 3.2] we get the following result

Proposition 10 *If a convolution operator is hyperbolic (resp. (ω)-hyperbolic) in x, then it admits a continuous linear right inverse in $C^\infty(\mathbb{R}\times\mathbb{R}^N)$ (resp. in $\mathcal{E}_{(\omega)}(\mathbb{R}\times\mathbb{R}^N)$).*

References

[1] S. Abdullah, *Hyperbolic operators in spaces of generalized distributions,* Rocky Mount. J. **3** (1986), 535-542.

[2] A. Beurling, *Quasi-analyticity and general distributions,* Lecture Notes 4 and 5, Amer. Math. Soc. Summer Institute, Stanford University, 1961.

[3] G. Björk, *Linear partial differential operators and generalized distributions*, Ark. Math. **6**, (1965) 351-407.

[4] J. Bonet, C. Fernández, R. Meise, *Characterization of ω-hypoelliptic convolution operators on ultradistributions,* Ann. Acad. Sci. Fenn. Math (to appear)

[5] J. Bonet, A. Galbis, *On the range of non-surjective convolution operators on Beurling spaces,* Glasgow Math. J. **38** (1996), 125-135.

[6] J. Bonet, A. Galbis, R. Meise, *On the range of convolution operators on non-quasianalytic ultradifferentiable functions,* Studia Math. **126** (1997), 171-198.

[7] R. W. Braun, *An extension of Komatsu's second structure theorem for ultradistributions,* J. Fac. Sci. Univ. Tokyo **40** (1993), 411-417.

[8] R. W. Braun, R. Meise, B. A. Taylor, *Ultradifferentiable functions and Fourier analysis,* Result. Math. **17** (1990), 206-237.

[9] R. W. Braun, R. Meise, D. Vogt, *Existence of fundamental solutions and surjectivity of convolution operators on classes of ultra-differentiable functions,* Proc. London Math. Soc. **61** (1990), 334-370.

[10] P. Domański, D. Vogt, *A splitting theorem for the space of smooth functions,* J. Func. Anal. **153** (1998), 203-248.

[11] L. Ehrenpreis, *Solution of some problems of division, Part IV. Invertible and elliptic operators,* Amer. J. Math. **82** (1960), 522-588.

[12] L. Ehrenpreis, *Solutions of some problems of division V. Hyperbolic operators,* Amer. J. Math. **84** (1962), 324-348.

[13] L. Hörmander, *On the range of convolution operators,* Annals of Math. **16** (1962), 148-170.

[14] L. Hörmander, *The Analysis of Linear Partial Differential Operators,* Vol. I, II. Springer-Verlag, Berlin, 1983.

[15] R. Meise, D. Vogt, *Characterization of convolution operators on spaces of C^∞-functions admitting a continuous linear right inverse,* Math. Ann. **279** (1987), 141-155.

[16] R. Meise, B. A. Taylor, D. Vogt, *Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse,* Ann. Inst. Fourier, Grenoble **40** (1990), 619-655.

[17] R. Meise, B. A. Taylor, D. Vogt, *Continuous linear right inverses for partial differential operators on non-quasianalytic classes and on ultradistributions,* Math. Nach. **180** (1996), 213-242.

[18] V. P. Palamodov, *A criterion for splitness of differential complexes with constant coefficients,* Geometrical and Algebraical Aspects in Several Complex Variables (C. A. Berenstein and D. Struppa, eds.), EditEl, 1991, pp. 265-291.

[19] W. Rudin, *Real and Complex Analysis,* McGraw-Hill, 1974.

[20] S. Sohr, *Stetige lineare Rechtsinverse für lineare partielle Differentialoperatoren mit konstanten Koeffizienten,* Diplomarbeit, Paderborn, September 1994.

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