ONE-PARAMETER SUBGROUPS AND MINIMAL SURFACES IN THE HEISENBERG GROUP[1]

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Abstract. *The purpose of this paper is to study the minimal surfaces of the Heisenberg group H_3 which are the product of two one-parameter subgroups. Minimal surfaces with harmonic Gauss map into the Grassmann bundle of H_3 are also characterized.*

1. INTRODUCTION

In a recent paper ([1]), Y-J. Dai, M. Shoji and H. Urakawa make new contributions to the theory of harmonic maps from a Riemannian manifold M into a homogeneous space N. In particular, they determine all those harmonic maps from the standard Euclidean space $\mathbb{R}^m$ into the 3-dimensional noncompact Lie groups $SE(2)$, $E(1,1)$, $SL(2,\mathbb{R})$, H_3 that are of the form

$$f(x_1, x_2, \cdots, x_m) = \exp(x_1X_1)\exp(x_2X_2)\cdots\exp(x_mX_m),$$

where $X_1, X_2, \cdots, X_m$ are tangent vectors at the identity element and the above groups are endowed with a left invariant metric (see [3]).

On the other hand, in a earlier report ([6]) devoted to the harmonic maps into Grassmann bundles, to stress on how much the nonconstant sectional curvature of the ambient space modifies the Ruh-Vilms type results [5] on the Gauss map, the second author considered the following example.

In the Heisenberg group H_3 of all real matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with the left-invariant metric

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2,$$

the surface $z = 0$ is at the same time minimal and the product of two one-parameter subgroups. But its Gauss map into the Grassmann bundle $G_2(TH_3)$ is neither harmonic nor vertically harmonic.

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This fact suggested us to begin a detailed study of the minimal surfaces of H_3 which are the product of two one-parameter subgroups.

In Section 2 we recall some results about Gauss maps into a Grassmann bundle, principally to introduce some formulas used later on.

Section 3 is devoted to the study of the minimal surfaces of H_3 which are the product of two one-parameter subgroups and to a characterization of the minimal surfaces with vertically harmonic or harmonic Gauss map. Such a characterization is related to the geometry of the one-parameter subgroups. For this reason, in Section 4, we compute the Frenet formulas for these curves.

2. GAUSS MAPS INTO GRASSMANN BUNDLES

Let M be an m-dimensional submanifold, isometrically immersed in an n-dimensional Riemannian manifold (N, g).

The Gauss map Γ of M into the Grassmann bundle $G_m(TN)$ of the m-planes of the tangent bundle TN, to each point $x \in M$ associates the subspace T_xM of T_xN.

The Grassmann bundle $G_m(TN)$ is the bundle on N associated to the principal bundle $O(N)$ of the orthonormal frames on N, with the Grassmannian $G_m(\mathbb{R}^n)$ of the m-planes of $\mathbb{R}^n$ as fibre; $G_m(TN)$ is endowed with a suitable Sasaki-like metric such that the projection $\pi : G_m(TN) \longrightarrow N$ is a Riemannian submersion with totally geodesic fibres (see [2] and [6] for details).

Any tangent vector field on $G_m(TN)$ decomposes into a vertical component, which is tangent to the fibres, and a horizontal component, orthogonal to the fibres. Hence the Gauss map $\Gamma : M \longrightarrow G_m(TN)$ is said to be harmonic, or vertically harmonic, if its tension field $\tau(\Gamma)$, or the vertical component of $\tau(\Gamma)$, vanishes.

Let $\{e_i, e_\alpha\}$ be a local orthonormal Darboux frame on M, where $i = 1, \cdots, m$; $\alpha = m+1, \cdots, n$, and let $h_{ij}^\alpha = g(h(e_i, e_j), e_\alpha) = g(\nabla_{e_i} e_j, e_\alpha)$ be the components of the second fundamental form. We denote by $H = \frac{1}{m} g^{ij} h_{ij}^\alpha e_\alpha$ the mean curvature vector field of M.

Standard computations (see [6], [7]) show that Γ is vertically harmonic if

$$m \nabla_k^\perp H^\alpha - R_{jkj\alpha}^N = 0, \tag{1}$$

where the covariant derivative and the components of the curvature tensor field of N are taken with respect to this frame; for example

$$R_{jkj\alpha}^N = \sum_{j=1}^m R^N(e_j, e_k, e_j, e_\alpha) = \sum_{j=1}^m g\left(\left(\nabla_{[e_j, e_k]} - \left[\nabla_{e_j}, \nabla_{e_k}\right]\right) e_j, e_\alpha\right). \tag{2}$$

The Gauss map Γ is harmonic if, in addition to (1), the following conditions are satisfied:

$$R_{\alpha kji}^N h_{kj}^\alpha = 0 \tag{3}$$

$$mH^{\alpha} + \lambda^2 R^N_{\beta kj\alpha} h^{\beta}_{kj} = 0, \tag{4}$$

where λ is a positive real constant fixing the metric of $G_m(TN)$ (see [7]).

If M is a minimal surface and N has dimension 3, conditions (1) take the expression

$$R^N_{1213} = 0 \qquad\qquad R^N_{2123} = 0. \tag{5}$$

Consequently (3) is automatically satisfied and (4) becomes

$$R^N_{3kj3} h_{kj} = 0. \tag{6}$$

3. MINIMAL SURFACES IN H_3 WHICH ARE THE PRODUCT OF TWO ONE-PARAMETER SUBGROUPS

We consider the Heisenberg group

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$$

with the group multiplication induced by the standard matrix product. The orthonormal basis

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of the tangent space at the identity, determines on H_3 a left-invariant metric

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2,$$

where

$$\omega^1 = dx, \qquad \omega^2 = dy, \qquad \omega^3 = dz - xdy$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad e_3 = \frac{\partial}{\partial z}.$$

The corresponding Lie brackets are

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = [e_2, e_3] = 0,$$

and the Levi-Civita connection forms are given by

$$\omega^2_1 = -\frac{1}{2}\omega^3, \qquad \omega^3_1 = -\frac{1}{2}\omega^2, \qquad \omega^3_2 = \frac{1}{2}\omega^1.$$

Then we have

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \qquad \nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = \frac{1}{2} e_3,$$
$$\nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -\frac{1}{2} e_2, \qquad \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2} e_1, \tag{7}$$

and therefore the nonvanishing components of the curvature tensor are

$$R_{1212} = -\frac{3}{4}, \qquad R_{1313} = \frac{1}{4}, \qquad R_{2323} = \frac{1}{4}.$$

Let P be an arbitrary point of the surface $S = \exp uX \exp vY$, where

$$X = aE_1 + bE_2 + cE_3, \qquad Y = \alpha E_1 + \beta E_2 + \gamma E_3$$

are two linearly independent vectors tangent to H_3 at the identity.

Then P has coordinates

$$P(u,v) = \begin{cases} x = ua + v\alpha, \\ y = ub + v\beta, \\ z = uc + v\gamma + \frac{1}{2}u^2 ab + \frac{1}{2}v^2\alpha\beta + uva\beta. \end{cases} \tag{8}$$

It is easy to see that S is a commutative subgroup of H_3 if and only if $L \equiv a\beta - \alpha b = 0$. Differentiating (8) with respect to u and to v we obtain

$$\begin{aligned} P_u &= a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + (c + abu + a\beta v)\frac{\partial}{\partial z} = ae_1 + be_2 + (c + vL)e_3 \\ P_v &= \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + (\gamma + a\beta u + \alpha\beta v)\frac{\partial}{\partial z} = \alpha e_1 + \beta e_2 + \gamma e_3. \end{aligned} \tag{9}$$

It follows that the components of the metric induced on S by that of H_3 are

$$\begin{aligned} g_{11} &= \|P_u\|^2 = a^2 + b^2 + (c + vL)^2, \\ g_{12} &= \langle P_u, P_v\rangle = a\alpha + b\beta + (c + vL)\gamma, \\ g_{22} &= \|P_v\|^2 = \alpha^2 + \beta^2 + \gamma^2, \end{aligned}$$

and that the vector field N normal to S is given by

$$N = (b\gamma - \beta(c + vL))e_1 - (a\gamma - \alpha(c + vL))e_2 + Le_3.$$

Now we can compute the components of the second fundamental form of S, that are:

$$\begin{cases} h_{11} = \langle \nabla_{P_u} P_u, \dfrac{N}{\|N\|} \rangle = \dfrac{(c+vL)\{\gamma(a^2+b^2)-(a\alpha+b\beta)(c+vL)\}}{\|N\|}, \\ h_{12} = \langle \nabla_{P_u} P_v, \dfrac{N}{\|N\|} \rangle = \dfrac{\{\gamma^2(a^2+b^2)-(\alpha^2+\beta^2)(c+vL)^2+L^2\}}{2\|N\|}, \\ h_{22} = \langle \nabla_{P_v} P_v, \dfrac{N}{\|N\|} \rangle = \dfrac{\{(a\alpha+b\beta)\gamma^2-\gamma(\alpha^2+\beta^2)(c+vL)\}}{\|N\|}. \end{cases} \tag{10}$$

It is not difficult to check that the mean curvature $H = \frac{1}{2} g^{ij} h_{ij}$ of S vanishes if and only if

$$\gamma L^2 = 0, \qquad (a\alpha + b\beta)L^2 = 0, \tag{11}$$

that is, when

(i) $L = 0$,

or

(ii) $L \neq 0, \gamma = 0, a\alpha + b\beta = 0$.

The example in the introduction is in Case (ii), for $a = 1, b = c = 0, \alpha = \gamma = 0, \beta = 1$.

Therefore, if X is fixed, the vector Y such that $\exp uX \exp vY$ is a minimal surface of H_3 verifies one of the following two conditions:

1. X and Y have proportional orthogonal projections on the plane $\{E_1, E_2\}$;
2. Y is a vector of the plane $\{E_1, E_2\}$ orthogonal to X.

Now we suppose that the Gauss map of the minimal surface $S = \exp uX \exp vY$ into the Grassmann bundle $G_2(TH_3)$ is vertically harmonic, and rewrite (5) in the form

$$R(P_u, P_v, P_u, N) = 0, \qquad R(P_v, P_u, P_v, N) = 0.$$

Since

$$R(P_v, P_u, P_v, N) = (c + vL)L(\alpha^2 + \beta^2) - L\gamma(a\alpha + b\beta),$$

we see that this component vanishes for every $v \in \mathbb{R}$ if and only if $L = 0$; in this case we also have

$$R(P_u, P_v, P_u, N) = 0.$$

For $a = b = 0$ (and α, β not both zero), one has $L = 0$ and the curve $\exp uX$ is a geodesic of H_3; in this case, for any vector Y, the Gauss map of the minimal surface $S = \exp uX \exp vY$ into $G_2(TH_3)$ is harmonic, i.e., condition (6) is also satisfied.
This fact is easily verified by using the orthonormal basis of H_3

$$\begin{cases} v_1 = e_3 \\ v_2 = \dfrac{1}{\sqrt{\alpha^2+\beta^2}}(\alpha e_1 + \beta e_2) \\ v_3 = \dfrac{1}{\sqrt{\alpha^2+\beta^2}}(-\beta e_1 + \alpha e_2). \end{cases}$$

Suppose again $L = 0$, but $(a,b) \neq (0,0)$. Making use of the orthonormal basis

$$\begin{cases} v_1 &= \dfrac{P_u}{\| P_u \|} = \dfrac{1}{\sqrt{a^2+b^2+c^2}}(ae_1 + be_2 + ce_3) \\ v_2 &= \dfrac{1}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}(-ace_1 - bce_2 + (a^2+b^2)e_3) \\ v_3 &= \dfrac{N}{\| N \|} = \dfrac{1}{\sqrt{a^2+b^2}}(be_1 - ae_2), \end{cases}$$

we obtain

$$\begin{cases} h(v_1,v_1) &= \dfrac{c\sqrt{a^2+b^2}}{a^2+b^2+c^2}v_3 \\ h(v_1,v_2) &= \dfrac{a^2+b^2-c^2}{2\left(a^2+b^2+c^2\right)}v_3, \\ h(v_2,v_2) &= -\dfrac{c\sqrt{a^2+b^2}}{a^2+b^2+c^2}v_3, \end{cases}$$

It follows that (6), that is, $R(v_3,v_i,v_j,v_3)h^3(v_i,v_j) = 0$, is satisfied if and only if

$$c\sqrt{a^2+b^2}(a^2+b^2-c^2) = 0.$$

Then we have two cases:

1. $c = 0$, and the curve $\exp uX$ is a geodesic of H_3, or
2. $a^2+b^2-c^2 = 0$; this condition is equivalent to the vanishing of the torsion of $\exp uX$ (see Section 4).

Hence we have the following

Theorem 1. *Let S be a surface of H_3 of the form*

$$S = \exp uX \exp vY$$

where $X = aE_1 + bE_2 + cE_3$ and $Y = \alpha E_1 + \beta E_2 + \gamma E_3$ are two linearly independent vectors tangent to H_3 at the identity and let Γ be the Gauss map of a surface S into the Grassmann bundle $G_2(TH_3)$. Then

1. *S is a minimal surface with Γ vertically harmonic if and only if* $\exp uX$ *and* $\exp vY$ *commute (that is, if and only if $L \equiv a\beta - \alpha b = 0$)*
2. *S is a minimal surface with Γ harmonic if and only if the following conditions are satisfied*
   (a) $\exp uX$ *and* $\exp vY$ *commute*
   (b) the one-parameter subgroup $\exp uX$ *(or* $\exp vY$*) either is a geodesic of H_3, or has torsion equal to zero at every point ($a^2+b^2-c^2 = 0$).*

4. ONE-PARAMETER SUBGROUPS OF H_3 AND FRENET FORMULAS

As observed above, there are properties of a minimal surface $S = \exp uX \exp vY$ of H_3 that are related to the geometry of the curve $\sigma(u) = \exp uX$. In this section we derive some results for the curve $\sigma(u)$, whose curvature and torsion are necessarily constant with respect to the transitive action of the Heisenberg group.

If we put $X = aE_1 + bE_2 + cE_3$, the point $P(u) = \exp uX$ has coordinates given by

$$P(u) = \begin{cases} x = ua, \\ y = ub, \\ z = uc + \dfrac{1}{2}u^2 ab, \end{cases}$$

and therefore

$$\frac{dP(u)}{du} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + (c + abu)\frac{\partial}{\partial z} = ae_1 + be_2 + ce_3,$$

where $\{e_1, e_2, e_3\}$ is as usual the left-invariant orthonormal frame associated with the basis $\{E_1, E_2, E_3\}$ of the tangent space of H_3 at the identity.

The unit vector field tangent to σ is given by

$$T = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \left(ae_1 + be_2 + ce_3\right), \tag{12}$$

and its covariant derivative is

$$\nabla_T T = \frac{c}{a^2 + b^2 + c^2} \left(be_1 - ae_2\right). \tag{13}$$

Thus σ is a geodesic of H_3, that is $\nabla_T T = 0$, if and only if either $c = 0$, or $a = b = 0$; if σ is not a geodesic, supposing $c > 0$, its curvature k is given by

$$k = \| \nabla_T T \| = \frac{c\sqrt{a^2 + b^2}}{a^2 + b^2 + c^2}, \tag{14}$$

and one obtains the first Frenet formula

$$\nabla_T T = kN, \tag{15}$$

where

$$N = \frac{1}{\sqrt{a^2 + b^2}} \left(be_1 - ae_2\right)$$

is the principal normal vector.

By a straightforward calculation one gets

$$\nabla_T N + kT = \frac{a^2+b^2-c^2}{2\sqrt{a^2+b^2}\sqrt{(a^2+b^2+c^2)^3}}\left(ace_1 + bce_2 - (a^2+b^2)e_3\right), \tag{16}$$

or

$$\nabla_T N = -kT - \tau B,$$

where

$$B = \frac{1}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\left(ace_1 + bce_2 - (a^2+b^2)e_3\right) \tag{17}$$

is the binormal vector field and the torsion τ is given by

$$\tau = -\frac{a^2+b^2-c^2}{2(a^2+b^2+c^2)}. \tag{18}$$

This result, which can also be deduced directly from the Frenet formula $\nabla_T B = -\tau N$, completes the proof of Theorem 1, in Section 3.

Previous calculations suggest some remarks.
First, if $\sigma(u) = \exp uX$ is not a geodesic, then k and τ are always related by the formula

$$k^2 + \tau^2 = \frac{1}{4}. \tag{19}$$

The second observation concerns in particular the geometric meaning of the vanishing of the torsion of $\sigma(u)$. This meaning is not trivial because in the Heisenberg group do not exist, even locally, totally geodesic or, in general, totally umbilical surfaces (see [7]), and is given by the following

Proposition 1. *If the one-parameter subgroup $\sigma(u) = \exp uX$ is not a geodesic, its normal surface (i.e., the ruled surface generated by the lines normal to $\sigma(u)$) $S_1 = \exp_{\sigma(u)} vN$, where N is the principal normal vector, verifies the following conditions:*

1. *S_1 is a minimal surface and coincides with the product surface $\exp uX \exp vN$, where N is the principal normal vector of $\sigma(u)$ at the origin;*
2. *S_1 is flat along $\sigma(u)$ if and only if the torsion of $\sigma(u)$ vanishes.*

We shall now determine explicitly the surface S_1. If u is fixed, a parametrization for the geodesic $\exp_{\sigma(u)} vN$ is given by a triplet $\{x(v), y(v), z(v)\}$ of functions satisfying the differential system (see [4])

$$\begin{cases} x'' + y'(z' - xy') = 0 \\ y'' - x'(z' - xy') = 0 \\ z' - xy' = N_3. \end{cases} \tag{20}$$

Here N_3 is exactly the third component of the vector N with respect to the basis $\{e_1, e_2, e_3\}$.

The initial conditions for the solution $\{x(v), y(v), z(v)\}$ are given by

$$\begin{cases} x(0) = au \\ y(0) = bu \\ z(0) = cu + \dfrac{1}{2}abu^2 \end{cases} \tag{21}$$

and

$$N = \frac{1}{\sqrt{a^2+b^2}}(be_1 - ae_2) = \begin{cases} x'(0) &= \dfrac{b}{\sqrt{a^2+b^2}} \\ y'(0) &= \dfrac{-a}{\sqrt{a^2+b^2}} \\ z'(0) &= \dfrac{-a^2u}{\sqrt{a^2+b^2}} \end{cases} \tag{22}$$

Therefore we get $N_3 = 0$ and the general point P of S_1 has coordinates given by

$$P(u,v) = \begin{cases} x(u,v) &= au + \dfrac{bv}{\sqrt{a^2+b^2}} \\ y(u,v) &= bu - \dfrac{av}{\sqrt{a^2+b^2}} \\ z(u,v) &= cu + \dfrac{1}{2}abu^2 - \dfrac{a^2uv}{\sqrt{a^2+b^2}} - \dfrac{abv^2}{2(a^2+b^2)} \end{cases} \tag{23}$$

It is easy to see that the first condition in Proposition 1 is fulfilled. As a consequence of (23) we get

$$\begin{aligned} P_u &= ae_1 + be_2 + (c - v\sqrt{a^2+b^2})e_3, \\ P_v &= N = \frac{1}{\sqrt{a^2+b^2}}(be_1 - ae_2). \end{aligned} \tag{24}$$

A vector $\mathcal{N}_1$ normal to S_1 at P is given by

$$\mathcal{N}_1 = (ac - av\sqrt{a^2+b^2})e_1 + (bc - bv\sqrt{a^2+b^2})e_2 - (a^2+b^2)e_3.$$

With respect to the Levi-Civita connection ∇ on H_3 one has $\nabla_{P_v}P_v = 0$ (obviously) and

$$\begin{aligned} \nabla_{P_u}P_u &= (bc - bv\sqrt{a^2+b^2})e_1 - (ac - av\sqrt{a^2+b^2})e_2 \\ \nabla_{P_u}P_v &= \frac{1}{2\sqrt{a^2+b^2}}\Big\{(-ac + av\sqrt{a^2+b^2})e_1 \\ &\quad - (bc - bv\sqrt{a^2+b^2})e_2 - (a^2+b^2)e_3\Big\}. \end{aligned} \tag{25}$$

It follows that the components of the second fundamental form of S_1 are

$$\begin{aligned} h_{uu} &= h_{vv} = 0, \\ h_{uv} &= -\frac{\sqrt{a^2+b^2}}{2\|\mathcal{N}_1\|}\Big\{(c^2 - a^2 - b^2 - 2vc\sqrt{a^2+b^2} + v^2(a^2+b^2))\Big\}. \end{aligned} \tag{26}$$

so that, by (18), one has $h = 0$ along the curve $\sigma(u)$ corresponding to $v = 0$ if and only if the torsion of $\sigma(u)$ vanishes.

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