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THE EXISTENCE OF A STRAIGHT LINE OF PIECEWISE RIEMANNIAN 2-MANIFOLDS

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Abstract. *In this paper, we study properties of* piecewise Riemannian 2-manifolds *which are combinatorial 2-manifolds such that each 2-simplex is a geodesic triangle of some Riemannian 2-manifold. We will introduce the* total excess $e(X)$ *of a piecewise Riemannian 2-manifold* X *and prove the following generalizations of results of Cohn-Vossen and the second author obtained for Riemannian 2-manifolds.*

Let X *be a piecewise Riemannian 2-manifold without boundary having one end.*

Theorem A. *If* X *admits total excess and contains a straight line, then* $e(X) \leqq 2\pi(\chi(X) - 1)$*, where* $\chi(X)$ *is the Euler characteristic of* X.

Theorem B. *If* X *admits total excess* $e(X)$ *which is smaller than* $2\pi(\chi(X) - 1)$*, then* X *contains a straight line.*

It should be noted that X may or may not contain a straight line when $e(X) = 2\pi(\chi(X) - 1)$.

1. INTRODUCTION

For a 2-dimensional Riemannian manifold M, the total curvature $C(M)$ is defined as the integral of the Gaussian curvature over M. The Gauss-Bonnet Theorem states that if M is compact, then $C(M) = 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M, and hence $C(M)$ is a topological invariant. For a noncompact surface M, $C(M)$ is no longer a topological invariant and reflects a certain "asymptotic property" of Gaussian curvature of M. The geometry of the total curvature has been studied by many authors, and several attempts have been made to generalize the concept of total curvature for some classes of metric spaces which need not have Riemannian metrics (see [2],[8],[9] and [10] for example). In particular, Machigashira and the second author in [10] defined the total excess for length spaces which are topological 2-manifolds with discrete topological singularities. So it would be a natural attempt to find metric or polyhedral analogues of those results concerning total curvature on Riemannian 2-manifolds.

In this paper, we will consider a *piecewise Riemannian 2-manifold* which is a topological 2-manifold with a triangulation such that each 2-simplex is a geodesic triangle in some Riemannian 2-manifold. As an analogue of total curvature, we will introduce the total excess in the sense of [10] and generalize some results. It is essential in our argument that the set of vertices "with nonzero angle excess", defined in Section 2, forms a countable discrete set.

The paper is organized as follows. Sections 2 and 3 are preliminary sections. Some basic definitions and fundamental results are reviewed in Section 2. Also two auxiliary lemmas,

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which are piecewise Riemannian analogues of results on Riemannian manifolds (see [7], [12] and [13]), are proved in Section 3 for later use. In Section 4, we study the problem of the existence of a straight line and prove Theorems A and B, which are piecewise Riemannian analogues of those of [7] and [12].

Basically most of our proofs proceed similarly to those of Riemannian 2-manifolds, but several nontrivial modifications are necessary in our context. The authors hope that the technique developed in this paper will find applications to further investigations on this subject.

2. PRELIMINARIES

We begin with reviewing relevant basic terminologies. For a metric space (X, d), a curve $\alpha : I \to X$ is called a *geodesic* if it is locally distance minimizing, that is, for each point $t \in I$, there exists a neighborhood U of t such that $d(\alpha(s_1), \alpha(s_2)) = |s_1 - s_2|$ for any points $s_1, s_2 \in U$. In what follows we assume that α is parametrized as being proportional to arc length. If the above equality holds for any points $s_1, s_2 \in I$, then we call α a *minimizing geodesic*. In particular, a minimizing geodesic defined on $[0, \infty)$ is called a *ray* and that on $(-\infty, \infty)$ a *straight line*. We sometimes identify a geodesic with their images. For a geodesic segment $\alpha : [a, b] \to X$ on a compact interval $[a, b]$, let

$$\mathring{\alpha} := \alpha|_{(a,b)} : (a, b) \to X,$$

namely the set of all interior points of α. As usual, the points $\alpha(a)$ and $\alpha(b)$ are called the end points of α.

Let X be a topological 2-manifold with a triangulation such that each 2-simplex is a geodesic triangle in some Riemannian 2-manifold. We introduce a natural metric d on X as follows: for any pair of points $x, y \in X$,

$$d(x, y) := \inf\{\, l(c) \mid c \text{ is a piecewise smooth curve from } x \text{ to } y\},$$

where $l(c)$ is the length of c.

Definition 2.1. We call such a space (X, d) a *piecewise Riemannian 2-manifold.*

A piecewise Riemannian 2-manifold X is said to be *piecewise flat* if each 2-simplex is isometric to a 2-simplex in the Euclidean plane $\mathbf{R}^2$.

For a point p on a piecewise Riemannian 2-manifold X, we denote by $\mathcal{R}_p$ the set of all minimizing geodesics emanating from p. For $\alpha, \beta \in \mathcal{R}_p$ we define the *angle* at p as follows. For an arbitrarily constant k, let $\tilde{\angle}_k(\alpha(s)p\beta(t))$ be the angle at the point corresponding to p of the triangle in $M(k)$ which corresponds to the triangle $\Delta(\alpha(s)p\beta(t))$, where $M(k)$ is the 2-dimensional space form of constant sectional curvature k. Then the limit

$$\angle_p(\alpha, \beta) := \lim_{s,t \to 0} \tilde{\angle}_k(\alpha(s)p\beta(t))$$

exists, which is independent of the choice of k. We call it the *angle* at p subtended by α and β. It is easily seen that the angle $\angle_p$ is a pseudo-metric on $\mathcal{R}_p$ and induces an equivalence relation $\sim$ defined as follows: $\alpha \sim \beta$ if and only if $\angle_p(\alpha, \beta) = 0$. The completion of the metric space $(\mathcal{R}_p/_\sim, \angle_p)$ is denoted by $(\Sigma_p, \angle_p)$ and is called the *space of directions* at p. For a subset Y of X, let

$$\mathcal{R}_p^Y := \{\alpha \in \mathcal{R}_p \mid \alpha([0, \epsilon]) \subset Y \text{ for some } \epsilon > 0\}.$$

The *space of directions with respect to* Y, denoted by Σ_p^Y, is the completion of the metric space $\mathcal{R}_p^Y/_\sim$.

If p is in the interior of X, then Σ_p is homeomorphic to S^1, the unit circle in the Euclidean plane $\mathbf{R}^2$. Furthermore, if p is not a vertex of the triangulation of X, then Σ_p is isometric to S^1.

For a piecewise Riemannian 2-manifold X and a interior point $p \in X$, let $k(p) = 2\pi - L(\Sigma_p)$, where L is the one-dimensional Hausdorff measure on Σ_p. $k(p)$ is called the *angle excess* at the point p in this paper. It is clear that, if p is not a vertex, then $k(p) = 0$. Note that, when X is piecewise flat, $k(p)$ is called the curvature at p in [8]. However, we would like to avoid the use of the terminology "curvature" here to prevent a possible confusion with the Gaussian curvature at a point in the interior of a 2-simplex.

For a Riemannian manifold without boundary, each geodesic is locally extended in a unique way, but this does not hold for a piecewise Riemannian manifold. Suppose that a piecewise Riemannian 2-manifold X has a minimizing geodesic α with an end point p. If $k(p) > 0$, then it is easily seen that α cannot be extended beyond p. On the other hand, if $k(p) < 0$, there are infinitely many minimizing geodesic-extensions beyond p. In this sense, a point with nonzero angle excess is "singular". We define the *positive singular set* $\mathrm{Sing}^+(X)$ and the *negative singular set* $\mathrm{Sing}^-(X)$ of X respectively by

$$\mathrm{Sing}^{\pm}(X) := \{p \in \overset{\circ}{X} \mid k(p) \gtrless 0\},$$

where $\overset{\circ}{X}$ is the interior points of X, and the union of these two sets is called the *singular set* and denoted by $\mathrm{Sing}(X)$, that is, $\mathrm{Sing}(X) = \mathrm{Sing}^+(X) \cup \mathrm{Sing}^-(X)$. Clearly, $\mathrm{Sing}(X)$ is a subset of the vertices of the triangulation of X. It is also clear that there is no positive singular point on the interior of any minimizing geodesic.

Now we define the total excess of X as follows. Let $C(\Delta)$ be the total curvature of the Riemannian 2-manifold Δ with boundary, and $e_{reg}(X) := \sum_{\Delta:\text{2-simplex}} C(\Delta)$ provided the sum is absolutely convergent, $e_{sing}(X) := \sum_{p \in \mathrm{Sing}(X)} k(p)$ if the sum converges absolutely. Then the *total excess* $e(X)$ of X is defined by

$$e(X) := e_{reg}(X) + e_{sing}(X),$$

when the sum of the right hand side makes sense.

In other words, for a 2-simplex Δ of X with the Gaussian curvature G, let $C_{\pm}(\Delta) = \int_\Delta G_{\pm} d\Delta$, where $G_+ = \max\{G, 0\}$ and $G_- = \min\{G, 0\}$, and $e_{reg}^{\pm}(X) := \sum_{\Delta:\text{2-simplex}} C_{\pm}(\Delta)$. We also define that $e_{sing}^{\pm}(X) := \sum_{p \in \mathrm{Sing}^{\pm}(X)} k(p)$, respectively. Under these notations, we may describe $e(X)$ as the sum of two quantities $e^{\pm}(X) := e_{reg}^{\pm}(X) + e_{sing}^{\pm}(X)$ provided the sum makes sense:

$$e(X) = e^+(X) + e^-(X).$$

We illustrate typical cases. If M is a Riemannian 2-manifold triangulated by geodesic triangles, then $e_{sing}(M) = 0$ and $e(M) = C(M)$, and if M is a piecewise flat PL 2-manifold, then $e_{reg}(M) = 0$ and $e(M) = \sum_{p\in \mathrm{Sing}(M)} k(p)$, the total curvature of M in [8].

Remark. Any piecewise Riemannian 2-manifold is a good surface in the sense of [10], and the above definition coincides with the one given in [10].

The above remark implies that the following theorems hold. They play the fundamental role in our argument.

A curve $c : [a,b] \to X$ is called a *broken geodesic* if there is a subdivision $a = x_0 < \ldots < x_n = b$ such that $c|_{[x_{i-1},x_i]}$ is a geodesic segment. The point $c(x_i)(i = 0, \ldots, n)$ is called a *vertex* of the broken geodesic c.

Theorem 2.1 (The generalized Gauss-Bonnet Theorem [10, Theorem 3.1]) Let X be a piecewise Riemannian 2-manifold without boundary and Y a compact domain of X such that ∂Y consists of simple closed broken geodesics without self-intersection. Then

$$e(Y) = 2\pi\chi(Y) - \sum_{p\in\partial Y} \theta^Y(p),$$

where $\theta^Y(p) = \pi - L(\Sigma_p^Y)$.

Remark. For a Riemannian 2-manifold X and its compact domain Y, the nontrivial contribution to the sum of the above equality is made only at the vertices of the broken geodesics. However, in our setting, a geodesic may pass through points of negative singularity and those singular points may contribute to the sum. Nevertheless, if $p \in \partial Y \setminus \mathrm{Sing}(X)$ is not a vertex of ∂Y, the boundary of Y, then $\theta^Y(p) = 0$. Since there are only finitely many singular points on ∂Y, the second term of the right side of the above equality makes sense.

In what follows, for brevity, $\sum_{p\in S} f(p)$ is often denoted by $\sum_S f$ for a function $f : S \to \mathbf{R}$ defined on a set S. For example, $\sum_{p\in\partial Y} \theta^Y(p)$ is abbreviated to $\sum_{\partial Y} \theta^Y$.

The following is an estimate of total excess for non-compact case.

A noncompact topological 2-manifold is said to be *finitely connected* if it is homeomorphic to a compact 2-manifold with finitely many points removed. If not, it is said to be *infinitely connected.*

Theorem 2.2 (Generalizations of the theorems of Cohn-Vossen and Huber [10, Theorem 4.1]) Let X be a piecewise Riemannian 2-manifold without boundary which admits total excess.

(1) If X is finitely connected, then $e(X) \leqq 2\pi\chi(X)$.
(2) If X is infinitely connected, then $e(X) = -\infty$.

Remark Theorem 4.1 in [10] requires the hypothesis $e^-_{sing}(X) > -\infty$, but the above theorem is valid without this assumption in our setting. See Remark after Theorem 4.1 in [10].

The above theorem implies that a piecewise Riemannian 2-manifold admits a total curvature if and only if $e^+(X) < \infty$. Therefore we have the following;

Proposition 2.3 Let M be a compact orientable piecewise flat 2-manifold without boundary and let $\widetilde{M}$ be the universal covering space of M with the induced triagulation and the induced metric. If $\tilde{M}$ is non-compact, then $\tilde{M}$ admits total excess if and only if $k(p) \leqq 0$ for any vertex p. Further, $e(\tilde{M})$ is finite if and only if M is a flat torus.

Proof. Let $\pi : \tilde{M} \to M$ be the covering map. If there exists a vertex $p \in M$ such that $k(p) > 0$, then each point $\tilde{p} \in \pi^{-1}(p)$ has the positive angle excess and hence $e^{+}_{sing}(X) = \infty$. The first assertion follows from this and the above comment. Similarly, if $k(p) < 0$ for a point p of M, then $e_{sing}(\tilde{M}) = -\infty$. Thus if $e(\tilde{M})$ is finite, then $k(p) = 0$ for any vertex p, and similarly we have $e^{\pm}_{reg}(\tilde{M}) = 0$. The generalized Gauss-Bonnet Theorem 2.7 implies that such a surface is homeomorphic to a flat torus. □

3. AUXILIARY LEMMAS

The following is an analogue of a lemma due to Cohn-Vossen [7] for Riemannian 2-manifolds. The proof of the Riemannian case depends on the first variation formula which is not available in our setting, and hence we need some modification of the proof.

In what follows, for a curve $\alpha : [a, b] \to X$, we define the inverse $\alpha^{-1} : [a, b] \to X$ of α by $\alpha^{-1}(t) := \alpha(a + b - t)$.

Lemma 3.1 Let X be a piecewise Riemannian 2-manifold without boundary. Fix a ray $\gamma : [0, \infty) \to X$ and $p \in X$ arbitrarily. For any $t \geqq 0$, let σ_t denote a minimizing geodesic segment from p to $\gamma(t)$. Then for any $\epsilon > 0$, there exists a sequence $\{t_j\}$ with $t_j \to \infty$ such that

$$\angle_{\gamma(t_j)}(\sigma_{t_j}^{-1}, (\gamma|\,[0, t_j])^{-1}) < \epsilon.$$

Proof. Let $f(t)$ be a distance from p to $\gamma(t)$, namely $f(t) := d(p, \gamma(t))$. The triangle inequality implies that f is Lipschitz continuous with the Lipschitz constant 1, so is differentiable almost everywhere. First we prove that

$$\operatorname*{ess\text{-}limsup}_{t\to\infty} f'(t) = 1. \qquad (*)$$

In fact, suppose that $\alpha := \text{ess-limsup}_{t\to\infty} f'(t) < 1$. Then, for $\eta := (1 - \alpha)/2 > 0$, there exists a positive number t_η such that $f'(s) < \alpha + \eta$ for almost all $s > t_\eta$. Then

$$\begin{aligned}
\lim_{t\to\infty} \frac{f(t)}{t} &= \lim_{t\to\infty} \frac{1}{t}\left\{\int_0^t f'(s)ds + f(0)\right\} \\
&< \lim_{t\to\infty} \frac{1}{t}\left\{\int_0^{t_\eta} f'(s)ds + \int_{t_\eta}^t (\alpha + \eta)ds + f(0)\right\} \\
&= \alpha + \eta < 1.
\end{aligned}$$

Similarly, if $\alpha > 1$, then $\lim_{t\to\infty} f(t)/t > 1$.

On the other hand, we have $t - f(0) \leqq f(t) \leqq t + f(0)$, and hence $f(t)/t$ tends to 1 as $t \to \infty$. This contradicts the above and completes the proof of $(*)$.

Now suppose that one cannot choose a sequence as in the conclusion. Then there exist two constants $\delta > 0$ and $t_0 > 0$ such that for any $t > t_0$,

$$\psi(t) := \angle_{\gamma(t)}(\sigma_t^{-1}, (\gamma|\,[0, t])^{-1}) > 2\delta.$$

We may assume that $\delta < \pi$, and we put $\epsilon := 1 - \sin((\pi - \delta)/2) > 0$. By the above property $(*)$, there exists a large number $t_1 > t_0$ such that

$$1 - \epsilon < f'(t)$$

for almost all $t > t_1$. Then for any $t > t_1$ and $s > 0$, we have

$$f(t+s) - f(t) = \int_t^{t+s} f'(u)du > \int_t^{t+s} (1-\epsilon)du = (1-\epsilon)s,$$

which implies

$$s\epsilon > f(t) + s - f(t+s).$$

Choose a large $t > t_1 > t_0$ as follows: $\gamma(t)$ is not a singular point and there exists a neighborhood U of $\gamma(t)$ such that each of the two components of $U \setminus \gamma$ is contained in a 2-simplex respectively. Such a choice is possible since the set of all points on γ without such neighborhood has measure zero. Let $q_s := \sigma_t(f(t) - s)$, a point on σ_t. Then three points q_s, $\gamma(t)$ and $\gamma(t+s)$ lie in a 2-simplex for sufficiently small s. Estimating $d(q_s, \gamma(t+s))$ by the Riemannian metric on the 2-simplex containing these points, we have

$$d(q_s, \gamma(t+s)) = 2s \sin \frac{\angle_{\gamma(t)}(\sigma_t^{-1}, \gamma|[t, t+s])}{2} + o(s^2).$$

Since $\angle_{\gamma(t)}(\sigma_t^{-1}, \gamma|[t, t+s]) = \pi - \psi(t) < \pi - 2\delta$, we have $d(q_s, \gamma(t+s)) < 2s\sin((\pi - \delta)/2)$ if s is sufficiently small. Now we derive a contradiction as follows:

$$\begin{aligned} 0 &\leqq d(p, q_s) + d(q_s, \gamma(t+s)) - d(p, \gamma(t+s)) \\ &< f(t) - s + 2s \sin \frac{\pi - \delta}{2} - f(t+s) \\ &= (f(t) + s - f(t+s)) - 2s(1 - \sin \frac{\pi - \delta}{2}) \\ &< s\epsilon - 2s\epsilon = -s\epsilon < 0. \end{aligned}$$

This completes the proof. □

Next we investigate a measure of rays, which plays an important role in proving Theorems A and B.

Let X be a piecewise Riemannian 2-manifold without boundary having one end. For a point $p \in X$, let R_p be the set of all points on rays emanating from p:

$$R_p = \cup\{\gamma([0, \infty)) |\ \gamma \text{ is a ray with } \gamma(0) = p\}.$$

It is easily seen that R_p is a closed subset of X and each component of $X \setminus R_p$ is bounded by rays.

Lemma 3.2 Assume that X has finite total excess. Fix a point $p \in X$, and let D be a component of $X \setminus R_p$ and γ_0 and γ_1 be the rays such that $\partial D = \gamma_0 \cup \gamma_1$. If $X \setminus D$ is homeomorphic to the closed half-plane, then

$$e(\overline{D}) = 2\pi(\chi(X) - 1) + L(\Sigma_p^{\overline{D}}) - \sum_{\partial D \setminus \{p\}} \theta^{\overline{D}},$$

where $\theta^{\overline{D}}(x) = \pi - L(\Sigma_x^{\overline{D}})$.

Proof. The basic scheme of the proof is similar to those for the Riemannian manifolds (cf. [11], [12, Lemma 3.1.1] and [13]).

First note that D is finitely connected with $\chi(X) = \chi(D)$. Since $\overline{D}$ has one end, two geodesics γ_0 and γ_1 define the same end of $\overline{D}$. Hence for any $t > 0$, there exists a curve $b_t : [0,1] \to \overline{D}$ such that

$$b_t(0) = \gamma_0(t),\ b_t(1) = \gamma_1(t) \ \text{and}\ d(p, b_t([0,1])) \geqq t.$$

Since there is no ray in D emanating from p, one can see that for any $\epsilon > 0$, there exists a number $t_\epsilon > 0$ such that

$$\angle_p(\alpha_t, \gamma_0) < \epsilon/2 \ \text{or}\ \angle_p(\alpha_t, \gamma_1) < \epsilon/2,$$

for any $t > t_\epsilon$, any $x_t \in b_t([0,1])$ and any geodesic α_t from p to x_t. Notice that, for $t > t_\epsilon$, the set

$$I_i := \left\{ s \in [0,1] \;\middle|\; \begin{array}{l} \text{there exists a geodesic } \alpha \text{ from } p \text{ to } b_t(s) \\ \text{such that } \angle_p(\alpha, \gamma_i) < \epsilon/2 \end{array} \right\}$$

is open in $[0,1]$ for $i = 0, 1$, and the union of these sets is $[0,1]$. The connectedness of $[0,1]$ implies that there is a point $x_t \in b_t([0,1])$ and some geodesics α_t and β_t from p to x_t on $\overline{D}$ such that

$$\angle_p(\alpha_t, \gamma_0) < \epsilon/2 \ \text{and}\ \angle_p(\beta_t, \gamma_1) < \epsilon/2. \tag{1}$$

Here note that if $\alpha_j(t) = \gamma_0(t)$ for some $t > 0$, then $\alpha_j([0,t]) = \gamma_0([0,t])$. Similarly if $\beta_j(t) = \gamma_1(t)$ for some $t > 0$, then $\beta_j([0,t]) = \gamma_1([0,t])$.

Then, for a decreasing sequence $\epsilon_j \to 0$, we can choose an increasing sequence $t_j := t_{\epsilon_j} \to \infty$ and a point $x_j \in b_{t_j}([0,1])$ and geodesics α_j and β_j from p to x_{t_j} satisfying the condition (1) for ϵ_j. Note that $\alpha_j \to \gamma_0$ and $\beta_j \to \gamma_1$ by the definition of D. In fact, since α_j is contained in $\overline{D}$, the ray α_∞ converging (a subsequence of) α_j is so. This implies that α_∞ is γ_0.

Let D_j be the compact set bounded by the union $\alpha_j \cup \beta_j$. By taking a subsequence if necessary, we may also assume that $\{D_j\}$ is monotone increasing and $\cup_j D_j \supset D$. Therefore by Theorem 2.1, we have

$$\begin{aligned} e(\overline{D}) &= \lim_{j\to\infty} e(D_j) \\ &= \lim_{j\to\infty} \left\{ 2\pi(\chi(D_j) - 1) + L(\Sigma_p^{D_j}) + L(\Sigma_{x_j}^{D_j}) - \sum_{\mathring{\alpha}_j \cup \mathring{\beta}_j} \theta^{D_j} \right\} \\ &\geqq 2\pi(\chi(D) - 1) + L(\Sigma_p^D) - \lim_{j\to\infty} \left\{ \sum_{\mathring{\alpha}_j \cup \mathring{\beta}_j} \theta^{D_j} \right\}. \end{aligned}$$

We compare the last term of the above with $\sum_{\partial D \setminus \{p\}} \theta^D$.

Let $t_0 := \sup\{t \in [0,\infty) \mid \alpha_j(t) = \gamma_0(t) \text{ for some } j\}$. We divide our argument into two cases concerning t_0.

Case 1. $t_0 = \infty$.

Let $\{t_i | i \in \mathbf{N}\}$ be a monotone divergent sequence. Then for any t_i, there is a number $j(i)$ such that $\alpha_{j(i)}(t_i) = \gamma_0(t_i)$. Note here that $\alpha_{j(i)}| [0, t_i] = \gamma_0| [0, t_i]$, by the choice of $\alpha_{j(i)}$.

On the other hand, since $e(X)$ is finite, it is clear that for any $\epsilon > 0$, there exists a compact set K such that $| e(X \setminus K)| < \epsilon$. This implies, for example, that $\lim_{i\to\infty} \sum_{\alpha_{j(i)}([t_i, m_{j(i)}))} \theta^{D_{j(i)}} = 0$, where $m_{j(i)} := d(p, x_{j(i)})$.

From these facts, we have

$$\begin{aligned}\lim_{i\to\infty} \sum_{\overset{\circ}{\alpha}_{j(i)}} \theta^{D_{j(i)}} &= \lim_{i\to\infty} \sum_{\alpha_{j(i)}((0,t_i])} \theta^{D_{j(i)}} + \lim_{i\to\infty} \sum_{\alpha_{j(i)}([t_i, m_{j(i)}))} \theta^{D_{j(i)}} \\ &= \lim_{i\to\infty} \sum_{\gamma_0((0,t_i])} \theta^{D} \\ &= \sum_{\gamma_0\setminus\{p\}} \theta^{D}.\end{aligned}$$

Case 2. t_0 is finite.

First we show that $\lim_{j\to\infty} \sum_{\alpha_j((t_0, m_j))} \theta^{D_j} = 0$, where $m_j := d(p, x_j)$.

For any $t > t_0$, there exists a tubular neighborhood N_t of $\gamma_0([0, t])$ such that

$$N_t \cap \mathrm{Sing}(X) \subset \gamma_0([0, t]),$$

because Sing(X) is a discrete set and $\gamma_0([0, t])$ is a compact set. Recall that the sequence $\{\alpha_j\}$ converges to γ_0. Then for any t, there is a number $j(t)$ such that $\alpha_j([0, t]) \subset N_t$ for any $j > j(t)$. Since N_t has singular points only on γ_0, there is no singular point on $\alpha_j(t_0, t]$ for any $j > j(t)$, that is, $\sum_{\alpha_j((t_0, t])} \theta^{D_j} = 0$ for any $j > j(t)$. Furthermore, together with a finiteness of $e(X)$, we can conclude that $\lim_{j\to\infty} \sum_{\alpha_j((t_0, m_j))} \theta^{D_j} = 0$.

Now we claim that $\theta^D(\gamma(t)) = 0$ for any $t > t_0$.

Suppose that $\theta^D(\gamma(t)) < 0$ for some $t > t_0$. Then there exists a point $x \in D$ such that $L(\Sigma^E_{\gamma_0(t)}) > \pi$, where E is a subset of D bounded by γ_x, $(\gamma_0| [0, t])^{-1}$ and $S(\gamma_0(t))$, where γ_x is a geodesic segment from $\gamma_0(t)$ to x and $S(\gamma_0(t))$ is a geodesic sphere centered at $\gamma_0(t)$ with a radius $d(\gamma_0(t), x)$. As $\cup_j D_j \supset D$, there is a sufficiently large j_0 such that $x \in D_{j_0}$. However, this implies that $\alpha_{j_0}([0, t]) = \gamma_0([0, t])$, since $N_{j(t)} \setminus \{p\}$ has no positive singular point. This contradicts the choice of t_0.

Therefore we have

$$\begin{aligned}\lim_{j\to\infty} \sum_{\alpha_j((0, m_j))} \theta^{D_j} &= \lim_{j\to\infty} \left\{ \sum_{\alpha_j((0, t_0])} \theta^{D_j} + \sum_{\alpha_j((t_0, m_j))} \theta^{D_j} \right\} \\ &= \sum_{\gamma_0((0, t_0])} \theta^{D} \\ &= \sum_{\gamma_0\setminus\{p\}} \theta^{D}.\end{aligned}$$

The same is valid for β_j and γ_1, and we have

$$e(D) \geqq 2\pi(\chi(X) - 1) + L(\Sigma^{\overline{D}}_p) - \sum_{\partial D\setminus\{p\}} \theta^{D}.$$

Next we will prove the reverse inequality.

From Lemma 3.1, for a decreasing sequence $\{\epsilon_j\}$ tending to 0, we obtain a divergent sequences $\{m_j\}$ and $\{n_j\}$ such that

$$\begin{cases} \eta_j = \angle_{\gamma_0(m_j)}(\gamma_{0j}^{-1}, \sigma_j^{-1}) < \epsilon_j, \\ \omega_j = \angle_{\gamma_1(n_j)}(\tau_j^{-1}, \gamma_{1j}^{-1}) < \epsilon_j, \end{cases}$$

where σ_j and τ_j are geodesic segments from x_j to $\gamma_0(m_j)$ and $\gamma_1(n_j)$ respectively, and $\gamma_{0j} := \gamma_0|[0, m_j]$ and $\gamma_{1j} := \gamma_1|[0, n_j]$. We may assume that $\sigma_j, \tau_j, \gamma_{0j}$ and γ_{1j} have no common points except their end points. Let E_j be the compact domain bounded by $\gamma_{0j}, \sigma_j, \tau_j$ and γ_{1j}.

Then, from Theorem 2.1, we can estimate $e(\overline{D})$ as follows:

$$\begin{aligned} & e(\overline{D}) \\ &= \lim_{j\to\infty} e(E_j) \\ &= \lim_{j\to\infty} \Big\{ 2\pi\chi(E_j) - (\pi - L(\Sigma_p^{E_j})) - (\pi - \eta_j) - (\pi - \omega_j) \\ &\qquad\qquad - (\pi - L(\Sigma_{x_j}^{E_j})) - \sum_{\overset{\circ}{\gamma}_{0j} \cup \overset{\circ}{\gamma}_{1j}} \theta^{E_j} - \sum_{\overset{\circ}{\sigma}_j \cup \overset{\circ}{\tau}_j} \theta^{E_j} \Big\} \\ &= 2\pi(\chi(D) - 1) + L(\Sigma_p^D) - \sum_{\partial D \setminus \{p\}} \theta^D - \lim_{j\to\infty} \Big\{ (2\pi - L(\Sigma_{x_j}^{E_j})) - \sum_{\overset{\circ}{\sigma}_j \cup \overset{\circ}{\tau}_j} \theta^{E_j} \Big\} \\ &\leqq 2\pi(\chi(D) - 1) + L(\Sigma_p^D) - \sum_{\partial D \setminus \{p\}} \theta^D. \end{aligned}$$

The last inequality comes from the finiteness of $e(X)$. This completes the proof. □

4. PROOFS OF MAIN THEOREMS

In this section, we prove Theorems A and B. If a piecewise Riemannian 2-manifold has at least two ends, then it always contains a straight line. Hence in the sequel, we confine our consideration to piecewise Riemannian 2-manifolds without boundary having one end. First we prove the following

Theorem A. Let X be a piecewise Riemannian 2-manifold without boundary having one end. If X admits total excess and contains a straight line, then $e(X) \leqq 2\pi(\chi(X) - 1)$.

Proof. The proof is a straightforward modification of that of Riemannian case via Lemma 3.2.

To prove this it is sufficient to consider the case that $e(X)$ is finite. Therefore, from Theorem 2.2, we assume that X is finitely connected and has the finite total excess. Fix a small number $\epsilon > 0$ arbitrarily. Then there exists an increasing sequence $\{K_j\}$ of compact domain of X bounded by simple closed broken geodesics without self-intersection satisfying the following three conditions:

(1) $\cup_{j=1}^{\infty} K_j = X$,

(2) $X \setminus K_j$ is homeomorphic to $S^1 \times \mathbf{R}$ for any j,
(3) $|e(X \setminus K_j)| < \epsilon$ for any j.

Let $\gamma : \mathbf{R} \to X$ be a straight line in X. We may assume that $K_1 \cap \gamma \neq \emptyset$. By the condition (2) above, the straight line γ separates $X \setminus K_j$ into two components A_j and B_j. For any j, we take a large t_j such that the points $\gamma(t_j)$ and $\gamma(-t_j)$ are on the "opposite sides" with respect to $K_j \cap \gamma$. Furthermore from Lemma 3.1, we can take t_j satisfying that

$$\angle_{\gamma(-t_j)}(P_j, Q_j) < \epsilon \text{ and } \angle_{\gamma(t_j)}(P_j^{-1}, Q_j^{-1}) < \epsilon,$$

where P_j and Q_j are curves in $\overline{A}_j$ and in $\overline{B}_j$, respectively, from $\gamma(-t_j)$ to $\gamma(t_j)$ of minimal length.

Let D_j be a compact domain bounded by P_j and Q_j. It is easily seen that, for any point $x \in \partial D_j \setminus \{\gamma(\pm t_j)\}$, we have $L(\Sigma_x^{X\setminus D_j}) \geqq \pi$, that is, $\theta^{X\setminus D_j}(x) \leqq 0$, since P_j and Q_j have minimal lengths. Hence $\theta^{D_j}(x) = k(x) - \theta^{X\setminus D_j}(x) \geqq k(x)$. Since $e(X)$ is finite, for any sufficiently large j, we have $\sum_{\partial D_j \setminus \{\gamma(\pm t_j)\}} k > -\epsilon$.

Then applying Theorem 2.1, we obtain

$$\begin{aligned} e(D_j) &= 2\pi\chi(D_j) - \theta^{D_j}(\gamma(t_j)) - \theta^{D_j}(\gamma(-t_j)) - \sum_{\partial D_j \setminus \{\gamma(\pm t_j)\}} \theta^{D_j} \\ &\leqq 2\pi(\chi(D_j) - 1) + 2\epsilon - \sum_{\partial D_j \setminus \{\gamma(\pm t_j)\}} \theta^{D_j} \\ &\leqq 2\pi(\chi(D_j) - 1) + 2\epsilon - \sum_{\partial D_j \setminus \{\gamma(\pm t_j)\}} k(v) \\ &\leqq 2\pi(\chi(D_j) - 1) + 3\epsilon. \end{aligned}$$

This implies that $e(X) \leqq 2(\pi\chi(X) - 1) + 3\epsilon$. Since ϵ is arbitrary, the conclusion follows. □

In the rest of this section, we study the converse of the above result.

Theorem B. Let X be a finitely connected piecewise Riemannian 2-manifold without boundary having one end. If X admits total excess $e(X)$ which is smaller than $2\pi(\chi(X) - 1)$, then X contains a straight line.

Proof. It is sufficient to show that there exists a compact set K in X through which there is a ray γ emanating from any point $p \in X \setminus K$.

Indeed, let us suppose that there exists such compact set K. Then for a divergent sequence $\{p_j\}$ on $X \setminus K$, there is a ray γ_j from p_j such that $\gamma_j \cap K \neq \emptyset$. Since K is compact, we can choose a subsequence of $\{\gamma_j\}$ which converges to a straight line γ.

We divide our argument into the following two cases.

Case 1. $e^-(X)$ is finite.

Let $\epsilon := \{2\pi(\chi(X) - 1) - e(X)\} / 2 > 0$. There exists a compact domain K bounded by finite number of simple closed broken geodesics without self-intersection satisfying the following two conditions:

(1) $e^-(K) < e^-(X) + \epsilon$,
(2) $X \setminus K$ is homeomorphic to $S^1 \times \mathbf{R}$.

We will show that K is the required set. Suppose that there exists a point $p \in X \setminus K$ from which there is no ray through K. Let E be the closure of the component of $X \setminus R_p$ containing K and α and β be rays from p such that $\partial E \subset \alpha \cup \beta$.

If E is bounded, then there are two non-negative numbers s_1, s_2 such that $\partial E = \alpha([s_1, s_2]) \cup \beta([s_1, s_2])$. Let $\theta_i = \angle_{p_i}(\alpha([s_1, s_2]), \beta([s_1, s_2]))$, where $p_i = \alpha(s_i)$ and $i = 1, 2$. Then

$$\begin{aligned} e(E) &= 2\pi\chi(X) - (\pi - \theta_1) - (\pi - \theta_2) - \sum_{\alpha([s_1,s_2])\cup\beta([s_1,s_2])\setminus\{p_1,p_2\}} \theta^E \\ &\geqq 2\pi(\chi(X) - 1). \end{aligned}$$

If E is unbounded, then there is a non-negative number s such that $\partial E = \alpha([s, \infty)) \cup \beta([s, \infty))$. Note that there is no ray emanating from $p_0 := \alpha(s)$ on E except α and β. Hence by Lemma 3.2,

$$\begin{aligned} e(E) &= 2\pi(\chi(X) - 1) + L(\Sigma_{p_0}^E) - \sum_{\alpha\cup\beta\setminus\{p_0\}} \theta^E \\ &\geqq 2\pi(\chi(X) - 1), \end{aligned}$$

where the above inequality follows from the fact that $L(\Sigma_{p_0}^E) \geqq 0$ and $\theta^E(q) \leqq 0$ for any $q \in \partial D \setminus \{p_0\}$.

On the other hand, since $e^+(E) \leqq e^+(X)$ and $e^-(E) \leqq e^-(K) < e^-(X) + \epsilon$, we have that $e(E) < e(X) + \epsilon < 2\pi(\chi(X) - 1)$, which is a contradiction.

Case 2. $e^-(X) = -\infty$.

Since X admits total excess, we have $e^+(X) < \infty$. Take sufficiently large $R > e^+(X)$ and choose a compact set K bounded by finitely many simple closed broken geodesics without self-intersection satisfying the following two conditions:

(1) $e^-(K) < 2\pi(\chi(X) - 1) - R$,

(2) $X \setminus K$ is homeomorphic to $S^1 \times \mathbf{R}$.

We will prove that K is the desired compact set in this case. As in Case 1 we suppose that there exists a point $p \in X \setminus K$ from which there is no ray through K. Then taking a closed domain E and two rays α and β in the same way as in Case 1, we have that $e(E) \geqq 2\pi(\chi(X) - 1)$ similarly. On the other hand, we have

$$\begin{aligned} e(E) &= e^+(E) + e^-(E) \leqq e^+(X) + e^-(K) \\ &< e^+(X) + 2\pi(\chi(X) - 1) - R \\ &< 2\pi(\chi(X) - 1). \end{aligned}$$

This is a contradiction, and we complete the proof of the theorem. □

Remark The Euclidean plane $\mathbf{R}^2$ is a trivial example with straight lines such that $e(\mathbf{R}^2) = 2\pi(\chi(\mathbf{R}^2) - 1) = 0$. On the other hand, the second author constructed in [11] a Riemannian 2-manifold M which is homeomorphic to $\mathbf{R}^2$ and satisfies $C(M) = e(M) = 0$ and does not contain a straight line. This indicates that the hypothesis $e(X) < 2\pi(\chi(X) - 1)$ above cannot be replaced by $e(X) \leqq 2\pi(\chi(X) - 1)$ to conclude the existence of a straight line. The idea used in that construction can be applied to construct a piecewise flat example, but we omit the detail here.

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