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SUPPORT FUNCTIONAL AND DUAL CONE WITH APPLICATION

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Dedicated to Professor Dr. Wilhelm Stoll on the occasion of his 75th birthday

Abstract. *The theory of the Minkowski support functional and the dual cone is developed in general linear spaces. Properties of the support functional are then used to establish certain theorems of combinatorial geometry which hold in infinite dimensions.*

1. INTRODUCTION AND NOTATION

We start with some basic terminology. $\mathbb{K}$ will always denote the field of real or complex numbers. For a linear space E over $\mathbb{K}$, E^* will denote the algebraic dual of E. If E is a topological linear space over $\mathbb{K}$, then E' is its topological dual. By a *flat* in E we mean a translate of a linear subspace of E. For a subset S of E, S^a will denote its *algebraic hull*, i.e. the set of all points $y \in E$ for which a nondegenerate half-open real line segment $[x, y) \subseteq S$. Following [8] and some earlier works, for a convex subset S of E and a point u in S, we denote by S_u^R the union of all closed real halflines emanating from u in S and call the *characteristic cone* of S at u. The reader is referred to [11], [15] and [17] for other definitions and concepts used below.

The purpose of this paper is to provide the reader with a detailed list of properties of the Minkowski support functional and the dual cone. The theory is developed in general linear spaces over $\mathbb{K}$ (cf. [1], [12], [17] for a finite-dimensional case and [2,Chap.II.Exc.§6.9],[5] for some extensions), and then its sample is used to establish three theorems of combinatorial geometry valid in infinite dimensions. Further such applications, including the infinite-dimensional extension of Valentine's proof of Horn's generalization of Helly's theorem [16], [17] are expected.

2. DEFINITION AND PROPERTIES

In the present section we develop the theory of the support functional and the dual cone in full generality. The basic definition and some of the properties are known in the finite-dimensional setting [1,§4], [12], [17,V].

Definition 2.1. *Let S be a nonempty subset of a linear space E over $\mathbb{K}$ and F a linear subspace of E^*. Define the support functional of S on F by*

$$h_S(u) = \sup_{x\in S}(Re\langle u, x\rangle) < \infty \tag{1}$$

for $u \in F$, and the dual cone of S by

$$C(S) = \{(z, u) \in \mathbb{K} \times F : Re z \geq h_S(u)\} \tag{2}$$

Moreover, let $C(\varnothing) = \mathbb{K} \times F$.

Proposition 2.2. *The domain of definition* $\mathrm{Dom}(h_S)$ *of the support functional* h_S *given by* (1) *is a convex possibly degenerate cone in* F *with apex at* $\circ$. *The functional* h_S *is sublinear in* $\mathrm{Dom}(h_S)$.

Proof. Of course $\circ \in \mathrm{Dom}(h_S)$. If $u \in \mathrm{Dom}(h_S)$ and λ is a real nonnegative number, then $h_S(\lambda u) = \sup_{x \in S}(Re\langle \lambda u, x\rangle) = \lambda \sup_{x\in S}(Re\langle u,x\rangle) = \lambda h_S(u) < \infty$, i.e. $\lambda u \in \mathrm{Dom}(h_S)$, h_S is positively homogeneous in $\mathrm{Dom}(h_S)$ and $\mathrm{Dom}(h_S)$ is a cone at $\circ$. If $u, v \in \mathrm{Dom}(h_S)$ and α, β are real nonnegative numbers, then $h_S(\alpha u + \beta v) = \sup_{x\in S}(Re\langle \alpha u + \beta v, x\rangle) = \sup_{x\in S}(\alpha Re\langle u,x\rangle + \beta Re\langle v,x\rangle) \leq \alpha \sup_{x\in S}(Re\langle u,x\rangle) + \beta \sup_{x\in S}(Re\langle v,x\rangle) = \alpha h_S(u) + \beta h_S(v) < \infty$, i.e. $\alpha u + \beta v \in \mathrm{Dom}(h_S)$ which means, taking $\alpha + \beta = 1$, that $\mathrm{Dom}(h_S)$ is convex and h_S is subadditive, as desired. Observe finally that $\mathrm{Dom}(h_E) = \{\circ\}$. □

Proposition 2.3. *Let* $S, T, S_\alpha (\alpha \in \mathcal{A})$ *be subsets of a linear space* E *over* $\mathbb{K}$. *The following assertions are true:*

(*a*) *if* $S \subseteq T$, *then* $C(T) \subseteq C(S)$,

(*b*) $C(\bigcup_{\alpha\in\mathcal{A}} S_\alpha) = \bigcap_{\alpha\in\mathcal{A}} C(S_\alpha)$,

(*c*) *for every* x *in* E, $C(\{x\})$ *is a real algebraically closed halfspace in* $\mathbb{K} \times F$,

(*d*) $C(S) = \mathbb{K} \times F$ *if and only if* $S = \varnothing$,

(*e*) $C(E) = \{(z, \circ) \in \mathbb{K} \times F : Rez \geq 0\}$,

(*f*) $C(S)$ *is an algebraically closed convex cone in* $\mathbb{K} \times F$ *at* $(0, \circ)$,

(*g*) $C(S) = C(\mathrm{conv}S) = C(S^a)$,

(*h*) $h_S = h_{\mathrm{conv}S} = h_{S^a}$,

(*i*) *for every* x_0 *in* E *and* u *in* F, $h_{S+x_0}(u) = h_S(u) + Re\langle u, x_0\rangle$ *and* $\mathrm{Dom}(h_{S+x_0}) = \mathrm{Dom}(h_S)$,

(*j*) *if* S *is a flat in* E *of finite codimension* $n (0 \leq n \leq \dim E)$ *and* h_S *is defined on* E^*, *then* $\{Reu : u \in \mathrm{Dom}(h_S)\}$ *is an* n*-dimensional linear subspace of* E^*.

Proof. We will successively justify the assertions.

If $S = \varnothing$, then (a) holds by definition, so that let S and T be nonempty. For $(z, u) \in C(T)$, $Rez \geq h_T(u) = \sup_{x\in T}(Re\langle u,x\rangle) \geq \sup_{x\in S}(Re\langle u,x\rangle) = h_S(u)$, i.e. $(z,u) \in C(S)$ and (a) is established.

Let us prove that $\bigcap_{\alpha\in\mathcal{A}} C(S_\alpha) \subseteq C(\bigcup_{\alpha\in\mathcal{A}} S_\alpha)$. Suppose, without loss of generality, that no S_α is empty. Select $(z,u) \in C(S_\alpha)$ for all $\alpha \in \mathcal{A}$. If $Rez < \sup_{x\in\bigcup_{\alpha\in\mathcal{A}} S_\alpha}(Re\langle u,x\rangle)$, then for some point $y \in \bigcup_{\alpha\in\mathcal{A}} S_\alpha$, i.e. $y \in S_{\alpha_0}$ for some index $\alpha_0 \in \mathcal{A}$, we have $Rez < Re\langle u,y\rangle \leq \sup_{x\in S_{\alpha_0}}(Re\langle u,x\rangle)$, whence $(z,u) \notin C(S_{\alpha_0})$, a contradiction. Consequently, $(z,u) \in C(\bigcup_{\alpha\in\mathcal{A}} S_\alpha)$ implying the desired inclusion. Since, by (a), the reverse inclusion is immediate, (b) is established too.

Now observe that $C(\{x\}) = \{(z,u) \in \mathbb{K} \times F : Re(z - \langle u,x\rangle) \geq 0\}$ and $v = Re(z - \langle u,y\rangle)$ is a real-valued linear functional on $\mathbb{K} \times F$, so that $C(\{x\})$ is a real algebraically closed halfspace in $\mathbb{K} \times F$ (cf. [11,§16.3]) which establishes (c).

Let in turn $C(S) = \mathbb{K} \times F$. If $S \neq \varnothing$, say $x \in S$, then, by (a), $C(S) \subseteq C(\{x\})$, a contradiction since, by (c), $C(\{x\})$ is a halfspace in $\mathbb{K} \times F$. On the other hand, $C(\varnothing) = \mathbb{K} \times F$ by definition, and (d) is proved.

Let us consider (e). It follows from (b) that $C(E) = \bigcap_{x\in E} C(\{x\}) = \bigcap_{x\in E}\{(z,u) \in$

$\mathbb{K} \times F : Re(z - \langle u, x \rangle) \geq 0\}$. The inclusion $\{(z, \circ) \in \mathbb{K} \times F : Rez \geq 0\} \subseteq C(E)$ is therefore clear. Conversely, let $(z, u) \in C(E)$, so that in particular $Rez \geq Re\langle u, \circ \rangle = 0$. If u were nonzero, then $Re\langle u, x_0 \rangle > 0$ for some point $x_0 \in E$, $x_0 \neq \circ$. But $(z, u) \in C(\{x_0\})$, so that $Rez \geq Re\langle u, x_0 \rangle > 0$. Then however taking $x_1 = 2x_0 Rez / \langle u, x_0 \rangle$, we have $Re(z - \langle u, x_1 \rangle) = -Rez < 0$, a contradiction with $(z, u) \in C(\{x_1\})$. Hence, $u = \circ$ and (e) is justified.

For $S \neq \varnothing$, by assertions (b) and (c), $C(S)$ is the intersection of a nonempty family of algebraically closed halfspaces in $\mathbb{K} \times F$ containing $(0, \circ)$ on their boundaries, so that it is a convex cone at $(0, \circ)$. Take a point $q \in C(S)^a \sim C(S)$. By assumption, $[p, q) \subseteq C(S)$ for some point $p \in C(S)$, $p \neq q$. Any algebraically closed halfspace determining $C(S)$ contains $[p, q)$, so that also $[p, q]$, whence $q \in C(S)$, a contradiction. Hence, $C(S)^a \subseteq C(S)$. Since the reverse inclusion is obvious, $C(S)$ is algebraically closed, as desired.

We prove (g). First, we aim to show that $C(S) = C(S^a)$. Suppose for nontriviality that $S^a \neq S \neq \varnothing$ and select any point $x \in S^a \sim S$. Then there is a point $w_x \in S$ such that $[w_x, x) \subseteq S$ and, by (b), $C(S^a) = \bigcap_{x \in S^a} C(\{x\}) = \bigcap_{x \in S} C(\{x\}) \cap \bigcap_{x \in S^a \sim S} C(\{x\}) = \bigcap_{x \in S} C(\{x\}) \cap \bigcap_{x \in S^a \sim S} \bigcap_{t \in [w_x, x]} C(\{t\})$. Now observe that $\bigcap_{t \in [w_x, x]} C(\{t\}) \subseteq \bigcap_{t \in [w_x, x)} C(\{t\})$. To prove the reverse inclusion fix $(z, u) \in \bigcap_{t \in [w_x, x)} C(\{t\})$, i.e. for all $0 < \lambda \leq 1$ we have $Rez \geq Re\langle u, \lambda w_x + (1 - \lambda)x \rangle = \lambda Re\langle u, w_x \rangle + (1 - \lambda) Re\langle u, x \rangle$. The continuity of the right-hand side with respect to λ implies that the last inequality holds for all $0 \leq \lambda \leq 1$, whence $(z, u) \in \bigcap_{t \in [w_x, x]} C(\{t\})$, as desired. Consequently, $C(S^a) = \bigcap_{x \in S} C(\{x\}) \cap \bigcap_{x \in S^a \sim S} \bigcap_{t \in [w_x, x)} C(\{t\}) = \bigcap_{x \in S} C(\{x\}) = C(S)$ and we are done. The inclusion $C(\text{conv}S) \subseteq C(S)$ is obvious by (a). Now let $S \neq \varnothing$ and $(z, u) \in C(S)$, i.e. $Rez \geq Re\langle u, x \rangle$ for all $x \in S$. Select any point $h \in \text{conv}S$. By Carathéodory's theorem [17,Th.1.21], $h = \sum_{i=1}^{n} \alpha_i x_i$ for some $x_i \in S$, $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geq 0 (i = 1, \ldots, n)$, so that $Re\langle u, h \rangle = \sum_{i=1}^{n} \alpha_i Re\langle u, x_i \rangle \leq Rez \sum_{i=1}^{n} \alpha_i = Rez$ implying $(z, u) \in C(\text{conv}S)$. In consequence, $C(S) = C(\text{conv}S)$ and (g) is proved.

The inequalities $h_S \leq h_{S^a}$ and $h_S \leq h_{\text{conv}S}$ are immediate since $S \subseteq S^a$ and $S \subseteq \text{conv}S$. Suppose for example that $h_S(u) < h_{S^a}(u)$ for some $u \in F$. Then there is $z \in \mathbb{K}$ such that $h_S(u) \leq Rez < h_{S^a}(u)$, i.e. $(z, u) \in C(S) \sim C(S^a)$, a contradiction, because $C(S) = C(S^a)$ by (g). Hence, $h_S = h_{S^a}$, as desired. The proof of $h_S = h_{\text{conv}S}$ proceeds analogously. (h) is therefore established.

Observe that $h_{S+x_0}(u) = \sup_{x \in S+x_0}(Re\langle u, x \rangle) = \sup_{x - x_0 \in S}(Re\langle u, x - x_0 \rangle) + Re\langle u, x_0 \rangle = h_S(u) + Re\langle u, x_0 \rangle$ and $h_{S+x_0}(u)$, $h_S(u)$ are simultaneously finite or not, whence $\text{Dom}(h_{S+x_0}) = \text{Dom}(h_S)$ which ends the proof of (i).

By (i), we may assume that S is a subspace of E. It is immediate that $u \in \text{Dom}(h_S)$ if and only if Reu vanishes on S, i.e. $Reu \in S^{\perp}$. By [11,§9.3.(7)], $\dim S^{\perp} = n$ and the argument is finished. □

Proposition 2.4. *Let S and T be nonempty subsets of a linear space E over $\mathbb{K}$ and F a linear subspace of E^*. If E is endowed with the weak topology generated by F, then the following assertions are true:*

(*a*) $C(S) = C(\text{cl}S)$,

(*b*) $h_S = h_{\text{cl}S}$,

(*c*) *if S is compact, then for each u in F there exists x_u in S such that $h_S(u) = Re\langle u, x_u \rangle$,*

(*d*) *if $E' = F$ and F separates points in E, then $h_S = h_T$ if and only if $C(S) = C(T)$ if and only if* cl conv$S$ = cl conv$T$.

If F is endowed with the weak topology generated by E, then the following assertions are *true:*

(*e*) *for every x in E,* $C(\{x\})$ *is a real closed halfspace in* $\mathbb{K} \times F$,

(*f*) $C(S)$ *is a closed convex cone in* $\mathbb{K} \times F$ *at* $(0, \circ)$,

(*g*) h_S *is lower semicontinuous on* $\mathrm{Dom}(h_S)$,

(*h*) *if S is bounded and finite dimensional, then* h_S *is continuous on F.*

If each of spaces E, F is endowed with the weak topology generated by the other and $E' = F$*, then the following assertions are true:*

(*i*) *if* h_S *is continuous on F, then S is bounded and finite dimensional,*

(*j*) h_S *is strongly continuous on F if and only if S is bounded.*

Proof. We justify first the assertions (a)-(d).

The inclusion $C(\mathrm{cl}S) \subseteq C(S)$ follows from Proposition 2.3(a). Now suppose that S is nonempty and nonclosed, and fix $(z,u) \in C(S) = \bigcap_{x \in S} C(\{x\})$. Hence, $Rez \geq Re\langle u,x\rangle$ for all $x \in S$. Pick out an arbitrary point $x_0 \in \mathrm{cl}S \sim S$. Since u is continuous on E, easily, $Rez \geq Re\langle u,x_0\rangle$, whence $(z,u) \in \bigcap_{x \in \mathrm{cl}S} C(\{x\}) = C(\mathrm{cl}S)$ implying $C(S) \subseteq C(\mathrm{cl}S)$. This establishes (a).

The proof of the equality $h_S = h_{\mathrm{cl}S}$ proceeds as the proof of its algebraic analogue $h_S = h_{S^a}$ in Proposition 2.3(h) and is therefore omitted.

Since S is compact and u is continuous, by the maximum value theorem, there exists a point x_u in S such that $h_S(u) = \sup_{x \in S}(Re\langle u,x\rangle) = Re\langle u,x_u\rangle$ which establishes (c).

By assumption, F separates points in E, so that E is a locally convex topological linear space. Let first $h_S = h_T$. Then, by the definition of the dual cone, $C(S) = C(T)$. If $\mathrm{cl\,conv}S \neq \mathrm{cl\,conv}T$, then there is a point y such that, say, $y \in \mathrm{cl\,conv}S \sim \mathrm{cl\,conv}T$. By the separation theorem [11,§20.7.(1)], there is a closed real hyperplane in E separating y and $\mathrm{cl\,conv}T$ strictly, i.e. for some $v \in E' = F$ we have $Re\langle v,y\rangle > \sup_{x \in \mathrm{cl\,conv}T}(Re\langle v,x\rangle) = h_{\mathrm{cl\,conv}T}(v)$. But by (b) above and Proposition 2.3(h), $h_{\mathrm{cl\,conv}T} = h_T$, so that $Re\langle v,y\rangle > h_T(v)$ implying $h_S(v) = h_{\mathrm{cl\,conv}S}(v) = \sup_{z \in \mathrm{cl\,conv}S}(Re\langle v,z\rangle) \geq Re\langle v,y\rangle > h_T(v)$, a contradiction. Hence, also $\mathrm{cl\,conv}S = \mathrm{cl\,conv}T$. Conversely, let $C(S) = C(T)$. If $h_S \neq h_T$, then, e.g. $h_S(u) < h_T(u)$ for some $u \in F$. Then take any $z \in \mathbb{K}$ such that $h_S(u) \leq Rez < h_T(u)$, i.e. $(z,u) \in C(S) \sim C(T) = \varnothing$, a contradiction. Consequently, $h_S = h_T$. Finally, if $\mathrm{cl\,conv}S = \mathrm{cl\,conv}T$, then again by (b) and Proposition 2.3(h), $h_S = h_{\mathrm{cl\,conv}S} = h_{\mathrm{cl\,conv}T} = h_T$ and the proof of (d) is finished.

For fixed x in E, $(z,u) \mapsto Re(z - \langle u,x\rangle)$ is a real-valued continuous linear functional on $\mathbb{K} \times F$, so that $C(\{x\}) = \{(z,u) \in \mathbb{K} \times F : Re(z - \langle u,x\rangle) \geq 0\}$ is a real closed halfspace and (e) is proved.

It follows from (e) and Proposition 2.3(b) that , for $S \neq \varnothing$, $C(S)$ is the intersection of a nonempty family of closed halfspaces in $\mathbb{K} \times F$ with $(0, \circ)$ on their boundaries which proves (f).

Select $u_0 \in \mathrm{Dom}(h_S)$ and a real number $\gamma < h_S(u_0) = \sup_{x \in S}(Re\langle u_0,x\rangle)$. Hence, there exists $x_0 \in S$ for which $Re\langle u_0,x_0\rangle > \gamma$, so that $Re\langle u,x_0\rangle > \gamma$ for all u in a neighbourhood U of u_0 in F, implying $h_S(u) > \gamma$ for all $u \in U \cap \mathrm{Dom}(h_S)$ which means that h_S is lower semicontinuous on $\mathrm{Dom}(h_S)$, as required in (g).

Let S be bounded and finite dimensional. The continuous linear image of S is bounded [11,§15.6.(5)], so that $\mathrm{Dom}(h_S) = F$. Pick out an arbitrary linear functional $u_0 \in F$ and a number $\epsilon > 0$. By assumption, $S \subseteq \mathrm{conv}\{x_1, \ldots, x_m\}$, where x_j are affinely independent points in E. Take a neighbourhood $U = \bigcap_{j=1}^{m}\{u \in F : |\langle u - u_0, x_j\rangle| < \epsilon\}$ of u_0 in F. Then for any $u \in$

U we have by virtue of Proposition 2.2 and Proposition 2.3(h), $h_S(u) \leq h_S(u_0) + h_S(u-u_0) \leq h_S(u_0) + h_{\mathrm{conv}\{x_1,\ldots,x_m\}}(u-u_0) = h_S(u_0) + h_{\{x_1,\ldots,x_m\}}(u-u_0) = h_S(u_0) + \max_{1\leq j\leq m}(Re\langle u-u_0, x_j\rangle) < h_S(u_0)+\epsilon$. On the other hand, h_S is lower semicontinuous on F, by (g), so that there is a neighbourhood W of u_0 in F such that $h_S(u) > h_S(u_0) - \epsilon$ for all $u \in W$. Hence, $|h_S(u) - h_S(u_0)| < \epsilon$ for $u \in U \cap W$, i.e. $u_0 \in U \cap W \subseteq h^{-1}((h_S(u_0)-\epsilon, h_S(u_0)+\epsilon))$ which means that h_S is continuous at u_0, as claimed in (h).

To prove (i) suppose that h_S is continuous on F. We follow the idea of [5,Th.6], where the real case has been considered. Fix a number $\mu > 0$. Then there exists a neighbourhood $V = \bigcap_{j=1}^{n}\{v \in F : |\langle v, x_j\rangle| < \nu\}$ of $\circ \in F$ for some points $x_j \in E$, so that $h_S(V) \subseteq (-\mu, \mu)$. Now let $K = \sqrt{2}\ \mathrm{conv}\{\pm x_1, \ldots, \pm x_n, \pm ix_1, \ldots, \pm ix_n\}$. By Proposition 2.3(h), we have $h_K(v) = \sqrt{2}h_{\{\pm x_1,\ldots,\pm x_n, \pm ix_1,\ldots,\pm ix_n\}}(v) = \sqrt{2}\max_{1\leq j\leq n}(|Re\langle v,x_j\rangle|, |Im\langle v,x_j\rangle|)$ for every $v \in F$. If $h_K(v) < \nu$, then $|\langle v, x_j\rangle| < \nu$ for $j = 1,\ldots,n$, implying $h_S(v) < \mu$. Hence, $h_{\nu S} \leq h_{\mu K}$ on F. The rest of the argument proceeds as in (d). Suppose that $\nu S \not\subseteq \mu K$ and let $y \in \nu S \sim \mu K$. By the separation theorem [11,§20.7.(1)], there is a closed real hyperplane in E separating y and μK strictly, i.e. for some $v \in E' = F$ we have $Re\langle v, y\rangle > \sup_{x\in\mu K}(Re\langle v,x\rangle) = h_{\mu K}(v)$ implying $h_{\nu S}(v) > h_{\mu K}(v)$, a contradiction. Consequently, $\nu S \subseteq \mu K$ which means that S is bounded and finite dimensional. The proof of (i) is finished.

Let us establish (j). Assume first that h_S is strongly continuous on F. The argument is analogous to the one given in (i). Fix a number $\mu > 0$. Then there exists a strong neighbourhood $V = \{v \in F : \sup_{x\in B}|\langle v,x\rangle| < \nu\}$ of $\circ \in F$ for some bounded subset B of E such that $h_S(V) \subseteq (-\mu, \mu)$. Put $K = \sqrt{2}\ \mathrm{conv}(\pm B \cup \pm iB)$. Again, we have $h_K(v) = \sqrt{2}\ h_{\pm B\cup\pm iB}(v) = \sqrt{2}\sup_{x\in B}(|Re\langle v,x\rangle|, |Im\langle v,x\rangle|)$ for every $v \in F$. If $h_K(v) < \nu$, then $|\langle v,x\rangle| < \nu$ for all $x \in B$, i.e. $h_S(v) < \mu$. Arguing as in the preceding assertion, we conclude that $\nu S \subseteq \mu K$ which means that S is bounded. Conversely, suppose that S is bounded. Then $\mathrm{Dom}(h_S) = F$. Select any point $u_0 \in F$ and a number $\epsilon > 0$. In the strong topology on F, $U = \{u \in F : \sup_{x\in S}|\langle u-u_0, x\rangle| < \epsilon\}$ is a neighbourhood of u_0. For any $u \in U$, we have on one hand $h_S(u) \leq \sup_{x\in S}(Re\langle u_0,x\rangle) + \sup_{x\in S}(Re\langle u-u_0,x\rangle) < h_S(u_0)+\epsilon$. On the other hand, $h_S(u) = \sup_{x\in S}(Re\langle u_0,x\rangle + Re\langle u-u_0,x\rangle) > \sup_{x\in S}(Re\langle u_0,x\rangle - \epsilon) = h_S(u_0) - \epsilon$. Consequently, $|h_S(u) - h_S(u_0)| < \epsilon$ for $u \in U$, i.e. $u_0 \in U \subseteq h_S^{-1}((h_S(u_0)-\epsilon, h_S(u_0)+\epsilon))$ which means that h_S is strongly continuous at u_0. The proof of (j) is finished (cf. [5,Th.7], [13], [14,Cor.7E]). □

We now include for completeness the simplified version of the bipolar theorem which will be used in the subsequent proposition. The proof of the general case when $E' \neq F$ is not so obvious (cf. [10] for the real case).

Lemma 2.5. *Let S be a nonempty subset of a linear space E over $\mathbb{K}$ and F a linear subspace of E^* separating points in E. Endow E with the weak topology generated by F and let $E' = F$. Then*

$$\mathrm{cl\,conv}S = \bigcap_{u\in F}\{y \in E : Re\langle u,y\rangle \leq \sup_{x\in S}(Re\langle u,x\rangle)\}$$

Proof. Since F separates points in E, the weak topology generated by F is Hausdorff, i.e. E is a locally convex topological linear space over $\mathbb{K}$. Each $u \in F$ is continuous, so that $H_u = \{y \in E : Re\langle u,y\rangle \leq \sup_{x\in S}(Re\langle u,x\rangle)\}$ is a closed real halfspace in E containing S which implies $\mathrm{conv}S \subseteq \bigcap_{u\in F} H_u$. To justify the reverse inclusion, select a point $z \notin \mathrm{cl\,conv}S$. By the separation theorem [11, §20.7.(1)], there is a closed real hyperplane in E separating z and $\mathrm{cl\,conv}S$ strictly. But $E' = F$, so that there exists $v \in F$ for which $Re\langle v, z\rangle > \sup_{x\in S}(Re\langle v, x\rangle)$. This means however that $z \notin H_v$, i.e. also $z \notin \bigcap_{u\in F} H_u$ implying $\bigcap_{u\in F} H_u \subseteq \mathrm{cl\,conv}S$, as desired. □

Proposition 2.6. *Let $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of nonempty subsets of a locally convex topological linear space L over $\mathbb{K}$ at least one member of which is bounded. Endow $F = L'$ with the weak topology generated by L. Then*

(*a*) *if* $\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha \neq \varnothing$, *then* $C(\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha) = \mathrm{cl\,conv}(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha))$,
and the following assertions are equivalent:

(*b*) $\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha \neq \varnothing$,

(*c*) $\mathrm{cl\,conv}(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha)) \neq (\mathbb{K}, L')$,

(*d*) *all dual cones $C(S_\alpha)$ have a common closed real hyperplane of support not containing $\mathbb{K} \times \{\circ\}$.*

Proof. By virtue of [11,§20.2.(1)], $\langle L', L\rangle$ is a dual pair over $\mathbb{K}$. Endow L with the weak topology $\mathfrak{T}(L')$ generated by L'. Then, by [11,§20.2.(3)], $\langle L', L[\mathfrak{T}(L')]\rangle$ is also a dual pair ,compatible with $\langle L', L\rangle$. Besides, by [11,§20.7.(6)], the closures of convex sets respectively to L and $L[\mathfrak{T}(L')]$ coincide, so that from now on we can assume that L and L' are locally convex topological linear spaces with weak topologies generated by each other.

We prove (a). Suppose first that $\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha \neq \varnothing$. By Proposition 2.3(a), $C(\mathrm{cl\,conv}S_\alpha) \subseteq C(\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha)$ for each $\alpha \in \mathcal{A}$. But $C(\mathrm{cl\,conv}S_\alpha) = C(S_\alpha)$, by Proposition 2.4(d), so that $\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha) \subseteq C(\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha)$. Consequently, $\mathrm{cl\,conv}\,(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha)) \subseteq C(\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha)$ since, by Proposition 2.3(g), the right-hand side is a closed convex set in $\mathbb{K} \times L'$. To prove the reverse inclusion, select $(z, u) \in C(\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha)$ and suppose that $(z, u) \notin \mathrm{cl\,conv}(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha))$. By the separation theorem, there exists a closed real hyperplane $\mathfrak{h}$ in $\mathbb{K} \times L'$ strictly separating (z, u) and $\mathrm{cl\,conv}(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha))$. Let $\mathfrak{h}_0$ be a translate of $\mathfrak{h}$ through $(0, \circ)$. Then $(z, u) \notin \mathfrak{h}_0$ and $\mathfrak{h}_0$ bounds $\mathrm{cl\,conv}(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha))$. Since at least one set, say S_{α_0}, is bounded, so that, by Mackey's theorem [11,§20.11.(7)], also weakly bounded, $\mathrm{Dom}(h_{S_{\alpha_0}}) = L'$ [10,§15.6.(5)], $\mathfrak{h}_0$ cannot contain $\mathbb{K} \times \{\circ\}$, i.e. be vertical. Since $L'' = L$, $\mathfrak{h}_0$ is determined by the equation $l(y, v) = Rey - Re\langle v, x_0\rangle = 0$ for $(y, v) \in \mathbb{K} \times L'$ and fixed $x_0 \in L$. Suppose that $l(z, u) < 0$, otherwise we could replace l by -l. Hence, $Rez < Re\langle u, x_0\rangle$. Since $(z, u) \in C(\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha)$, we have $Rez \geq h_{\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha}(u)$ $= \sup_{x\in\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha}(Re\langle u, x\rangle)$. Consequently, $x_0 \notin \bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha$. Each $C(S_\alpha)$ is separated by $\mathfrak{h}_0$ from (z, u), so that $l(y, v) \geq 0$ for all $(y, v) \in C(S_\alpha)$. This means that $Rey \geq Re\langle v, x_0\rangle$ whenever $Rey \geq \sup_{x\in S_\alpha}(Re\langle v, x\rangle)$ implying the inequality $Re\langle v, x_0\rangle \leq \sup_{x\in S_\alpha}(Re\langle v, x\rangle)$ for all $v \in L'$. By Lemma 2.5, $x_0 \in \mathrm{cl\,conv}S_\alpha$ for all $\alpha \in \mathcal{A}$, i.e. $x_0 \in \bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha$, a contradiction. In consequence, $C(\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha) \subseteq \mathrm{cl\,conv}\,(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha))$ and the proof of (a) is finished.

It remains to establish the required equivalences. If $\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha \neq \varnothing$, then, by Proposition 2.3(d), $C(\bigcap_{\alpha\in\mathcal{A}} \mathrm{cl\,conv}S_\alpha) \neq (\mathbb{K}, L')$, so that, by (a), $\mathrm{cl\,conv}(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha)) \neq (\mathbb{K}, L')$ and the implication (b)⇒ (c) is proved. If $\mathrm{cl\,conv}(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha)) \neq (\mathbb{K}, L')$, then the separation theorem implies the existence of a closed real hyperplane in $\mathbb{K} \times L'$ bounding the closed convex cone $\mathrm{cl\,conv}(\bigcup_{\alpha\in\mathcal{A}} C(S_\alpha))$. Easily (cf. [11,§17.5.(6)]), its translate through $(0, \circ)$ supports this cone, so that also each $C(S_\alpha)$. Since $\mathrm{Dom}(h_{S_{\alpha_0}})$ is equal to L' for some member S_{α_0}, the translated hyperplane cannot contain $\mathbb{K} \times \{o\}$ which establishes the implication (c)⇒ (d). To finish the argument, suppose that there exists a closed real hyperplane $\mathfrak{h}_0$ through $(0, o)$ in $\mathbb{K} \times L'$ supporting all cones $C(S_\alpha)$ and not containing $\mathbb{K} \times \{o\}$. Since $L'' = L$, $\mathfrak{h}_0$ is determined by the equation $l(y, v) = Re y - Re\langle v, x_0\rangle = 0$ for some fixed $x_0 \in L$ and $(y, v) \in \mathbb{K} \times L'$. Since all $C(S_\alpha)$ lie above $\mathfrak{h}_0$, we have $\sup_{x \in S_\alpha}(Re\langle v, x\rangle) \geq Re\langle v, x_0\rangle$ for all $v \in L'$. By Lemma 2.5, $x_0 \in \bigcap_{\alpha \in \mathcal{A}} \operatorname{cl} \operatorname{conv} S_\alpha \neq \varnothing$. Hence, the implication (d)⇒(b) holds too which completes the argument. □

3. APPLICATIONS

In [17,V] and [16] Valentine applied the dual cone in order to prove certain theorems of combinatorial geometry in R^d. From a geometric point of view, the dual cone enables one to obtain some theorems with ease since the dual situation may become intuitively simpler. Here, we are concerned only with those combinatorial results which hold in infinite-dimensional setting and analyse them by using exclusively the support functional. Its application always reduces the problem to finite dimensional, i.e. usually to standard theorems of combinatorial geometry. An introductory lemma is needed for an analogue of de Santis theorem [17,Th.6.23], [16,Th.13].

Lemma 3.1 *Let S be a convex subset of a real linear space E and $M(S)$ the maximal flat contained in S. If $o \in M(S)$ and* $\operatorname{codim} M(S) < \infty$, *then*

$$M(S) = \bigcap_{u \in \operatorname{Dom}(h_S)} \{y \in E : \langle u, y\rangle = 0\} \tag{3}$$

Proof. Denote $c = \operatorname{codim} M(S) < \infty$, $0 \leq c \leq \dim E$, and consider h_S in E^*. If $c = 0$, then $S = M(S) = E$, so that $\operatorname{Dom}(h_S)$ consists of a zero functional and we are done. If $c = 1$, then by Proposition 2.3(j), $\operatorname{Dom}(h_S)$ is a one-dimensional subspace of E^*. Since every member of $\operatorname{Dom}(h_S)$ is bounded on S, i.e. zero on $M(S)$, we have $M(S) \subseteq \bigcap_{u \in \operatorname{Dom}(h_S)} \{y \in E : \langle u, y\rangle = 0\}$. Since for $c = 1$ the zero set of any nonzero linear functional in $\operatorname{Dom}(h_S)$ coincides with $M(S)$, the reverse inclusion holds and we are done. Assume in the sequel that $c \geq 2$ and let $\mathfrak{G}$ be the c-dimensional linear subspace of E complementary to $M(S)$ with the corresponding linear projection π of E onto $\mathfrak{G}$. By construction, $\pi(S)$ is a convex subset of $\mathfrak{G}$ not containing any straight line. Let $\pi(S)^R$ be the characteristic cone of $\pi(S)$ at o. It may be degenerate, i.e. $\pi(S)^R = \{o\}$ and does not contain any straight line. An easy argument reveals that $\operatorname{cl}\pi(S)$, the closure of $\pi(S)$ if $\mathfrak{G}$ is treated as Euclidean space R^c, does not contain any straight line either. By [11,§25.4.(2)], there is a hyperplane $\mathfrak{H}_1$ in $\mathfrak{G}$ having with $\operatorname{cl}\pi(S)$ only the apex o in common. Easily, other $c - 1$ such hyperplanes $\mathfrak{H}_2, \ldots, \mathfrak{H}_c$ through o may be found with $\{o\} = \bigcap_{j=1}^{c} \mathfrak{H}_j$. Then however $M(S) + \mathfrak{H}_j (j = 1, \ldots, c)$ is a hyperplane in E and $M(S) = \bigcap_{j=1}^{c}(M(S) + \mathfrak{H}_j)$. Each hyperplane $M(S) + \mathfrak{H}_j$ bounds the characteristic cone S^R of S at o, so that, by the extension theorem [11,§9.2.(1′)], there is a linear functional u_j on E vanishing on $M(S) + \mathfrak{H}_j$ and taking nonpositive values on S^R. Hence, it is bounded from above on S, i.e. $u_j \in \operatorname{Dom}(h_S)$. Moreover, $\bigcap_{u \in \operatorname{Dom}(h_S)} \{y \in E : \langle u, y\rangle = 0\} \subseteq \bigcap_{j=1}^{c} \{y \in E : \langle u_j, y\rangle = 0\} = M(S)$. The reverse inclusion is obvious, so that the required equality holds. □

Theorem 3.2. *A nonempty finite family $\mathcal{F}$ of convex sets in a real linear space E contains a common flat of codimension c, where $c \leq \dim E$ is a nonnegative integer, if and only if every subfamily of $c+1$ or fewer members of $\mathcal{F}$ contains a common flat of codimension c.*

Proof. We establish the sufficiency of the condition. Let $\mathcal{F}$ consist of at least $c+2$ members and $\mathcal{W}$ be the collection of all $(c+1)$-element subsets of $\mathcal{F}$. Consider all supporting functionals in E^*. By assumption, for every $F \in \mathcal{W}$, $\bigcap_{\mathfrak{s}\in F} \mathfrak{s}$ contains a flat of codimension c in E, so that, by Proposition 2.3(j), the convex cones $D_{\mathfrak{s}} = \mathrm{Dom}(h_{\mathfrak{s}})(\mathfrak{s} \in F)$ at $\circ$ are all contained in a c-dimensional linear subspace of E^*. Hence $\bigcup_{\mathfrak{s}\in F} D_{\mathfrak{s}}$ contains at most c linearly independent nonzero elements, so that easily $\bigcup_{\mathfrak{s}\in \mathcal{F}} D_{\mathfrak{s}}$ is also contained in a c-dimensional linear subspace G of E^*.

By Lemma 3.1, every $\mathfrak{s} \in \mathcal{F}$ contains a translate of the linear subspace $\bigcap_{u\in D_{\mathfrak{s}}} \{x \in E : \langle u,x\rangle = 0\}$, so that, since $D_{\mathfrak{s}} \subseteq G$, also a translate of $\bigcap_{u\in G}\{x \in E : \langle u,x\rangle = 0\} = G^{\perp}$. Of course, $\mathrm{codim} G^{\perp} = c$. Now let M be a c-dimensional linear subspace of E complementary to $G^{\perp}$ with a corresponding linear projection σ of E onto M. By initial assumption, every $c+1$ of the convex sets $\sigma(\mathfrak{s})(\mathfrak{s} \in \mathcal{F})$ have a point in common. Then, by Helly's theorem, there is a point $p \in \bigcap_{\mathfrak{s}\in\mathcal{F}} \sigma(\mathfrak{s})$, so that $p + G^{\perp}$ is a flat of codimension c in E contained in all members of $\mathcal{F}$. □

The following theorem was proved inductively in [4,Lemma 2.1] and used there to derive some Krasnosel'skii-type characterizations for cones.

Theorem 3.3. *A nonempty family $\mathcal{G}$ of flats in a linear space E over $\mathbb{K}$ has a nonempty intersection of codimension at most c, where $0 < g = \max_{\mathfrak{g}\in\mathcal{G}}\{\mathrm{codim}\mathfrak{g}\} \leq c < \infty$, if and only if every subfamily of $c - g + 2$ or fewer members of $\mathcal{G}$ has a nonempty intersection of codimension at most c.*

Proof. We establish the sufficiency of the condition. For nontriviality, let $\mathcal{G}$ contain at least $c-g+3$ members. Consider all supporting functionals below in E^* and let $\mathcal{U}$ be the collection of all $(c-g+2)$-element subsets of $\mathcal{G}$. For every $G \in \mathcal{U}$, $\bigcap_{\mathfrak{g}\in G} \mathfrak{g}$ is a flat of codimension $\leq c$ in E, so that, by Proposition 2.3(j), the linear subspaces $D_{\mathfrak{g}} = \mathrm{Dom}(h_{\mathfrak{g}})(\mathfrak{g} \in G)$ are all contained in a linear subspace of E^* of dimension $\leq c$ and $\dim D_{\mathfrak{g}} = \mathrm{codim}\mathfrak{g}$.

We claim that $\bigcup_{\mathfrak{g}\in\mathcal{G}} D_{\mathfrak{g}}$ is contained in a linear subspace H of E^* of dimension $\leq c$. Suppose not, i.e. $\dim\bigcup_{\mathfrak{g}\in\mathcal{G}} D_{\mathfrak{g}} > c$. Construct inductively a subcollection $\{D_{\mathfrak{g}_1}, \ldots, D_{\mathfrak{g}_k}\}$ of $\{D_{\mathfrak{g}} : \mathfrak{g} \in \mathcal{G}\}$ as follows. Let $\mathfrak{g}_1$ be a flat for which $\mathrm{codim}\mathfrak{g}_1 = g \leq c$. Given a subcollection $\{D_{\mathfrak{g}_1}, \ldots, D_{\mathfrak{g}_n}\}$ such that $\dim\bigcup_{j=1}^{n} D_{\mathfrak{g}_j} \leq c$, we add $D_{\mathfrak{g}_{n+1}}$, if exists, such that $D_{\mathfrak{g}_{n+1}} \not\subseteq \mathrm{aff}(\bigcup_{j=1}^{n} D_{\mathfrak{g}_j})$ and $\dim\bigcup_{j=1}^{n+1} D_{\mathfrak{g}_j} \leq c$. This process always stops with some number k of selected subspaces. Adding a new member increases the dimension of the union of subspaces at least by 1, so that $g + (k-1) \leq c$, i.e. $k \leq c - g + 1$ and, since $\dim\bigcup_{\mathfrak{g}\in\mathcal{G}} D_{\mathfrak{g}} > c$, we can always add a new subspace $D_{\mathfrak{g}_{k+1}}$ to increase the dimension of the union beyond c, a contradiction with the assumption that every collection of $c - g + 2$ members is contained in a subspace of dimension $\leq c$.

Now let $\mathfrak{g}^0$ be a linear subspace of E parallel to $\mathfrak{g}$. $D_{\mathfrak{g}}$ consists of all functionals vanishing on $\mathfrak{g}^0$, so that $\mathfrak{g}^0 \subseteq \bigcap_{u\in D_{\mathfrak{g}}}\{x \in E : \langle u,x\rangle = 0\}$ and the reverse inclusion follows from the extension theorem [11,§9.2.(1′)]. Since $D_{\mathfrak{g}} \subseteq H$, $F = \bigcap_{u\in H}\{x \in E : \langle u,x\rangle = 0\} \subseteq \mathfrak{g}^0$ for every $\mathfrak{g} \in \mathcal{G}$ and, by [11,§9.2.(7a)], $\mathrm{codim}F = \dim H \leq c$. Besides, $\mathrm{codim}F \geq$

$\max_{\mathfrak{g}\in\mathcal{G}}\{\mathrm{codim}\mathfrak{g}\} = g$. Now let M be a linear subspace of E complementary to F with a corresponding linear projection σ of E onto M. Of course, $m = \dim M = \mathrm{codim} F$ and $g \leq m \leq c$. By assumption, every $c - g + 2 \geq m - g + 2$ of flats $\sigma(\mathfrak{g})(\mathfrak{g} \in \mathcal{G})$ have a point in common. Moreover, F and $\sigma(\mathfrak{g})$ are complementary in $\mathfrak{g}$, so that $0 \leq \min_{\mathfrak{g}\in\mathcal{G}}\{\dim\sigma(\mathfrak{g})\} = \min_{\mathfrak{g}\in\mathcal{G}}\{\mathrm{codim}F\text{-}\mathrm{codim}\mathfrak{g}\} = m - g$. By a variant of Helly's theorem for finite-dimensional flats [3,Lemma 1] there is a point $p \in \bigcap_{\mathfrak{g}\in\mathcal{G}} \sigma(\mathfrak{g})$ and $p + F$ is a flat of codimension at most c contained in all members of $\mathcal{G}$. □

The last result of this section in R^d is due to Kołodziejczyk [9,Th.1].

Theorem 3.4. *A nonempty finite family $\mathcal{F}$ of convex sets in a real linear space E contains a common half-flat of codimension c, where $c < \dim E$ is a nonnegative integer, if and only if every subfamily of $2c + 2$ or fewer members of $\mathcal{F}$ contains a common half-flat of codimension c.*

Proof. Again, we are concerned with the sufficiency of the condition. Without loss of generality, we can assume that all half-flats are algebraically closed and that $\mathfrak{F}$ contains at least $2c + 3$ members. If $c = \dim E - 1$, i.e. we deal with halflines in a finite-dimensional linear space, then the conclusion follows from Katchalski's theorem [6,Th.C]. Hence, let $c < \dim E - 1$ in the sequel. Since $2c + 2 \geq c + 2$, by Theorem 3.2, $\mathcal{F}$ contains a common flat $\mathfrak{H}$ of codimension $c + 1$. Suppose, with no loss of generality, that $\circ \in \mathfrak{H}$ and let $\mathfrak{M}$ be a $(c + 1)$-dimensional linear subspace of E complementary to $\mathfrak{H}$ with a corresponding linear projection σ of E onto $\mathfrak{M}$. Pick out a subfamily of $2c + 2$ members of $\mathcal{F}$. Since it contains a common half-flat $\mathfrak{G}$ of codimension c as well as the flat $\mathfrak{H}$, it is immediate that it contains $\mathrm{conv}(\mathfrak{H} \cup \mathfrak{G})$, so that also a half-flat of codimension c whose relative algebraic boundary is a flat parallel to $\mathfrak{H}$. Thus, by initial assumption, every $2c + 2$ of convex sets $\sigma(S)(S \in \mathcal{F})$ in $\mathfrak{M}$ contain a common halfline. Again, by Katchalski's theorem, there is a halfline λ in $\mathfrak{M}$ common to all sets $\sigma(S)(S \in \mathcal{F})$. Then however $\lambda + \mathfrak{H}$ is a half-flat of codimension c in E common to all members of $\mathfrak{F}$ and the proof is complete. □

4. REMARK

One of primary goals of this paper was to prepare the reader for the proof of an important generalization of Helly's theorem due to Horn. It states in the simplest form that if every n members of a family of compact convex sets in R^d $(1 \leq n \leq d + 1)$ have a nonempty intersection, then every $(d - n)$-dimensional flat in R^d ($\dim\varnothing = -1$ by definition) is contained in a $(d - n + 1)$-dimensional flat which intersects all members of the family. Variants of this result including its converse have been established in R^d by the method of the so-called n-sets in [7,Th.II] and by the method of the dual cone in [16,Th.12], [17,Th.6.2]. The approach of [7] can be generalized to real locally convex topological linear spaces but the question remains whether the alternative approach based on the dual cone can also be extended to infinite dimensions. The detailed analysis of the finite-dimensional case reveals that the situation is very delicate. In a possible proof surely the Proposition 2.6 will interfere as it is in its proof in R^d.

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