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CHARACTERISATIONS OF KIEPERT, JARABEK AND FEUERBACH HYPERBOLAS

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Abstract. *We obtain several characterisations of the Kiepert, Jarabek, and Feuerbach hyperbolas of a scalene triangle ABC using the family of triangles with vertices on Euler lines of triangles BCP, CAP, and ABP for a variable point P in the plane and the notion of orthologic triangles.*

1. INTRODUCTION

Among conics which pass through the vertices A, B, C of the scalene triangle ABC and its orthocentre H the most interesting are Feuerbach, Kiepert, and Jarabek hyperbolas. These are equilateral hyperbolas that go through the incentre I, the centroid G, and the circumcentre O, respectively. They have been extensively studied in the past. The following are some more recent papers that consider them: [1], [2], [6], [7], [5], [11], [19], [18], and [22].

This paper presents new characterisations of the Kiepert, Jarabek, and Feuerbach hyperbolas associated to a scalene triangle ABC. We use the same method for all three hyperbolas. Our idea is, for a given real number $\lambda \neq 1$, to associate to every point P a triangle $P_a^\lambda P_b^\lambda P_c^\lambda$ and to look for triangles XYZ having the property that P lies on a hyperbola if and only if the triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and XYZ are orthologic. Hence, to characterise a hyperbola, choose for a given λ a suitable triangle XYZ and the triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ will make a family of triangles in the plane with the property that $P_a^\lambda P_b^\lambda P_c^\lambda$ and XYZ are orthologic if and only if the point P lies on the hyperbola. For example, when $\lambda = -3$, then $P_a^\lambda P_b^\lambda P_c^\lambda$ is the triangle $F_aF_bF_c$ formed by centers F_a, F_b, and F_c of the nine-point circles of the triangles BCP, CAP, and ABP. As an application of our main theorem a point P lies on the Kiepert hyperbola if and only if $F_aF_bF_c$ is orthologic to the first Brocard triangle $A_bB_bC_b$ of ABC. In other words, for the Kiepert hyperbola, the first Brocard triangle is a good selection for the triangle XYZ in our characterisation using the relation of orthology. Similarly, the orthic triangle $A_oB_oC_o$ (on feet of altitudes) and the extriangle $A_eB_eC_e$ (on excenters) are examples of the triangle XYZ for the Jarabek and Feuerbach hyperbolas.

Recall that triangles UVW and XYZ are *orthologic* provided the perpendiculars at vertices of UVW onto sides YZ, ZX, and XY of XYZ are concurrent. The point of concurrence of these perpendiculars is denoted by $[UVW, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto sides VW, WU, and UV of UVW are concurrent at the point $[XYZ, UVW]$.

In this definition and throughout this paper all triangles are nondegenerate, that is, their vertices are not collinear. The last assumption implies that in our approach we must exclude some points P so that ours are characterisations of three named hyperbolas without a small number of their points.

In order to describe triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ more precisely, we need the following definitions.

Let $\lambda \neq -1$ be a real number. For points X and Y, let $[X, Y, \lambda]$ equal X when $X = Y$ or a unique point P on the line XY such that $|XP| / |PY| = \lambda$ when $X \neq Y$.

For a triangle ABC, let $W(ABC)$ denote the complement in the plane of the union of the side lines BC, CA, AB. For a point P in $W(ABC)$, let G_a^P, G_b^P, G_c^P and O_a^P, O_b^P, O_c^P denote centroids and circumcentres of triangles BCP, CAP, ABP, respectively. Let

$$P_a^\lambda = [O_a^P, G_a^P, \lambda], \qquad P_b^\lambda = [O_b^P, G_b^P, \lambda], \qquad \text{and} \qquad P_c^\lambda = [O_c^P, G_c^P, \lambda],$$

and let $\mathcal{F}_\lambda$ denote the function which associates to a point P the triangle $P_a^\lambda P_b^\lambda P_c^\lambda$. Note that P_a^λ, P_b^λ, and P_c^λ are similarly placed points on Euler lines of the (scalene) triangles BCP, CAP, and ABP. It should also be said that, while λ can be any real number different from -1, it is fixed and not to be thought of as a parameter which describes the position of P on the hyperbola.

In the section 4 we shall prove that the points P_a^λ, P_b^λ, and P_c^λ are collinear if and only if P lies on a plane quartic denoted here by Q_λ. Hence, the domain of the function $\mathcal{F}_\lambda$ is the complement $W_\lambda(ABC)$ of Q_λ in $W(ABC)$. Let $V_\lambda(ABC)$ denote the complement of the circumcircle γ_0 of ABC in $W_\lambda(ABC)$.

Let γ be a curve in the plane. Let $\mathcal{F}$ be a function from a subset S of the plane that associates to each point P of S a triangle $\mathcal{F}(P)$. A triangle XYZ is $(\mathcal{F}, \gamma)$-*simple in* S provided XYZ is orthologic to $\mathcal{F}(P)$ if and only if a point P is in the set $\gamma \cap S$.

Let γ_F, γ_J, and γ_K denote the Feuerbach, Jarabek, and Kiepert hyperbola of the triangle ABC, respectively. With the above definitions and notation we can formulate the results of this paper as contributions to the following problem.

Problem. For $\gamma \in \{\gamma_F, \gamma_J, \gamma_K\}$, find triangles that are $(\mathcal{F}_\lambda, \gamma)$-simple in $V_\lambda(ABC)$.

Observe that when we know that a triangle XYZ is $(\mathcal{F}_\lambda, \gamma)$-simple in $V_\lambda(ABC)$ for $\gamma \in \{\gamma_F, \gamma_J, \gamma_K\}$, then we have the following characterisation of γ:

The hyperbola γ is the closure of all points P in $V_\lambda(ABC)$ such that the triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and XYZ are orthologic.

Our main result is the following theorem. The notation in it and the conditions M_i will be explained below.

Theorem 1. *Let $\lambda \neq -1$ be a fixed real number. For $X = K, J, F$, for $i \in I_X$, and for all triangles ABC which satisfy the condition M_i, any triangle LMN homothetic to the triangle $A_iB_iC_i$ is $(\mathcal{F}_\lambda, \gamma_X)$-simple in $V_\lambda(ABC)$.*

Let $I_K = \{b, br, K, Kk, Km, u, ub, v, vb, un, vn\}$, $I_J = \{h, o, r, t, tr, w, J, Jh, Jo, Jt\}$, and $I_F = \{e, ep, eq, er, k, kr, p, pe, pp, F, Fe, Fk, Fp\}$. The triangles XYZ which we prove in this paper to provide solutions to the above problem are of the form $A_iB_iC_i$ where i is an element of either I_K, I_J, or I_F. For each element i of these three sets we define a triangle $A_iB_iC_i$ by describing the vertex A_i. The vertices B_i and C_i have analogous descriptions.

Let A_e be the centre of the A-excircle, A_{ep} the projection of A_e onto BC, the point A_{er} is the reflection of A_e at BC, the vertex A_k is the second intersection of the bisector of the angle A with the circumcircle, A_{kr} is the reflection of A_k at BC, the point A_p is the projection of the incentre I onto BC, the vertex A_{pe} is the projection of A_p onto B_eC_e, the vertex A_{pp} is the projection of A_p onto AI, the point A_{eq} is the reflection of A_e at A_pI, A_F is a projection onto BC of any point different from the circumcentre O and two intersections of IO with the circumcircle of ABC on the line IO joining the incentre with the circumcentre, A_{Fp} is a projection onto B_pC_p of any point different from central point X_{65} [9] on line IO and two intersections of IO with the incircle of ABC, A_{Fe} and A_{Fk} are projections onto B_eC_e and B_kC_k of any point on IO different from the incentre I and two intersections of IO with the circumcircles of $A_eB_eC_e$ and ABC.

The point A_b is the projection of the Grebe-Lemoine point K onto the perpendicular bisector of BC, the point A_{br} is the reflection of A_b at BC, the vertex A_K is the projection of any point on the line KO different from the circumcentre O and two intersections of KO with the circumcircle of ABC onto BC, the vertex A_{Km} is the projection of any point different from the circumcentre O on the line KO onto the perpendicular bisector of BC, the vertex A_{Kk} is the projection of any point different from the circumcentre O and the Grebe-Lemoine point K on the line KO onto the line A_bK, vertices A_u and A_{um} are the vertex and the centre of the equilateral triangle build on BC towards inside, A_v and A_{vm} are the vertex and the centre of the equilateral triangle build on BC towards outside, A_{ub} and A_{vb} are projections of the Grebe-Lemoine points of $A_uB_uC_u$ and $A_vB_vC_v$ onto perpendicular bisectors of B_uC_u and B_vC_v.

A_h is the second intersection of altitude line AH with the circumcircle, A_o is the projection of A onto BC, the point A_r is the reflection of A at BC, the intersection of tangents to the circumcircle at B and C is A_t, the reflection of A_t at BC is A_{tr}, A_w is the intersection of common tangents of the A-excircle with B-excircle and C-excircle, A_{Jh} and A_{Jo} are the projections onto B_hC_h and B_oC_o of any point X on the Euler line HO of ABC different from the orthocentre H and two intersections of the Euler line with the circumcircle and the nine-point (or Feuerbach) circle of ABC, and A_J and A_{Jt} are the projections onto BC and B_tC_t of any point X on the Euler line of ABC different from the circumcentre O and two intersections of the Euler line with the circumcircle of ABC and the circumcircle of $A_tB_tC_t$, respectively.

The conditions M_i will be described in a separate section. They are certain relations among side lengths a, b, c of ABC besides the standard assumption that ABC is a scalene triangle. Observe that γ_K, γ_J, and γ_F are not defined when ABC is an equilateral triangle.

Let us observe that the claim that the first Brocard triangle $A_bB_bC_b$ is $(\mathcal{F}_\lambda, \gamma_K)$-simple in $V_\lambda(ABC)$ is analogous to the observation in [12] which shows that triangles ABC and $X_hY_hZ_h$ are orthologic and the point $[ABC, X_hY_hZ_h]$ traces the Kiepert hyperbola as h goes through reals, where X_h, Y_h, and Z_h are vertices of similar isosceles triangles built on sides of ABC.

Similarly, the claim that the pedal triangle $A_pB_pC_p$ is $(\mathcal{F}_\lambda, \gamma_F)$-simple in $V_\lambda(ABC)$ is analogous to the observation that triangles ABC and $P_hQ_hR_h$ are orthologic and the point $[ABC, P_hQ_hR_h]$ traces the Feuerbach hyperbola as h goes through reals, where P_h, Q_h, and R_h are intersections of circles concentric to the incircle with perpendiculars through the incentre to sides.

2. PRELIMINARIES ON COMPLEX NUMBERS

In our proofs we shall use complex numbers because they lead to the simplest expressions. Hence, our proofs are entirely algebraic. Every book on the use of complex numbers in geometry from the references below gives excellent and adequate introductions to this technique of proof. In this section we give only the most basic notions and conventions.

A point P in the Gauss plane is represented by a complex number p. This number is called the *affix* of P and we write $\tilde{P} = p$ or $P(p)$ to indicate this. The complex conjugate of p is denoted $\bar{p}$.

In the sections on the Kiepert and Jarabek hyperbolas, we follow the standard assumption that the vertices A, B, and C of the reference triangle are represented by numbers u, v, and w on the unit circle so that the circumcentre O of ABC is the origin. Hence, the affix of O is number 0 (zero) and complex conjugates of u, v, and w are $1/u$, $1/v$, and $1/w$.

Most interesting points, lines, circles, curves,... associated with the triangle ABC are expressions that involve symmetric functions of u, v, and w that we denote as follows.

$$\sigma = u + v + w, \quad \tau = v\,w + u\,w + u\,v, \quad \mu = u\,v\,w,$$

$$\sigma_a = -u + v + w, \quad \sigma_b = u - v + w, \quad \sigma_c = u + v - w,$$

$$\tau_a = -v\,w + w\,u + u\,v, \quad \tau_b = v\,w - w\,u + u\,v, \quad \tau_c = v\,w + w\,u - u\,v,$$

$$\mu_a = v\,w, \quad \mu_b = w\,u, \quad \mu_c = u\,v, \quad \delta_a = v - w,$$

$$\delta_b = w - u, \quad \delta_c = u - v, \quad \zeta_a = v + w, \quad \zeta_b = w + u, \quad \zeta_c = u + v.$$

For each $k \geq 2$, σ_k, σ_{ka}, σ_{kb}, and σ_{kc} are derived from σ, σ_a, σ_b, and σ_c with the substitution $u = u^k$, $v = v^k$, $w = w^k$. In a similar fashion we can define analogous expressions using letters τ, μ, δ, and ζ. We shall use corresponding small Latin letters to denote analogous symmetric functions in a, b, and c (lengths of sides of ABC). For example, $m = a\,b\,c$, $s = a + b + c$, $t = b\,c + c\,a + a\,b$, $z_a = b + c$, and $s_{2a} = b^2 + c^2 - a^2$.

Let us close these preliminaries with few words on analytic geometry that we shall use.

In triangle geometry lines play an important role so that we have special notation $[f,\, g,\, h]$ for the set of all points $P(p)$ that satisfy the equation $f\,p + g\,\bar{p} + h = 0$. When g is a complex conjugate of f and h is a real number, this set is a line.

Let $X(x)$, $Y(y)$, and $Z(z)$ be three points and let ℓ be a line $[f,\, g,\, h]$ in the plane. Then the line XY is $[\bar{x} - \bar{y},\, y - x,\, \bar{y}\,x - \bar{x}\,y]$, the parallel to ℓ through X is $[f,\, g,\, -g\,\bar{x} - f\,x]$ and the perpendicular to ℓ through X is $[f,\, -g,\, g\,\bar{x} - f\,x]$, where g is a complex conjugate of f. The conditions for points X, Y, and Z to be collinear and for lines $[f,\, g,\, h]$, $[k,\, m,\, n]$, and $[r,\, s,\, t]$ to be concurrent are

$$\begin{vmatrix} 1 & \bar{x} & x \\ 1 & \bar{y} & y \\ 1 & \bar{z} & z \end{vmatrix} = 0, \qquad \text{and} \qquad \begin{vmatrix} f & g & h \\ k & m & n \\ r & s & t \end{vmatrix} = 0.$$

3. DESCRIPTION OF CONDITIONS M_i

Let M_0 denote the condition that ABC is a scalene (i. e., not an equilateral) triangle while M_R means that ABC does not have a right angle.

Then M_i is M_0 for i equal to b, br, v, vb, un, vn, K, Kk, Km, J, e, ep, k, kr, p, pe, pp, F, Fp, Fe, and Fk, while M_i is the union of M_0 and M_R for i equal to h, o, t, tr, w, Jh, Jo, and Jt.

For the remaining indices i the condition M_i is the union of M_0 and the requirement that sides a, b, c do not satisfy the relation $N_i = 0$, where $N_u = N_{ub} = 5\sqrt{s\,s_a\,s_b\,s_c} - s_2\sqrt{3}$, $N_r = s_6 - m_2 - a^4\,z_{2a} - b^4\,z_{2b} - c^4\,z_{2c}$, $N_{er} = s^2\,(s^2 - 4\,t)^2 + 4\,m\,(3\,s^3 - 12\,s\,t + 7\,m)$, and

$$N_{eq} = s^2\,(s^2 - 4\,t)^2 + 5\,m\,(3\,s^3 - 12\,s\,t + 11\,m).$$

Observe that $N_u(28\sqrt{3}+2\sqrt{21},\ 28\sqrt{3}+2\sqrt{21},\ 2\sqrt{3}+4\sqrt{21})=0$,

$$N_r(3+2\sqrt{3},\ 3+2\sqrt{3},\ 6+3\sqrt{3}) = N_{er}(1,\ 1,\ \sqrt{2}) = N_{eq}(\sqrt{5}-1,\ \sqrt{5}-1,\ 2)=0,$$

which shows that none of these special conditions does reduce to M_0 in general.

4. PRELIMINARIES FOR PROOFS

Let us first determine the affixes of points P_a^{λ}, P_b^{λ}, and P_c^{λ}. Since the affix of G_a^P is $(p+\zeta_a)/3$ and the affix of O_a^P is $\mu_a(p\bar{p}-1)/n_a$, where $n_a = p+\mu_a\bar{p}-\zeta_a$, it follows that P_a^{λ} has the affix

$$\frac{3\,\mu_a(p\bar{p}-1)+\lambda\,n_a(\zeta_a+p)}{3(\lambda+1)\,n_a}.$$

The affixes of P_b^{λ} and P_c^{λ} are cyclic permutations of the affix of P_a^{λ}.

Observe that points P_a^{λ}, P_b^{λ}, P_c^{λ} are collinear if and only if the point P lies on a quartic Q_{λ} with equation $3\,\mu(2\lambda+3)(p\bar{p}-1)^2+\lambda^2\,n_a\,n_b\,n_c=0$.

The following theorem describes with complex numbers the condition for two triangles to be orthologic. The symmetric nature of the orthology condition $(XYZ,\ PQR)$ below immediately implies all properties of the orthology relation from the introduction.

Theorem 2. *Triangles XYZ and PQR with affixes of vertices x, y, z, p, q, and r are orthologic if and only if* $(XYZ,\ PQR)=0$, *where* $(XYZ,\ PQR)$ *is*

$$x(\bar{q}-\bar{r})+\bar{x}(q-r)+y(\bar{r}-\bar{p})+\bar{y}(r-p)+z(\bar{p}-\bar{q})+\bar{z}(p-q).$$

Proof. The line QR is $[\bar{q}-\bar{r},\ r-q,\ q\bar{r}-\bar{q}r]$ so that the perpendicular per_{QR}^X through X onto QR is the line $[\bar{q}-\bar{r},\ q-r,\ x(\bar{r}-\bar{q})+\bar{x}(r-q)]$. The perpendiculars per_{RP}^Y and per_{PQ}^Z through Y and Z onto RP and PQ have equations that are cyclic permutations of the above equation of per_{QR}^X. These three perpendiculars are concurrent if and only if $\Theta=0$, where Θ denotes the determinant

$$\begin{vmatrix} \bar{q}-\bar{r} & q-r & x(\bar{r}-\bar{q})+\bar{x}(r-q) \\ \bar{r}-\bar{p} & r-p & y(\bar{p}-\bar{r})+\bar{y}(p-r) \\ \bar{p}-\bar{q} & p-q & z(\bar{q}-\bar{p})+\bar{z}(q-p) \end{vmatrix}.$$

But, $\Theta=(XYZ,\ PQR)\,m$, where $m=p(\bar{q}-\bar{r})+q(\bar{r}-\bar{p})+r(\bar{p}-\bar{q})$. Since $m=0$ if and only if points P, Q, and R are collinear (and our assumptions exclude this possibility), we conclude that the triangles XYZ and PQR are orthologic if and only if $(XYZ,\ PQR)=0$. □

5. PROOF OF THEOREM 1 FOR $X = K$ AND $i = b$

In order to determine affixes of the vertices A_b, B_b, and C_b of the first Brocard triangle of ABC, we shall use the fact that they are projections of the Grebe-Lemoine point K onto the perpendicular bisectors of sides.

The point K is the intersection of lines A_mA_n and B_mB_n, where A_m is the midpoint of the side BC and A_n is the midpoint of the altitude AA_o joining the vertex A with the projection A_o of A onto BC. It follows that K has the affix $2(\tau_2-\mu\,\sigma)/(\sigma\,\tau-9\,\mu)$

The perpendicular bisector of the side BC has equation $p - \mu_a \bar{p} = 0$, so that the affix of A_b is $(\zeta_{2a}(u^2 + \mu_a) - 2\,\mu\,\zeta_a)\,/(\sigma\,\tau - 9\,\mu)$. The affixes of B_b and C_b are its cyclic permutations.

Since triangles $A_bB_bC_b$ and LMN are homothetic, there is a point $T(x)$ and a real number $\xi \neq -1$ such that

$$\tilde{L} = \frac{\tilde{A_b} + \xi x}{\xi + 1}, \qquad \tilde{M} = \frac{\tilde{B_b} + \xi x}{\xi + 1}, \qquad \tilde{N} = \frac{\tilde{C_b} + \xi x}{\xi + 1}.$$

Let us observe that points L, M, and N will be collinear if and only if ABC is an equilateral triangle because the the first Brocard triangle $A_bB_bC_b$ is similar to ABC.

The orthology condition for triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and LMN is

$$(P_a^\lambda P_b^\lambda P_c^\lambda,\, LMN) = \frac{\delta_a\,\delta_b\,\delta_c\,j_K\,j_0}{\mu\,(\lambda+1)\,(\xi+1)\,(\sigma\tau - 9\mu)\,n_a\,n_b\,n_c},$$

where $j_0 = p\,\bar{p} - 1$ and

$$j_K = (\tau^2 - 3\mu\sigma)p^2 - \mu^2(\sigma^2 - 3\tau)\bar{p}^2 + (4\mu\sigma^2 - \sigma\tau^2 - 3\mu\,\tau)p - \mu(4\tau^2 - \sigma^2\tau - 3\mu\,\sigma)\bar{p} + \tau^3 - \mu\,\sigma^3.$$

Notice that $j_0 = 0$ is the equation of the circumcircle of ABC while $j_K = 0$ is the equation of the Kiepert hyperbola of ABC since the vertices $A(u)$, $B(v)$, and $C(w)$, the orthocentre $H(\sigma)$, and the centroid $G(\sigma\,/\,3)$ satisfy it. This shows that LMN is $(\mathcal{F}_\lambda, \gamma_K)$-simple in $V_\lambda(ABC)$ whenever ABC is a scalene triangle. □

6. PROOF OF THEOREM 1 FOR $X = J$ AND $i = h$

In order to determine $\tilde{A_h}$, we shall find common solutions of the equation $j_0 = 0$ of the circumcircle and the equation $u\,p - \mu\,\bar{p} + \mu_a - u^2 = 0$ of the altitude line AH. Hence, $\tilde{A_h} = -\mu_a\,/\,u$, $\tilde{B_h} = -\mu_b\,/\,v$, and $\tilde{C_h} = -\mu_c\,/\,w$.

Next we find $\tilde{L}$, $\tilde{M}$, and $\tilde{N}$ as above. Let us observe that these points will be collinear if and only if $\delta_a\,\zeta_a\,\delta_b\,\zeta_b\,\delta_c\,\zeta_c = 0$. In other words, if and only if ABC has a right angle.

The orthology condition for triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and LMN is

$$(P_a^\lambda P_b^\lambda P_c^\lambda,\, LMN) = \frac{-j_J\,j_0\,\delta_a\,\delta_b\,\delta_c}{\mu\,(\lambda+1)\,(\xi+1)\,n_a\,n_b\,n_c},$$

where $j_J = \sigma\,p^2 - \mu\,\tau\,\bar{p}^2 + (\tau - \sigma^2)\,p + (\tau^2 - \mu\,\sigma)\,\bar{p}$.

The equation of the Jarabek hyperbola of ABC is $j_J = 0$ since the vertices $A(u)$, $B(v)$, and $C(w)$, the orthocentre $H(\sigma)$, and the circumcentre $O(0)$ satisfy it. □

7. PROOF OF THEOREM 1 FOR $X = F$ and $i = e$

In contrast with the previous two sections, in order to avoid square roots, here we shall assume that the vertices A, B, and C of the base triangle have affixes u^2, v^2, and w^2, with the same assumption about u, v, and w.

This time $\tilde{L} = (\tau_a + \xi x)\,/(\xi + 1)$, $\tilde{M} = (\tau_b + \xi x)\,/(\xi + 1)$, and $\tilde{N} = (\tau_c + \xi x)\,/(\xi + 1)$ because it is well-known [13] that $\tilde{A_e} = \tau_a$, $\tilde{B_e} = \tau_b$, and $\tilde{C_e} = \tau_c$.

Let us observe that points L, M, and N will be collinear if and only if

$$\frac{4\,\delta_a\,\delta_b\,\delta_c}{\mu\,(\xi+1)^2} = 0.$$

But, this product clearly can never be zero.

The orthology condition for triangles $P_a^\lambda P_b^\lambda P_c^\lambda$ and LMN is

$$(P_a^\lambda P_b^\lambda P_c^\lambda,\, LMN) = \frac{-2\,\delta_a\,\delta_b\,\delta_c\, j_F\, j_0}{\mu\,(\lambda+1)\,(\xi+1)\,(p+\mu_{2a}\,\bar p-\zeta_{2a})\,(p+\mu_{2b}\,\bar p-\zeta_{2b})\,(p+\mu_{2c}\,\bar p-\zeta_{2c})},$$

where

$$j_F = \tau\,p^2 - \mu_3\,\sigma\,\bar p^2 + (\mu\,\sigma + 2\,\mu\,\sigma^2 - \sigma^2\,\tau)\,p + \mu\,(\sigma\,\tau^2 - 2\,\sigma^2\,\mu - \mu\,\tau)\,\bar p + \tau^3 - \sigma^3\,\mu.$$

Observe that $j_F = 0$ is the equation of the Feuerbach hyperbola of ABC since the vertices $A(u^2)$, $B(v^2)$, and $C(w^2)$, the orthocentre $H(\sigma_2)$, and the incentre $I(-\tau)$ satisfy it. □

The proofs of all other cases of the theorem are almost identical to one of the above three proofs so we leave to the reader to fill in the details.

8. CHARACTERISATIONS WITH REFLECTIONS OF P_a^λ, P_b^λ, AND P_c^λ

In this final section we shall give characterisations of the three named hyperbolas using as vertices of variable triangles reflections of points P_a^λ, P_b^λ, and P_c^λ at suitable lines.

For a real number $\lambda \neq -1$, let Q_a^λ, Q_b^λ, and Q_c^λ denote the reflections of points P_a^λ, P_b^λ, and P_c^λ at the sides B_bC_b, C_bA_b, and A_bB_b of the first Brocard triangle of ABC and let $\mathcal{G}_\lambda$ denote the function which associates to a point P the triangle $Q_a^\lambda Q_b^\lambda Q_c^\lambda$. One can show that the points Q_a^λ, Q_b^λ, and Q_c^λ are collinear if and only if P lies on a quintic (curve of order five) R_λ. Hence, the domain of the function $\mathcal{G}_\lambda$ is the complement $W_\lambda(ABC)$ of R_λ in $W(ABC)$. Let $V_\lambda(ABC)$ denote the complement of the circumcircle γ_0 of ABC in $W_\lambda(ABC)$.

Theorem 3. *The first Brocard triangle of a scalene triangle ABC is $(\mathcal{G}_\lambda, \gamma_K)$-simple in $V_\lambda(ABC)$.*

Proof. In section 5 we computed affixes of the vertices A_b, B_b, and C_b of the first Brocard triangle of ABC. Hence, we can now determine equations of its sides and find that the affix of the reflection Q_a^λ of P_a^λ at B_bC_b is the quotient $k\,/(3\,(\tau^2 - 3\,\mu\,\sigma)(\sigma\,\tau - 9\,\mu))$, where k denotes the polynomial $k_1\,(\lambda+3)\,p\,\bar p + \lambda\,\mu^2\,k_2\,\bar p^2 + \lambda\,k_3\,(p+\zeta_a) + k_4\,(p+(\lambda+1)\,\mu_a\,\bar p) + k_5$, with

$$k_1 = u\,\mu\,k_2,\; k_2 = (\tau-\sigma_2)(\sigma\,\tau-9\,\mu),\; k_4 = 3\,(\zeta_{2b}\,v^2 - 2\,\mu\,\zeta_b + \mu_b\,\zeta_{2b})(\zeta_{2c}\,w^2 - 2\,\mu\,\zeta_c + \mu_c\,\zeta_{2c}),$$

$$k_3 = (\mu_a - \zeta_{2a})\,u^6 + \zeta_a\,(3\,\zeta_{2a} - \mu_a)\,u^5 - 9\,\mu_a\,(\mu_a + \zeta_{2a})\,u^4 +$$

$$\zeta_a\,(2\,\zeta_{4a} - 7\,\mu_a\,\zeta_{2a} + 24\,\mu_{2a})\,u^3 - \mu_a\,(\zeta_{2a} - \mu_a)(\zeta_{2a} + 5\,\mu_a)\,u^2 + 3\,\mu_a^2\,\zeta_a\,(\zeta_{2a} - 3\,\mu_a)\,u + 3\,\mu_{4a},$$

$$k_5 = 3\,(\mu_a\,\zeta_{2a} - 4\,\mu_a^2 - \zeta_{4a})\,u^5 - 3\,\mu_a\,\zeta_a\,(5\,\mu_a + \zeta_{2a})\,u^4 +$$

$$3\,(4\,\mu_a\,\zeta_{4a} - 9\,\mu_a^2\,\zeta_{2a} - \zeta_{6a} - 4\,\mu_a^3)\,u^3 + 6\,\mu_a^2\,\zeta_a\,(\zeta_{2a} - \mu_a)\,u^2 - 3\,\mu_a^2\,\zeta_a^2\,(\zeta_{2a} - 3\,\mu_a)\,u - 3\,\mu_a^4\,\zeta_a.$$

The affixes of reflections Q_b^λ and Q_c^λ are cyclic permutations of the affix of Q_a^λ. The orthology condition for triangles $Q_a^\lambda Q_b^\lambda Q_c^\lambda$ and $A_b B_b C_b$ is

$$(Q_a^\lambda Q_b^\lambda Q_c^\lambda,\, A_b B_b C_b) = \frac{\delta_a\, \delta_b\, \delta_c\, j_K\, j_O}{\mu\,(\lambda+1)\,(\sigma\tau - 9\mu)\, n_a\, n_b\, n_c},$$

which completes our proof. □

For a real number $\lambda \neq -1$, let Q_a^λ, Q_b^λ, and Q_c^λ denote reflections of points P_a^λ, P_b^λ, and P_c^λ at the sides B_oC_o, C_oA_o, and A_oB_o of the orthic triangle of ABC and let $\mathcal{G}_\lambda$ denote the function which associates to a point P the triangle $Q_a^\lambda Q_b^\lambda Q_c^\lambda$. One can show that the points Q_a^λ, Q_b^λ, and Q_c^λ are collinear if and only if P lies on a quintic (curve of order five) R_λ. Hence, the domain of the function $\mathcal{G}_\lambda$ is the complement $W_\lambda(ABC)$ of R_λ in $W(ABC)$. Let $V_\lambda(ABC)$ denote the complement of the circumcircle γ_0 of ABC in $W_\lambda(ABC)$.

Theorem 4. *The orthic triangle of a scalene triangle ABC is $(\mathcal{G}_\lambda, \gamma_J)$-simple in $V_\lambda(ABC)$.*

Proof. The affix $(u\,\sigma - \mu_a)/(2\,u)$ of the projection A_o of A onto BC is the common solution of the equations of AH from the section 6 and the equation $n_a = 0$ of the side line BC. Of course, affixes of B_o and C_o are cyclic permutations of the affix of A_o. As above, we can now determine the affix of the reflection Q_a^λ of P_a^λ at B_oC_o as the quotient $k/(3\,(\lambda+1)\,\mu_a\, n_a)$, where k denotes the polynomial

$$k_1\,(\lambda+3)\,p\,\bar p + \lambda\,\mu^2\, k_2\,\bar p^2 + \lambda\, k_3\,(p - \zeta_a) + k_4\,(p + (\lambda+1)\,\mu_a\,\bar p) - k_5,$$

with

$$k_1 = u\,\mu\, k_2, \quad k_2 = -2, \quad k_3 = \zeta_a\, u^2 - 3\,\delta_a^2\, u + 3\,\mu_a\,\zeta_a,$$
$$k_4 = 3\,(\zeta_a\, u^2 - \delta_a^2\, u + \mu_a\,\zeta_a), k_5 = 3\,(\zeta_{2a}\, u^2 - \delta_a^2\,\zeta_a\, u + \mu_a\,\zeta_a^2).$$

The affixes of reflections Q_b^λ and Q_c^λ are cyclic permutations of the affix of Q_a^λ. The orthology condition for triangles $Q_a^\lambda Q_b^\lambda Q_c^\lambda$ and $A_oB_oC_o$ is

$$(Q_a^\lambda Q_b^\lambda Q_c^\lambda,\, A_o B_o C_o) = \frac{\delta_a\, \delta_b\, \delta_c\, j_J\, j_O}{2\,\mu\,(\lambda+1)\, n_a\, n_b\, n_c}.$$

□

Remark. Instead of reflecting at the sides of the orthic triangle $A_oB_oC_o$ one can reflect at sides of triangles $A_hB_hC_h$ and $A_tB_tC_t$ and use any of these three triangles in the above theorem.

For a real number $\lambda \neq -1$, let Q_a^λ, Q_b^λ, and Q_c^λ denote reflections of points P_a^λ, P_b^λ, and P_c^λ at the angle bisectors AI, BI, and CI of the triangle ABC and let $\mathcal{G}_\lambda$ denote the function which associates to a point P the triangle $Q_a^\lambda Q_b^\lambda Q_c^\lambda$. One can show that the points Q_a^λ, Q_b^λ, and Q_c^λ are collinear if and only if P lies on a quintic (curve of order five) R_λ. Hence, the domain of the function $\mathcal{G}_\lambda$ is the complement $W_\lambda(ABC)$ of R_λ in $W(ABC)$. Let $V_\lambda(ABC)$ denote the complement of the circumcircle γ_0 of ABC in $W_\lambda(ABC)$.

Theorem 5. *The extriangle $A_eB_eC_e$ of a scalene triangle ABC is $(\mathcal{G}_\lambda, \gamma_F)$-simple in $V_\lambda(ABC)$.*

Proof. Under the assumptions made in section 7, we determine the affix of the reflection Q_a^λ of P_a^λ at B_eC_e as the quotient $k\,/(3\,(\lambda+1)\,\mu_a\,(p+\mu_a^2\,\bar{p}-\zeta_{2a}))$, where k denotes the polynomial $\mu^2\,(\lambda+3)\,p\,\bar{p}+\lambda\,\mu^2\,\mu_a^2\,\bar{p}^2+\lambda\,k_3\,(p-\zeta_{2a})+k_4\,(p+(\lambda+1)\,\mu_a^2\,\bar{p})-3\,\mu_a\,k_5$, with

$$k_3=(3\,\mu_a+\zeta_{2a})\,u^2-3\,\mu_a^2,\quad k_4=3\,\mu_a\,(u^2-\mu_a),\quad k_5=(\mu_a+\zeta_{2a})\,u^2-\mu_a\,\zeta_{2a}.$$

The affixes of reflections Q_b^λ and Q_c^λ are cyclic permutations of the affix of Q_a^λ. The orthology condition for triangles $Q_a^\lambda Q_b^\lambda Q_c^\lambda$ and $A_eB_eC_e$ is

$$(Q_a^\lambda Q_b^\lambda Q_c^\lambda,\,A_eB_eC_e)=\frac{\delta_a\,\delta_b\,\delta_c\,j_F\,j_0}{\mu\,(\lambda+1)\,(p+\mu_{2a}\,\bar{p}-\zeta_{2a})\,(p+\mu_{2b}\,\bar{p}-\zeta_{2b})\,(p+\mu_{2c}\,\bar{p}-\zeta_{2c})}.$$

□

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