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AN EXTENSION THEOREM FOR BILINEAR FORMS

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Apart from the fundamental interest in a *weak* characterization of quadratic forms the possibility to extend quadratic forms defined on subspaces of a vector space (P,K) to a common form on P is helpfull in the representation of higher dimensional geometries as well as in characterizing quadrics by their plane sections. In the case that all vector subspaces endowed with a quadratic form are three-dimensional, there are extension theorems, first under additional conditions by J.Tits [11], F.Buekenhout [1], H.-J.Kroll [4], K.J.Dienst, H.Mäurer [2], E.M.Schröder [7], H.Mäurer [6] and finally in the most general form by E.M.Schröder [8],[10].

Apparently there are only two papers investigating under which conditions quadratic forms defined on two-dimensional vector subspaces posess a common extension. In [9], (8.7) E.M.Schröder showed how to extend quadratic forms defined on two-dimensional vector subspaces of a given three-dimensonal vector space by employing the hexagram-condition. In [5] euclidean spaces of arbitrary dimension are described by an axiom system relating only to incidence and congruence. Taking euclidean planes for granted, it is required that every plane of the space is an euclidean plane. By a further axiom all planes are made compatible. Based on the method of [4] it is possible to construct a quadratic form which describes the cogruence relation. For this construction H.Karzel gave a direct proof in [3]. Implicitly in his proof in the case of characteristic $\neq 2$ anisotropic symmetric bilinear forms are extended from two-dimensional vector subspaces to the vector space containing them.

In this paper we prove an extension theorem for symmetric bilinear forms defined on two-dimensional vector subspaces. In the case of characteristic 2 our proof is not valid for quadratic forms but only for bilinear forms with sufficiently many anisotropic vectors. On the other side in the case of a characteristic $\neq 2$ Satz (3.2) in [8] is a direct conclusion of our theorem.

Gratefully at the end of section 1 we repeat a comment of this paper's referee, that our theorem might be deduced from a general representation theorem for polarities. But in section 2 the extension will be constructed directly.

1. The Extension Theorem

From now on P will be an at least two-dimensional vector space over a commutative field K and $\mathfrak{L}_2$ the set of all two-dimensional vector subspaces. On each vector subspace $E \in \mathfrak{L}_2$ let $q_E : E \times E \to K$ be a symmetric bilinear form. For vectors $a, b \in P$ let $\langle a,b \rangle$ be the vector subspace spanned by them. For linearly independent vectors a, b we write

$$q(a,b) := q_{\langle a,b \rangle}(a,b).$$

If all bilinear forms $q_E, E \in \mathfrak{L}_2$, can be extended to a common form on P , then the following conditions are necessary :

V1 If $a,b,c \in P$ are linearly independent and $0 = q(a,b) = q(a,c)$, then $0 = q(a,b+c)$.

V2 $I := \{x \in P : \exists\, E \in \mathfrak{L}_2$ with $x \in E$ and $q_E(x,x) = 0\} = \{x \in P : \forall\, F \in \mathfrak{L}_2$ with $x \in F$, $q_F(x,x) = 0\}$.

To manage the extension in the case of characteristic 2 we need anisotropic vectors in every non-trivial plane. With $V := P \setminus I$ we make use of the following property:

V3 If $E \in \mathfrak{L}_2$ with $E \cap V = \emptyset$, then $q_E = 0$.

Theorem. *If the bilinear forms $q_E, E \in \mathfrak{L}_2$, satisfy the conditions* **V1, V2** *and* **V3**, *then there is a bilinear form $f : P \times P \to K$ and for every vector subspace $C \in \mathfrak{L}_2$ a scalar $\kappa_C \in K^* := K \setminus \{0\}$, such that for all $a,b \in C$, $f(a,b) = \kappa_C q_C(a,b)$.*

In the following this theorem will be proved.

In the case $V = \emptyset$ for every $C \in \mathfrak{L}_2$ the form q_C is trivial, either by itself if $charK \neq 2$ or because of **V3** if $2 = 0$. Therefore the null-form $f : P \times P \to 0$ is their common extension. In the other trivial case $K = \mathbb{Z}_2$ obviously all κ_C must be chosen equal to 1. So from now on let $V \neq \emptyset$ and $|K| \geq 3$. By using **V2** the conclusion of **V1** can be extended:

(1.1) *Let $B,C,D \in \mathfrak{L}_2$ and $a,b,c \in P$ with $a,b \in C$, $c,a \in B$, $a,b+c \in D$ and $0 = q_C(a,b) = q_B(a,c)$. Then $0 = q_D(a,b+c)$.* □

(1.2) *For $a \in P$ the set $a^\perp := \{x \in P \setminus Ka : q(a,x) = 0\} \cup (Ka \cap I)$ is either a hyperplane or equal to P.*

Proof. (1) Because of **V1, V2** and (1.1) the set $a^\perp$ is a vector subspace.

(2) Let $a \in V$. For every $x \in P \setminus Ka$ we have $q(a, x - q(a,x) q_{\langle x,a \rangle}(a,a)^{-1} a) = 0$, therefore $P = Ka \oplus a^\perp$.

(3) Let $a \in I$ and $b \in P$ with $q(a,b) \neq 0$. We have $Ka - b \overset{\mathbf{V3}}{\subset} V$ for $charK = 2$ and $\{\alpha a - b : \alpha \neq \frac{q_{\langle a,b \rangle}(b,b)}{2q(a,b)}\} \subset V$ for $charK \neq 2$. Therefore let $\alpha, \beta \in K$ with $\alpha \neq \beta$ and $\alpha a - b, \beta a - b \in V$. Then $(\alpha a - b)^\perp \cap (\beta a - b)^\perp \overset{(1.1)}{\subset} a^\perp$. Because of $(Ka - b) \cap a^\perp = \emptyset$ and (2) the set $Ka \oplus ((\alpha a - b)^\perp \cap (\beta a - b)^\perp)$ is a hyperplane. □

The mapping $a \to a^\perp$ has the property

$$\textit{If } a \in b^\perp \textit{ then } b \in a^\perp$$

and therefore induces a (possibly degenerate) polarity on the projective derivation of (P,K). If one can prove that this polarity can be represented by a 0-symmetric semibilinear form $f : P \times P \to K$, then for every $C \in \mathfrak{L}_2$ the forms q_C and $f|_{C\times C}$ only differ by a factor κ_C.

The following proof of the extension theorem does not make use of a representation theorem for polarities.

2. Proof of the Extension Theorem

First we show that for vector subspaces $A,B,C \in \mathfrak{L}_2$ and vectors $a,b,c \in P$ with $a,b \in C$, $b,c \in A$, $c,a \in B$ the scalar

$$\Delta(a,b,c) := q_C(a,a)q_A(b,b)q_B(c,c) - q_C(b,b)q_A(c,c)q_B(a,a)$$

vanishes. This is obvious for $\{a,b,c\} \cap I \neq \emptyset$ or for linearly dependent a,b,c . Just as obvious:

(2.1) *Let $a,b,c \in V$ be linearly independent and $\xi \in K^*$. Then $\Delta(a,b,c) = 0$ if and only if $\Delta(\xi a,b,c) = 0$.* □

(2.2) *Let $a,b,c \in V$ be linearly independent, $q(b,c)q(c,a) \neq 0$ and $\langle a,b \rangle$ non-degenerate* [1]. *Then $\Delta(a,b,c) = 0$.*

Proof. Because of (2.1) let

(1) $q(a,c-a) = q(b,c-b) = 0$.

As q_C is non-degenerate there is a vector $s \in C$ with

(2) $q_C(a,a-s) = q_C(b,b-s) = 0$.

Therefore

(3) $q_C(q_C(b,b)a - q_C(a,a)b, s) = 0$.

From the quoted equations and (1.1) we get:

(4) $q(a,c-s) = q(b,c-s) = 0$ by (1),(2),
(5) $q(q_C(b,b)a - q_C(a,a)b, c-s) = 0$ by (4),
(6) $q(c, q_C(b,b)a - q_C(a,a)b) = 0$ by (5),(3).

After multiplying

(7) $q(c, q(a,c)c - q_B(c,c)a) = 0$

with $q_C(b,b)$ and (6) with $q_B(c,c)$ and using **V1** we obtain

[1] $\langle a,b \rangle = C$ and q_C are called *non-degenerate*, if the radical $Rad\, q_C := \{x \in C : \forall\, y \in C,\ q_C(x,y) = 0\} = \{0\}$.

$$0 = q(c, q_C(b,b)q(a,c)c - q_B(c,c)q_C(a,a)b) \overset{(1)}{=} q_C(b,b)q_B(a,a)q_A(c,c) - \\ - q_B(c,c)q_C(a,a)q_A(b,b).$$

□

(2.3) *Let $a,b,c \in V$ be linearly independent.*

(1) *If $\langle c,a\rangle$ is degenerate, then $q(c,a) \neq 0$.*

(2) *If there is a vector $d \in \langle b,c\rangle \cap V$ with*

(a) $\Delta(a,b,d) = 0$ *and* (b) $\Delta(a,d,c) = 0$, *then* $\Delta(a,b,c) = 0$.

(3) *If $q(a,b) = 0$ and $\langle a,b\rangle \cap V = K^*a \cup K^*b$, then $K = \mathbb{Z}_3$.*

Proof.(1) Since $c,a \in V$, $Rad\ q_B$ is one-dimensional and $c,a \notin Rad\ q_B$.

(2) $$\frac{q_C(a,a)q_A(b,b)}{q_C(b,b)} \overset{(a)}{=} \frac{q_A(d,d)q_{\langle d,a\rangle}(a,a)}{q_{\langle d,a\rangle}(d,d)} \overset{(b)}{=} \frac{q_A(c,c)q_B(a,a)}{q_B(c,c)}.$$ □

(2.4) *For all $a,b,c \in P$, $\Delta(a,b,c) = 0$.*

Proof. Let $a,b,c \in V$ be linearly independent. We discuss the cases not considered in (2.2):

(1) Let $K \neq \mathbb{Z}_3$, $q(b,c) = 0$, $q(c,a) \neq 0$ and $q(a,b) = 0$.

By (2.3.3) there is a $\delta \in K^*$ with $b - \delta c \in V$, and $q(a, b-\delta c) \overset{\mathbf{V1}}{\neq} 0$. Since $b,c \in V$, $q(b,b-\delta c)q(c,b-\delta c) \neq 0$. By (2.3.1) neither $\langle a,b\rangle$ nor $\langle b-\delta c,c\rangle$ are degenerate. By (2.2) we obtain $\Delta(a,b,b-\delta c) = 0$ and $\Delta(b-\delta c,c,a) = 0$ and so $\Delta(a,b,c) = 0$ by (2.3.2).

(2) Let $K \neq \mathbb{Z}_3$ and $q(b,c) = q(c,a) = q(a,b) = 0$.

Because of (2.3.3) there is a $\delta \in K^*$ with $b - \delta c \in V$ and by (1) we obtain $\Delta(b,a,b-\delta c) = 0$ and $\Delta(c,a,b-\delta c) = 0$.

(3) If $q(a,b) = 0$ and $q(b,c)q(c,a) \neq 0$, then C is non-degenerate by (2.3.1) and so

$$\Delta(a,b,c) \overset{(2.2)}{=} 0.$$

(4) If A,B are degenerate and C non-degenerate, then $0 \overset{(2.3.1)}{\neq} q(b,c)q(c,a)$.

(5) Let A,B,C be degenerate.

Because of (2.1) let $a-b \in Rad\ q_C$ and $c-a \in Rad\ q_B$. Then $q(a,b-c) = 0$ by **V1**.

If $D := \langle a,b-c\rangle$ is non-degenerate, then $b-c \in V$ and $\Delta(a,b-c,c) = 0$, $\Delta(a,b-c,b) = 0$ by (4).

If D is degenerate, then $b-c \in I$ and so $0 = q_A(b,b) - 2q(b,c) + q_A(c,c)$. Since A is degenerate, $0 = q_A(b,b)q_A(c,c) - q(b,c)^2$. Together we obtain $q_A(b,b) = q_A(c,c)$. Since $a-b \in Rad\ q_C$ we have $q_C(a,a) = q_C(b,b)$, and $c-a \in Rad\ q_B$ implies $q_B(c,c) = q_B(a,a)$.

(6) Let $K = \mathbb{Z}_3$ and $q(b,c) = 0,\ q(c,a) \neq 0,\ q(a,b) = 0$.
If $b - c \in V$, then as in the proof of (1), $\Delta(a,b,c) = 0$.
Let
(a) $a - b \in I$ as well as (b) $b - c \in I$.
Then $q_C(a,a) = -q_C(b,b)$ and $q_A(b,b) = -q_A(c,c)$. We will show $q_B(a,a) = q_B(c,c)$, and therefore $\Delta(a,b,c) = 0$.
Let $\delta \in \{1,-1\}$ with
(c) $q(\delta a - c, c) = 0$.
From the quoted equations and (1.1) we get:
(d) $q(\delta a - c, b) = 0$ because of $q(a,b) = q(b,c) = 0$,
(e) $q(\delta a - c, b - c) = 0$ by (c), (d),
(f) $q(\delta a - b, b - c) = 0$ by (e), (b),
(g) $q_{\langle \delta a - b, b - c\rangle} = 0$ by (f), (a), (b).
Therefore $0 \neq q(\delta a - c, \delta a - c) = q(\delta a - c, \delta a) = q_B(a,a) - q_B(c,c)$.

(7) Let $K = \mathbb{Z}_3$ and
(a) $q(b,c) = q(c,a) = q(a,b) = 0$.
Further let *either* $a - b, b - c \in V$ and $\delta := 1$ *or* $a - b, b - c \in I$ and $\delta := -1$.
Then
(b) $q_C(a,a) = \delta q_C(b,b)$ as well as (c) $q_A(b,b) = \delta q_A(c,c)$.
From the quoted equotations and (1.1) we get:
(d) $q(a - b, a + \delta b + c) = 0$ by (b), (a)
(e) $q(b - c, a + \delta b + c) = 0$ by (c), (a)
(f) $q(a - c, a + \delta b + c) = 0$ by (d), (e)
$q(a - c, a + c) = 0$ by (f), (a). □

Let $e \in V$ be a fixed vector. For $\alpha \in K$ let $f(\alpha e, \alpha e) := \alpha^2$. For $a \in P \setminus Ke$ and $E := \langle a, e\rangle$ let $f(a,a) := \dfrac{q_E(a,a)}{q_E(e,e)}$. Then (2.4) implies

(2.5) *For $C \in \mathfrak{L}_2$ and all $a,b \in C \cap V$:* $\dfrac{f(a,a)}{q_C(a,a)} = \dfrac{f(b,b)}{q_C(b,b)} =: \kappa_C$. □

For $C \in \mathfrak{L}_2$ with $C \cap V = \emptyset$ let $\kappa_C := 1$.

(2.6) *If $C \in \mathfrak{L}_2$ and $a \in C,\ \beta \in K$, then $\kappa_C q_C(a, \beta a) = \beta f(a,a)$.* □

For $C \in \mathfrak{L}_2$ and $a,b \in C$, $f(a,b) := \kappa_C q_C(a,b)$ is welldefined because of (2.5), (2.6).

(2.7) *If $C \in \mathfrak{L}_2$ and $a,b \in C$, then $f(a,b) q_C(a,a) = f(a,a) q_C(a,b)$.* □

(2.8) *$f : P \times P \to K$ is a symmetric bilinear form.*

Proof. Let $a,b,c \in P$ and $\lambda \in K$. By definition $f(a,\lambda b) = \lambda f(a,b)$ and $f(a,b) = f(b,a)$.

(1) If a,b,c are linearly dependent, then $f(a,b+c) = f(a,b) + f(a,c)$.

From now on let a,b,c be linearly independent and $B := \langle c,a\rangle, C := \langle a,b\rangle, D := \langle a,b+c\rangle$.

(2) Because of $q_C(a,\ f(a,\ b)a - f(a,\ a)b) \overset{(2.7)}{=} 0$ and $q_B(a,\ f(a,\ c)a - f(a,\ a)c) = 0$ and (1.1), we get $q_D(a,\ (f(a,\ b) + f(a,\ c))a - f(a,\ a)(b+c)) = 0$. This and $q_D(a, f(a,b+c)a - f(a,a)(b+c)) \overset{(2.7)}{=} 0$ further implies

$$(f(a,b) + f(a,c) - f(a,b+c))q_D(a,a) = 0.$$

Therefore for $a \in V$ we obtain $f(a,b+c) = f(a,b) + f(a,c)$.

(3) If two of the scalars $f(a,b), f(a,c), f(a,b+c)$ vanish, then $f(a,b+c) \overset{\mathbf{V1}}{=} f(a,b) + f(a,c)$.

(4) If $B \cup C \subset I$, then $q_B = q_C = 0$ by **V3** and so $q(a,b+c) \overset{\mathbf{V1}}{=} 0$.

(5) Let $2 \neq 0$, $a \notin V$ and $C \cap V \neq \emptyset$. Then there is a $\beta \in K$ with $d := \beta a + b \in V$, and

$$\begin{aligned} f(a+d+c,a+d+c) &\overset{(1)}{=} f(a,a) + 2f(a,d+c) + f(d+c,d+c) \overset{(1)}{=} \\ &= f(a,a) + 2f(a,d+c) + f(d,d) + 2f(d,c) + f(c,c) = \\ &= f(d,d) + 2f(d,a+c) + f(a,a) + 2f(a,c) + f(c,c). \end{aligned}$$

Because of $2 \neq 0$ we get

$$f(a,d+c) - f(a,c) - f(a,d) = f(d,a+c) - f(d,c) - f(d,a) \overset{(2)}{=} 0.$$

Therefore $f(a,b+c) = f(a,b+c) + f(a,\beta a) \overset{(1)}{=} f(a,\beta a+b+c) = f(a,\beta a+b) + f(a,c) \overset{(1)}{=}$ $= f(a,\beta a) + f(a,b) + f(a,c) = f(a,b) + f(a,c)$.

(6) Let $2 = 0$, $a \notin V$, $C \cap V \neq \emptyset$ and $B \cap V \neq \emptyset$. Then $C \setminus V = Ka$ since $q_C(b,b) = $ $= q_C(\beta a+b, \beta a+b)$ for $\beta \in K$.

If $D \cap V \neq \emptyset$, then $b, a+b, c, b+c \in V$, and

$$\begin{aligned} f(a,b+c) &= f(a-b+b,b+c) \overset{(2)}{=} f(a-b,b+c) + f(b,b+c) \overset{(2)}{=} \\ &= f(a-b,b) + f(a-b,c) + f(b,b) + f(b,c) \overset{(2)}{=} f(a,b) - f(b,b) + f(a,c) - f(b,c) + \\ &\quad f(b,b) + f(b,c). \end{aligned}$$

If $D \cap V = \emptyset$, then $q_D = 0$ by **V3** and so $f(a,b+c) = 0$ and $b+c \notin V$. Since $b,c \in V$, $\langle b,c\rangle \setminus V = K(b+c)$. Therefore $b+\delta c \in V$ for $\delta \in K \setminus \{1\}$, and as in the case $D \cap V \neq \emptyset$ we have $f(a,b+\delta c) = f(a,b) + \delta f(a,c)$. Because of (3) let $f(a,c) \neq 0$. Then $f(a,b+\delta c) \neq 0$ by **V1**. For all $\delta \in K \setminus \{1\}$ we obtain $f(a,b) \neq \delta f(a,c)$, and therefore $f(a,b) = f(a,c)$. $\square$

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