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# Codimension Two Homogeneous Submanifolds of Space Forms

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**Abstract.** In this paper we study codimension two homogeneous submanifolds of Space Forms for which the index of minimum relative nullity is small. Such submanifolds have been studied in the case that they are immersed into the Euclidean space. Under this assumption on the relative nullity, we investigate the rigidity of the immersion, which in turn implies that the submanifold is the orbit of an isometric action in the ambient space. We also study the non-rigid case, that is, we completely classify the codimension two non-rigid immersions of Riemannian homogeneous manifolds into the sphere and into the Hyperbolic space.

**Keywords:** Rigidity, homogeneous manifolds, isometric immersion

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## Introduction

Let $M^n$ be a connected $n$-dimensional Riemannian manifold and $I(M)$ be the Lie group of all isometries of $M$. $M$ is called a *Riemannian homogeneous manifold* if $I(M)$ acts transitively on $M$. The study of isometric immersions of Riemannian homogeneous manifolds started with Kobayashi in [12], proving the classical result that a compact homogeneous hypersurface of Euclidean space is a round sphere. The non-compact case was studied by Nagano and Takahashi, [14], and Harle, [10]. In [19] and [20], Takahashi classified homogeneous hypersurfaces of the hyperbolic space. Such a classification can be found in Section 5 of this article.

For $n \geq 4$, the classification of homogeneous hypersurfaces of the sphere follows from the work of Hsiang and Lawson in [11], Uchida in [24], and from a result of Harle in [10]. In fact, in [11], they classify compact linear groups

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of cohomogeneity two that act in Euclidean spaces by isometries. This classification was completed by Uchida in [24] (see also Straume, [18], who gives the complete description of compact linear groups of cohomogeneity 2 and 3 acting in Euclidean spaces). Now, from Harle's theorem we get that the immersion $f : M \to S^{n+1}$ is rigid. Then, if $g$ is an isometry of $M$, $\bar{f} = f \circ g$ is another isometric immersion of $M$ that, being congruent to $f$, identifies the isometry $g$ with an isometry of $\mathbf{R}^{n+2}$. This means that the immersion is equivariant and the group $I(M)$ can be realized as a subgroup of rigid motions of $\mathbf{R}^{n+2}$. As the authors point out in [11], the complete classification also implies that homogeneous hypersurfaces of the sphere are orbits of the isotropy representation of a Riemannian symmetric pair (see also [21]).

On [4], the present authors started the study of codimension two isometric immersions of Riemannian homogeneous manifolds. In that paper we restricted ourselves to the case that the ambient space is the Euclidean space. Our first step was to investigate the equivariance of the immersion, which in turn implies that its image is the orbit of an isometric action in the ambient space. As explained above, such a property can be established by studying the rigidity of the immersion. To this end, in our previous article we used result of Dajczer, [7], on flat bilinear forms combined with a rigidity theorem of do Carmo and Dajczer in higher codimensions (see [3]). It turns out that the same techniques can be applied to the case that the ambient space has constant curvature, and consequently, an analogous rigidity result can be obtained. Before stating it, we point out that from now on, $\mathbf{Q}_c^N$ denotes a complete and simply connected Riemannian manifolds of constant curvature $c$, *i.e.*, the hyperbolic space or the Euclidean space or the round sphere.

**1 Theorem.** *Let $f : M^n \to \boldsymbol{Q}^{n+2}$ be an isometric immersion of a Riemannian homogeneous manifold such that the minimum index of relative nullity $\bar{\nu} = \min_{x \in M} \nu_f(x) \leq n-5$. Then either $f$ is rigid or for every point $p$ in $M$ there exist orthonormal vectors $\xi, \eta \in T_pM^\perp$ such that $rank A_\eta \leq 2$ and if $g \in I(M)$, $\xi$ can be oriented so that $g_\star \circ A_\xi = A_\xi \circ g_\star$.*

In this paper we study the non-rigid case. Notice that if $g_\star \circ A_\xi = A_\xi \circ g_\star$ for some $g \in I(M)$, then the eigenvalues of $A_\xi$ in $p$ and $g(p)$ are the same. Since $I(M)$ acts on $M$ transitively, for the sake of brevity, in this paper we will refer to the property $g_\star \circ A_\xi = A_\xi \circ g_\star$ as $A_\xi$ *is constant.* In addition, we observe that the Gauss equation together with homogeneity of $M$ and the fact that $A_\xi$ is constant imply that either $rank A_\eta \leq 1$ for all points of $M$ or $rank A_\eta \equiv 2$.

We start by studying the case that $A_\xi$ is constant and $rank A_\eta \leq 1$. We prove that each point has neighborhood that can be realized as a hypersurface of a space form. Using the classification of homogeneous hypersurface we prove Theorem 6 of section 2. As in the classification of codimension two submanifolds of Euclidean space, the hardest case is $\mathrm{rank} A_\eta \equiv 2$. We then study separately submanifolds of the sphere and of hyperbolic space in sections 4 and 5, respectively.

The main results of this paper are the following two theorems. They follow from Theorem 1 above, Theorem 6, Theorem 16, and Theorem 19.

**2 Theorem.** *Let $f : M^n \to S_c^{n+2}$ be an isometric immersion of a homogeneous Riemannian manifold such that $\bar{\nu} = \min_{x \in M} \nu_f(x) \leq n - 5$. Then one of the following occurs:*

*(a) $f(M^n)$ is the orbit of an isometric action in $S_c^{n+2}$.*

*(b) $M^n$ can be isometrically immersed in $S_c^{n+1}$ as an isoparametric hypersurface.*

*(c) $f(M^n)$ is a Riemannian product $\Sigma^2 \times S_{c_1}^{n-2}$, where $\Sigma^2$ is a surface of constant curvature contained in a 3-sphere and $c < c_1$.*

*(d) $f(M^n)$ is a Riemannian product $\Sigma^3 \times S_{c_1}^{n-3}$, where $c < c_1$ and $\Sigma^3$ is a homogeneous hypersurface a 4-dimensional sphere.*

**3 Theorem.** *Let $f : M^n \to \boldsymbol{H}_c^{n+2}$ be an isometric immersion of a homogeneous Riemannian manifold such that $\bar{\nu} = \min_{x \in M} \nu_f(x) \leq n - 5$. Then one of the following occurs:*

*(a) $f(M^n)$ is the orbit of an isometric action in $\boldsymbol{H}_c^{n+2}$.*

*(b) $\tilde{M}$, the universal covering of $M$, can be isometrically immersed in $\boldsymbol{H}_c^{n+1}$ as an isoparametric hypersurface.*

*(c) $\tilde{M}$ is a Riemannian product $\Sigma^2 \times N^{n-2}$, where $\Sigma^2$ is a surface of constant curvature isometrically immersed in a 3-dimensional space form and $N^{n-2}$ is isometric to one of the following:*

  *(i) a sphere $S_{c_1}^{n-2}$.*
  *(ii) the hyperbolic space $\boldsymbol{H}_{c_1}^{n-2}$, $c < c_1 < 0$.*
  *(iii) the Euclidean space.*

*(d) $\tilde{M}$ is a Riemannian product $\Sigma^3 \times \mathbf{H}_{c_1}^{n-3}$, where $c < c_1$ and $\Sigma^3$ is a homogeneous hypersurface of a 4-dimensional sphere.*

*(e) $M$ is a cohomogeneity one manifold such that all orbits are flat spaces.*

We remark that if $f(M^n)$ is the orbit of an isometric action in $S_c^{n+2}$ then $f$ is not necessarily an isoparametric immersion. If so, $f(M^n)$ would be the orbit of the isotropy representation of a symmetric space, by a theorem of Torbergsson (see [23] or [16]). However, in [11] we find the classification of compact linear groups of cohomogeneity three acting in the Euclidean space, and there are four cases that are not isotropy representations. For the case that the ambient space is the hyperbolic space, we have not found in the mathematical literature a characterization of codimension two orbits of isometric actions. The ones that are isoparametric submanifolds have been classified by B. Wu in [25].

Before finishing this section, we remark that due to the rigidity problem for codimensions greater than 1, we have to assume that $\bar{\nu} = \min_{x \in M} \nu_f(x) \leq n-5$, which in turn implies that $n \geq 5$. But, notice that homogeneous Einstein manifolds of dimension less than five are well known, and isometric immersions of Einstein manifolds of dimension $n \geq 5$ in space forms naturally satisfy the condition $\bar{\nu} \leq n-5$. Therefore our results can be used to study isometric immersions of homogeneous Einstein manifolds in $S^{n+2}$ and $\mathbf{H}^{n+2}$.

We also point out that hypersurfaces of cohomogeneity one of the hyperbolic space and of spheres have not been extensively studied. Their principal orbits are codimension two homogeneous submanifolds of the ambient space. In addtion, if $\gamma(t)$ denotes the normal geodesic through a point $x = \gamma(0)$, the vector $\xi = \gamma'(0)$ is a normal direction of the immersion of the orbit into the ambient space. We then have that $A_\xi$ is constant. The rigidity of the immersion of the cohomogeneity one hypesurface can be established by the rank of its shape operator, say $A_\eta$. A classical result states that if $rank A_\eta \geq 3$, the immersion is rigid. Therefore if $rank A_\eta \leq 2$, ou results can be applied. We hope that they will be useful in this regard.

## 1 Preliminaries

Let $f : M^n \to \bar{M}^{n+k}$ be an isometric immersion, $s$ a integer $1 \leq s \leq k$ and $U^s$ an $s$-dimensional subspace of $T_pM^\perp$. Let $\pi : T_pM^\perp \to U^s$ be the orthogonal projection. Consider the bilinear form

$$\alpha_{U^s} : T_pM \times T_pM \to U^s$$

given by

$$\alpha_{U^s} = \pi \circ \alpha,$$

where $\alpha$ is the second fundamental form of the immersion. The $s$-nullity of the immersion $f$ at $p$ is defined as

$$\nu_s(p) = max\{\dim N(\alpha_{U^s}) \mid U^s \subset T_pM^\perp\},$$

where $N(\alpha_{U^s})$ denotes the nullity space of the bilinear form $\alpha_{U^s}$.

The concept of $s$-nullity was introduced by do Carmo and Dajczer in [3] to study rigidity of isometric immersion of high codimension. Theorem 1.4 of [3] states *that an isometric immersion* $f : M^n \to R^{n+k}$ *such that* $k \leq 5$, $\nu_s(p) \leq n - (2s+1)$ *for all* $p \in M$ *and for all* $s$, $1 \leq s \leq k$, *is rigid.* In that paper, the authors observe that the theorem remains true when the ambient space is a space form $\mathbf{Q}_c^{n+k}$. In fact, using algebraic arguments, the assumptions on the $s$-nullity imply that if $f, \bar{f} : M^n \to \mathbf{R}^{n+k}$ are isometric immersions, then for each $p$ in $M$ the immersions induce a map $T$ between the normal bundles of $f$ and $\bar{f}$ restricted to a neighborhood $V$ of $p$. The isometry $T$ preserves the metric and the second fundamental form. These arguments depend only on the second fundamental form and the dimension of the normal space and therefore can be used when the ambient space has constant curvature. A theorem of Nomizu, [15] implies that $T$ also preserves the normal connection. It follows from the fundamental theorem for submanifolds that there exists a unique isometry $\Phi : \mathbf{Q}_c^{n+k} \to \mathbf{Q}_c^{n+k}$ such that $\bar{f}|_V = \Phi \circ f|_V$ and $\Phi|_{TM_f^\perp} = T$. The uniqueness of $\Phi$ for each neighborhood implies that $\bar{f} = \Phi \circ f$ and do Carmo-Dacjzer's theorem stated above. This theorem applied to codimension 2 gives the following result.

**4 Lemma.** *Let* $f : M^n \to \mathbf{Q}_c^{n+2}$, *be an isometric immersion of a homogeneous Riemannian manifold and* $p$ *a point in* $M$ *such that* $\nu_f(p) = \bar{\nu} \leq n - 5$. *Then either* $f$ *is rigid or there exists* $\bar{\eta} \in T_pM^\perp$ *such that* $rank A_{\bar{\eta}} \leq 2$.

This lemma and Lemma 2.3 of [4] are proved in the same fashion. For completeness, we repeat here the argument that shows how do Carmo-Dacjzer's theorem is used in the proof. We suppose that there does not exist $\bar{\eta} \in T_pM^\perp$ satisfying the condition $rank A_{\bar{\eta}} \leq 2$. This implies that $\nu_1(p) \leq n-3$. Since $\nu_2(p) = \nu_f(p) \leq n-5$, there exists a neighborhood $U$ of $p$ such that $\nu_1(q) \leq n-3$ and $\nu_2(q) \leq n-5$ for all $q$ in $U$. We have now the hypotheses of do Carmo-Dajczer's theorem, which also hold for every neighborhood $U' \subset U$ that contains $p$. Therefore $f|_{U'}$ is rigid, and Proposition 2.2 of [4], which holds when the ambient space is a space form $\mathbf{Q}_c^N$, implies that $f$ is rigid.

**5 Lemma.** *With the same hypotheses, let* $p$ *be such that* $\nu_f(p) = \bar{\nu} \leq n-5$ *and* $q$ *an arbitrary point of* $M$. *Consider an isometry* $g$ *of* $M$ *such that* $g(p) = q$. *Then for all* $X, Y, Z, W \in T_pM$ *one of the following occurs:*

*(a)* $\langle \alpha(X,Y), \alpha(Z,W) \rangle = \langle \alpha(g_\star(X), g_\star(Y)), \alpha(g_\star(Z), g_\star(W)) \rangle$

*(b) There exist orthonormal bases* $\{\xi, \eta\}$ *and* $\{\tilde{\xi}, \tilde{\eta}\}$ *of* $T_pM^\perp$ *and* $T_qM^\perp$ *respectively, such that* $rank A_\eta \leq 1$, $rank A_{\tilde{\eta}} \leq 1$ *and* $A_\xi = A_{\tilde{\xi}}$, i.e.,

$$\langle A_\xi X, Y \rangle = \langle A_{\tilde{\xi}} g_\star(X), g_\star(Y) \rangle$$

(c) *There exist orthonormal bases* $\{\xi, \eta\}$ *and* $\{\tilde{\xi}, \tilde{\eta}\}$ *of* $T_pM^\perp$ *and* $T_qM^\perp$ *respectively, such that* $rank A_\eta = rank A_{\tilde{\eta}} = 2$ *and* $A_\xi = A_{\tilde{\xi}}$.

Lemmas 4 and 5 are the main tools for proving the rigid result for homogeneous submanifolds of space forms stated in the Introduction, namely, Theorem 1. Their proofs are analogous to their corresponding results in Section 2 of [4]. Likewise, the initial steps for proving the next result are in Section 3 of [4].

**6 Theorem.** *Let* $f : M^n \to \boldsymbol{Q}_c^{n+2}$ *be an isometric immersion of a Riemannian homogeneous manifold such that* $\bar{\nu} = k \leq n - 4$. *Suppose that for each* $x \in M$ *there exists an orthonormal frame* $\{\xi, \eta\}$ *of the normal space* $T_xM^\perp$ *such that* $rank A_\eta \leq 1$ *and* $A_\xi$ *is constant. Then:*

(a) *If* $c = 0$, *then* $M = M_1^m \times \boldsymbol{R}^k$, *where* $M_1$ *is isometric to a sphere* $S^m$ *or is covered by the Riemannian product* $S^{m-1} \times \boldsymbol{R}$.

(b) *If* $c > 0$, *then each point of* $M$ *has a neighborhood that can be realized as an open part of an isoparametric hypersurface of* $S_c^{n+1}$.

(c) *If* $c < 0$, *then the universal cover* $\tilde{M}$ *can be isometrically immersed in* $\boldsymbol{H}_c^{n+1}$ *as an isoparametric hypersurface.*

Proof. Part (a) is proved in [4]. Furthermore, the same arguments used in the beginning of the proof of Theorem 2 of [4] can be repeated here to conclude that for every point $p$ of $M$ there is an open set $U$ containing $p$ that isometrically immerses in codimension 1 with second fundamental form given by $A_\xi$.

For the case $c > 0$, we observe that since the eigenvalues of $A_\xi$ are constant, $U$ is a *local isoparametric submanifold* of $\mathbf{R}^{n+2}$, with second fundamental form given by $A_\xi$ and $A_\zeta$, the latter from the immersion $S^{n+1} \to \mathbf{R}^{n+2}$. A result of Terng (see Theorem 3.4 in [22]) states that there exists a complete isoparametric submanifold $N$ of $\mathbf{R}^{n+2}$ which includes $U$. Since the immersion $N \to \mathbf{R}^{n+2}$ has an umbilical direction given by $\zeta$, $N$ lies in a sphere $S^{n+1}$. This implies (b).

If $c < 0$, from the classification of isoparametric hypersurfaces of the hyperbolic space, (see [2]), we get that the number of distinct eigenvalues of $A_\xi$ is $g \leq 2$, and for $g = 2$, $\lambda_1\lambda_2 = c$. Therefore, if $g = 1$, $U$ and hence $M$, has constant curvature, and the result in (c) is obvious. If $g = 2$, $U$ is a Riemannian product and each factor has constant curvature. Using the homogeneity of $M$ we conclude that its universal cover $\tilde{M}$ splits into a Riemannian product of $S^k \times \mathbf{H}^{n-k}$ and again we have (c). $\boxed{QED}$

## 2 Rank $\mathbf{A}_\eta = 2$

Throughout this section $f$ will be an isometric immersion of an $n-$dimensional homogeneous Riemannian manifold $M$ into $\mathbf{Q}\,_c^{n+2}$ where $c = -1, 1$, for simplicity. We will assume that for each point of $M$ we can choose smooth orthonormal sections $\xi, \eta$ of the normal bundle such that $A_\xi$ is constant and $rank A_\eta = 2$ and $\bar{\nu} \leq n-5$. In addition, the homogeneity of $M$ and the Gauss equation imply that the distribution $\operatorname{Ker} A_\eta$ is invariant by isometries.

With these assumptions we first make the following considerations: given $g \in I\,(M)$, we have another immersion $\tilde{f} : M^n \to \mathbf{Q}_c^{n+2}$, $\tilde{f} = f \circ g$, and the isometry $\tau : T^\perp f \to T^\perp \tilde{f}$ given by $\tau\eta(p) = \eta(g(p))$ and $\tau\xi(p) = \xi(g(p))$. If $f$ is not equivariant, since $A_\xi$ is constant, there exists $g \in I\,(M)$ such that $A_\eta \neq A_{\tau\eta}$. Notice that we can apply here the same arguments used to prove Lemma 6 of [9], since they involve only the Codazzi Equation and the fact that $rank A_\eta = 2$. Such arguments imply that $\nabla^\perp_X \eta = 0$, $\forall\ X \in \operatorname{Ker} A_\eta$.

**7 Lemma.** *The distribution* $\operatorname{Ker} A_\eta$ *is involutive and its leaves are homogeneous manifolds.*

Proof. Write the Codazzi equation for $A_\eta$ and $X, Y \in \operatorname{Ker} A_\eta$. In this case we will have

$$A_\eta[X,Y] = A_{\nabla^\perp_X \eta} Y \ - \ A_{\nabla^\perp_Y \eta} X = 0$$

and then $[X,Y] \in \operatorname{Ker} A_\eta$. Now, the second part of this lemma has the same proof of Lemma 4.4 of [4]. $\boxed{QED}$

**8 Lemma.** *The leaves of the distribution* $\operatorname{Ker} A_\eta$ *are totally geodesic if and only if* $\xi$ *and* $\eta$ *are parallel sections of the normal bundle.*

The lemma above is proved as Lemma 4.6 of [4]. The key point for proving results for the case of $rank A_\eta = 2$ is to conclude that the leaves of the distribution $\operatorname{Ker} A_\eta$, that will be denoted by $N$, are totally geodesic in $M$. The first steps for both cases, $c = -1, 1$, are the same.

We start by considering a maximal leave through a point $p$, denoted by $N_p$, and the Codazzi equation

$$\nabla_Z A_\eta X - A_{\nabla^\perp_Z \eta} X - A_\eta(\nabla_Z X) = \nabla_X A_\eta Z - A_{\nabla^\perp_X \eta} Z - A_\eta(\nabla_X Z)$$

where $X \in \operatorname{Ker} A_\eta$ and $Z \in \operatorname{Im} A_\eta$. Taking inner product with $Y \in \operatorname{Ker} A_\eta$, we get

$$\langle \nabla^\perp_Z \eta, \xi \rangle \langle \alpha(X,Y), \xi \rangle = \langle \nabla_X Y, A_\eta Z \rangle. \tag{1}$$

If for all $Z \in \operatorname{Im} A_\eta$, $\langle \nabla^\perp_Z \eta, \xi \rangle = 0$, then $\langle \nabla_X Y, A_\eta Z \rangle = 0$ and, since $rank A_\eta = 2$, we obtain that $N_p$ is totally geodesic in $M$.

Let us then suppose that for two linearly independent vector fields $Z_1, Z_2$ of $\operatorname{Im} A_\eta$, we have $\langle \nabla^\perp_{Z_i} \eta, \xi \rangle \neq 0$ on a neighborhood $U$ of $p$. Then a suitable linear combination of them will give a vector field $Z$ such that $\langle \nabla^\perp_Z \eta, \xi \rangle = 0$ and hence $\langle \nabla_X Y, A_\eta Z \rangle = 0$ for all points of $U$. If for some isometry $h$ such that $h(p) = p$, $h_\star(A_\eta Z)$ and $A_\eta Z$ are linearly independent we have $\langle \nabla_X Y, W \rangle = 0$

for all $W \in \operatorname{Im} A_\eta$. Using the homogeneity of $N_p$ and of $M$ we conclude that $N_p$ is totally geodesic in $M$. Since we are supposing $\langle \nabla^\perp_Z \eta, \xi \rangle \neq 0$ for some $Z$, from (1) we get that $\langle \alpha(X,Y), \xi \rangle = 0$ for all $X, Y \in \operatorname{Ker} A_\eta$. This contradicts our assumption on the index of relative nullity.

Therefore if $\langle \nabla^\perp_Z \eta, \xi \rangle \neq 0$ for some $Z$ we conclude that such an isometry does not exist. This implies that there exists a one-dimensional distribution $\mathcal{T} \subset \operatorname{Im} A_\eta$, which is invariant by isometries and with the property that $\langle \nabla_X Y, Z \rangle = 0$, for all $X, Y \in \operatorname{Ker} A_\eta$ and $Z \in \mathcal{T}$.

In the rest of this paper we denote $Z_1$ a unit local vector field orthogonal to $\mathcal{T}$ and $Z_2$ a unit local vector field in $\mathcal{T}$. Then we have

$$\langle \nabla_X Y, Z_2 \rangle = 0, \qquad \forall X, Y \in \operatorname{Ker} A_\eta,$$

and from (1)

$$\langle \nabla_X Y, Z_1 \rangle = \langle \alpha(X,Y), \xi \rangle \langle \nabla^\perp_W \eta, \xi \rangle \tag{2}$$

where $A_\eta W = Z_1$. Notice that, $Z_1$ is (locally) invariant by isometries and since $\langle \alpha(X,Y), \xi \rangle$ is constant, we conclude that $\langle \nabla^\perp_W \eta, \xi \rangle$ is also constant.

Let us consider the immersion

$$g = f_{|N_p} : N_p \to \mathbf{Q}^{\,n+2},$$

with second fundamental form and normal connection denoted by $\bar{\alpha}$ and $\bar{\nabla}^\perp$ respectively. From the above we conclude that if $\langle \nabla^\perp_W \eta, \xi \rangle \neq 0$ then the vector $\beta$ given by

$$\beta = \langle \nabla^\perp_W \eta, \xi \rangle \xi - Z_1$$

is in the normal space of the immersion $g$ and is orthogonal to the first normal space $N_1(g)$. Moreover, the vector $\zeta \in N_1(g)$ given by

$$\zeta = \xi + \langle \nabla^\perp_W \eta, \xi \rangle Z_1$$

is such that $\|\zeta\|$ is constant. Observe that from (2) we obtain

$$\bar{\alpha}(X,Y) = \langle \alpha(X,Y), \xi \rangle (\xi + \langle \nabla^\perp_W \eta, \xi \rangle Z_1). \tag{3}$$

Since $\langle \alpha(X,Y), \xi \rangle$ is constant we have $g$ is a $1-$regular immersion and $\dim N_1(g) = 1$. In addition, our asumptions on the nullity of $f$ and (3) imply that $N_1(g)$ is parallel (see [6], Proposition 4.5). It follows that the codimension of $g$ can be reduced to 1. Therefore,

$$\bar{\nabla}^\perp_X \xi = -\langle \nabla^\perp_W \eta, \xi \rangle \bar{\nabla}^\perp_X Z_1, \tag{4}$$

and since $Z_1$ is a unit vector field, we conclude that

$$-\langle \bar{\nabla}^\perp_X \xi, Z_1 \rangle = \langle A_\xi X, Z_1 \rangle = 0, \quad \forall \ X \in \operatorname{Ker} A_\eta. \tag{5}$$

Furthermore, from the fact that $N_1(g)$ is parallel we conclude that if $c=-1$ then $N_p$ is a homogeneous hypersurface of Hyperbolic Space $\mathbf{H}^{n-1}$ which is totally geodesic in $\mathbf{H}^{n+2}$, and if $c=1$ then $N_p$ is a homogeneous hypersurface of a totally geodesic sphere $S^{n-1}\subset S^{n+2}$.

**9 Lemma.** *If $A_\xi$ has two eigenvectors in* $\operatorname{Ker} A_\eta$ *corresponding to two distinct non-zero eigenvalues then $N$ is totally geodesic in $M$.*

Proof. Let $X_i,\ i=1,2$ denote such eigenvectors with corresponding eigenvalues $\lambda_i,\ i=1,2$. Using Lemma 6.2(a) of [4] we conclude that $\nabla_{X_i}X_i,\ i=1,2$ are also eigenvectors of $A_\xi$ corresponding to $\lambda_i$. Since (3) implies that the $X_i's$ are also eigenvectors of $\bar{A}_\zeta$, from the fact that the eigenspaces of $\bar{A}_\zeta$ are autoparallel distributions, if $\langle\nabla_{X_i}X_i,Z_1\rangle\neq 0$ for $i=1,2$, then $Z_1$ is an eigenvector of $A_\xi$ with eigenvalue $\lambda_i$. Since we are supposing that $\lambda_1\neq\lambda_2$ we conclude that for one of them, say $Z_1$, we have $\langle\nabla_{X_1}X_1,Z_1\rangle=0$. This substituted in (1) implies $\langle\nabla^\perp_Z\eta,\xi\rangle=0$ for all $Z\in\operatorname{Im}A_\eta$, because $\lambda_1\neq 0$. Therefore $\xi$ and $\eta$ are normal parallel sections and hence $N$ is totally geodesic. $\boxed{QED}$

**10 Lemma.** *Let $X_1$ and $X_2$ be eigenvectors of $A_\eta$ corresponding to non-zero eigenvalues $\delta_1$ and $\delta_2$ respectively. Then we have:*

*(a)* $\langle\nabla_{X_1}X_1+\nabla_{X_2}X_2,X\rangle=0,\ \forall\, X\in\operatorname{Ker}A_\eta$.

*(b)* $\langle\nabla_{Z_1}Z_1+\nabla_{Z_2}Z_2,X\rangle=0,\ \forall\, X\in\operatorname{Ker}A_\eta$.

Proof. Consider the Codazzi equation

$$\nabla_X A_\eta X_i - A_\eta\nabla_X X_i - \langle\nabla^\perp_X\eta,\xi\rangle A_\xi X_i = \nabla_{X_i}A_\eta X - A_\eta\nabla_{X_i}X - \langle\nabla^\perp_{X_i}\eta,\xi\rangle A_\xi X,$$

where $X\in\operatorname{Ker}A_\eta$ and $i=1,2$. Taking inner product with $X$ we have

$$X(\delta_1)=\delta_1\langle\nabla_{X_1}X_1,X\rangle - a_1\langle A_\xi X,X_1\rangle\ ,\quad X(\delta_2)=\delta_2\langle\nabla_{X_2}X_2,X\rangle - a_2\langle A_\xi X,X_2\rangle,$$

where $a_1=\langle\nabla^\perp_{X_1}\eta,\xi\rangle$ e $a_2=\langle\nabla^\perp_{X_2}\eta,\xi\rangle$. Since the Gauss equation implies that $\delta_1\delta_2$ is constant, we conclude that

$$0=\delta_1\delta_2\langle\nabla_{X_1}X_1+\nabla_{X_2}X_2,X\rangle - \langle A_\xi(a_1\delta_2X_1+a_2\delta_1X_2),X\rangle.$$

Notice that $a_1\delta_2X_1+a_2\delta_1X_2$ is a multiple of $Z_1$. In fact, $\nabla^\perp_{(-a_2X_1+a_1X_2)}\eta=0$ and hence $A_\eta(-a_2X_1+a_1X_2)$ is a multiple $Z_2$. On the other hand, $A_\eta(-a_2X_1+a_1X_2)=-a_2\delta_1X_1+a_1\delta_2X_2$ and the latter vector is orthogonal to $a_1\delta_2X_1+a_2\delta_1X_2$. Since $A_\xi Z_1$ is orthogonal to $\operatorname{Ker}A_\eta$ the equation above implies

$$\langle\nabla_{X_1}X_1+\nabla_{X_2}X_2,X\rangle=0,\ \ \forall\, X\in\operatorname{Ker}A_\eta.$$

Writing $Z_1$ and $Z_2$ as linear combination of $X_1$ and $X_2$ we obtain (b). $\boxed{QED}$

**11 Lemma.** *If the shape operator* $\bar{A}_\zeta$ *has one eigenvalue* $\bar{\lambda}$ *of multiplicity at least* 2 *then* $A_\xi Z_2$ *is orthogonal to its eigenspace.*

Proof. If the leaves $N$ are totally geodesic in $M$ then the immersion $f$ has zero normal curvature, by Lemma 8. From the Ricci equation we conclude that the operators $A_\xi$ and $A_\eta$ commute and thus, for an eigenvector $X_i$ of $A_\eta$ and $X \in \operatorname{Ker} A_\eta$, we have

$$\langle A_\eta \circ A_\xi X, X_i\rangle = \delta_i \langle A_\xi X, X_i\rangle = \langle A_\xi \circ A_\eta X, X_i\rangle = 0,$$

implying that $A_\xi(\operatorname{Ker} A_\eta) \subset \operatorname{Ker} A_\eta$.

If $N$ is not totally geodesic in $M$, let $E_{\bar{\lambda}}$ denote the eigenspace of $\bar{A}_\zeta$ corresponding to $\bar{\lambda}$ and $L$ be the distribution spanned by the orthogonal projection of $A_\xi Z_2$ onto $E_{\bar{\lambda}}$. Notice that $\dim L \leq 1$ and $L$ is invariant by isometries, since $A_\xi$ commutes with isometries and *span*$\{Z_2\}$ is (locally) invariant by them. We suppose that $\dim L = 1$ and we will get a contradiction. Let $V$ denote a unit vector field in $L$. Observe that the Gauss equation for the immersion $M \to \mathbf{Q}_c^{n+2}$ implies $\langle R(X, V)Z_1, Z_2\rangle = 0$. Applying the Ricci equation to the immersion $i : N_{\bar{\lambda}} \to M$ and using the facts that $\langle \nabla_X Y, Z_2\rangle = 0$ and $E_{\bar{\lambda}}$ is auto-parallel, we conclude that $i$ has flat normal bundle. Let $\bar{R}^\perp$ denote the normal curvature tensor of $i : N_{\bar{\lambda}} \to M$ and let us consider $X \in E_{\bar{\lambda}}$ orthogonal to $V$. Since $\langle \nabla_X Z_1, Z_2\rangle$ and $\langle \nabla_V Z_1, Z_2\rangle$ are constant, we have

$$0 = \langle \bar{R}^\perp(X, V)Z_1, Z_2\rangle = \langle \bar{\nabla}^\perp_{[X,V]} Z_1, Z_2\rangle, \ \ \forall X \in \operatorname{Ker} A_\eta,$$

that substituted in (4) gives $\langle A_\xi Z_2, [X, V]\rangle = 0$. Then $[X, V]$ is orthogonal to $V$ implying that $\nabla_V V \perp \operatorname{Ker} A_\eta$, since $E_{\bar{\lambda}}$ is auto-parallel. Consider now $X \in E_{\bar{\lambda}}$ and orthogonal to $V$. Then (3) and (5) imply that $X$ is also an eigenvector of $A_\xi$ with corresponding eigenvalue $\lambda = \bar{\lambda}/\|\zeta\|^2$. Then we have that $\nabla_X X$ is an eigenvector of $A_\xi$ corresponding to $\lambda$, by Lemma 6.2(a) of [4]. Since $E_{\bar{\lambda}}$ is auto-parallel and we are supposing that $\langle \nabla_X X, Z_1\rangle \neq 0$, we obtain that

$$\langle \nabla_X X, V\rangle A_\xi V + \langle \nabla_X X, Z_1\rangle A_\xi Z_1 = \lambda \langle \nabla_X X, V\rangle V + \lambda \langle \nabla_X X, Z_1\rangle Z_1.$$

Since $\langle A_\xi V, Z_1\rangle = 0$, taking inner product with $V$ and $Z_1$ we get that $\langle A_\xi V, V\rangle = \langle A_\xi Z_1, Z_1\rangle = \lambda$. Moreover, taking inner product with $Z_2$ we obtain

$$\langle A_\xi Z_2, V\rangle\langle \nabla_X X, V\rangle + \langle A_\xi Z_2, Z_1\rangle\langle \nabla_X X, Z_1\rangle = 0 \ \ \forall X \in E_{\bar{\lambda}}, \ X \perp V,$$

and using (2) we conclude

$$\langle A_\xi Z_2, V\rangle\langle \nabla_X X, V\rangle + \lambda\langle \nabla^\perp_W \eta, \xi\rangle\langle A_\xi Z_2, Z_1\rangle = 0. \tag{6}$$

Consider now a (local) vector field $X \in E_{\bar{\lambda}}$, which is orthogonal to $V$ and invariant by isometries. We compute

$$\begin{aligned}\langle R(X,Z_1)Z_2,X\rangle &= -\langle \nabla_{Z_1}Z_2,\nabla_X X\rangle + \langle \nabla_{Z_1}X,Z_1\rangle\langle \nabla_{Z_1}Z_2,X\rangle \\ &\quad +\langle \nabla_{Z_1}X,Z_2\rangle\langle \nabla_{Z_2}Z_2,X\rangle \\ &= -\langle \nabla_{Z_1}Z_2,\nabla_X X\rangle \\ &\quad -\langle \nabla_{Z_1}Z_2,X\rangle\Big[\langle \nabla_{Z_1}Z_1,X\rangle + \langle \nabla_{Z_2}Z_2,X\rangle\Big] \\ &= -\langle \nabla_{Z_1}Z_2,\nabla_X X\rangle,\end{aligned}$$

where the last equality follows from Lemma 10(b). Since $\langle R(X,Z_1)Z_2,X\rangle = \lambda\langle A_\xi Z_1,Z_2\rangle$, the Gauss equation implies

$$\lambda\langle A_\xi Z_1,Z_2\rangle = -\langle \nabla_{Z_1}Z_2,\nabla_X X\rangle. \tag{7}$$

On the other hand, $\langle R(V,Z_1)Z_2,V\rangle = \langle R(V,Z_2)Z_1,V\rangle$, and computing these curvatures we have

$$\begin{aligned}\langle R(V,Z_1)Z_2,V\rangle &= -\langle \nabla_{Z_1}Z_2,\nabla_V V\rangle \\ &\quad +\langle \nabla_V Z_1,Z_2\rangle\Big[\langle \nabla_{Z_1}Z_1,V\rangle - \langle \nabla_{Z_2}Z_2,V\rangle\Big] \\ \langle R(V,Z_2)Z_1,V\rangle &= \langle \nabla_V Z_1,Z_2\rangle\Big[\langle \nabla_{Z_1}Z_1,V\rangle - \langle \nabla_{Z_2}Z_2,V\rangle)\Big],\end{aligned}$$

which implies that $\langle \nabla_{Z_1}Z_2,\nabla_V V\rangle = 0$. Since $\nabla_V V \perp \operatorname{Ker} A_\eta$, we obtain that $\langle \nabla_{Z_1}Z_2,Z_1\rangle = 0$. Now we consider the Codazzi equation

$$\nabla_X A_\xi Z_1 - A_\xi \nabla_X Z_1 = \nabla_{Z_1}A_\xi X - A_\xi\nabla_{Z_1}X - \langle \nabla^\perp_{Z_1}\xi,\eta\rangle A_\eta X$$

and, taking inner product with $V$, we conclude that $\langle \nabla_{Z_1}Z_2,X\rangle = 0$. Since $E_{\bar{\lambda}}$ is auto-parallel, from (7) we conclude that

$$\lambda\langle A_\xi Z_1,Z_2\rangle = -\langle \nabla_{Z_1}Z_2,V\rangle\langle \nabla_X X,V\rangle. \tag{8}$$

Now (6) and (8) imply

$$\langle \nabla_X X,V\rangle\Big[\langle A_\xi Z_2,V\rangle - \langle \nabla^\perp_W\eta,\xi\rangle\langle \nabla_{Z_1}Z_2,V\rangle\Big] = 0. \tag{9}$$

Therefore, if $\langle \nabla_X X,V\rangle \neq 0$, then

$$\langle A_\xi Z_2,V\rangle = \langle \nabla^\perp_W\eta,\xi\rangle\langle \nabla_{Z_1}Z_2,V\rangle \tag{10}$$

On the other hand, considering the Codazzi equation

$$\nabla_V A_\xi Z_1 - A_\xi\nabla_V Z_1 = \nabla_{Z_1}A_\xi V - A_\xi\nabla_{Z_1}V - \langle \nabla^\perp_{Z_1}\xi,\eta\rangle A_\eta V,$$

taking inner product with $V$, we conclude that $\langle \nabla_V Z_1, Z_2\rangle = -2\langle \nabla_{Z_1} Z_2, V\rangle$. Using (2) we conclude that

$$\langle A_\xi Z_2, V\rangle = -2\langle \nabla^{\perp}_{W}\eta, \xi\rangle\langle \nabla_{Z_1} Z_2, V\rangle, \tag{11}$$

which together with (10) implies $\langle A_\xi Z_2, V\rangle = 0$.

If $\langle \nabla_X X, V\rangle = 0$, then from (6), and since we are supposing that $\xi$ is not a parallel section, we get that $Z_1$ is an eigenvector of $A_\xi$ with eigenvalue $\lambda$. We will show that this fact together with $\dim L = 1$ give a contradiction. In fact, from the Gauss equation we get that $\langle R(V, Z_2)Z_1, V\rangle = \lambda\langle A_\xi Z_1, Z_2\rangle = 0$, and from the computation above for $\langle R(V, Z_2)Z_1, V\rangle$ we get that $\langle \nabla_{Z_1} Z_1, V\rangle - \langle \nabla_{Z_2} Z_2, V\rangle = 0$ (notice that $\langle \nabla_V Z_1, Z_2\rangle \neq 0$ by (8) ). Combining this fact with Lemma 10 we conclude that $\langle \nabla_{Z_1} Z_1, V\rangle = \langle \nabla_{Z_2} Z_2, V\rangle = 0$. Further, the Codazzi equation

$$\nabla A_\xi Z_i - A_\xi \nabla_X Z_i = \nabla_{Z_i} A_\xi X - A_\xi \nabla_{Z_i} X, i = 1, 2,$$

and Lemma 10(b) gives

$$\langle \nabla_{Z_1} Z_1, X\rangle = 0 \quad \text{and} \quad \langle \nabla_{Z_1} Z_2, X\rangle = 0.$$

Now we compute the curvatures

$$\begin{aligned}
K(V, Z_1) &= -\langle \nabla_V Z_1, Z_2\rangle\langle \nabla_{Z_1} Z_2 + \nabla_{Z_2} Z_1, V\rangle - \langle \nabla_{Z_1} Z_2, V\rangle\langle \nabla_{Z_2} Z_1, V\rangle \\
&\quad -\langle \nabla_V V, Z_1\rangle^2 \\
K(X, Z_1) &= -\langle \nabla_{Z_1} Z_1, \nabla_X X\rangle - \langle \nabla_X X, Z_1\rangle^2 - \langle \nabla_{Z_1} Z_1, X\rangle^2 \\
&\quad -\langle \nabla_{Z_2} Z_1, X\rangle\langle \nabla_{Z_1} Z_2, X\rangle \\
&= -\langle \nabla_X X, Z_1\rangle^2.
\end{aligned}$$

On the other hand, the Gauss equation gives

$$K(V, Z_1) = c + \langle A_\xi V, V\rangle\langle A_\xi Z_1, Z_1\rangle = c + \lambda^2 = K(X, Z_1),$$

and since $-\langle \nabla_X X, Z_1\rangle = -\langle \nabla_V V, Z_1\rangle$, we obtain

$$\langle \nabla_V Z_1, Z_2\rangle\langle \nabla_{Z_1} Z_2 + \nabla_{Z_2} Z_1, V\rangle + \langle \nabla_{Z_1} Z_2, V\rangle\langle \nabla_{Z_2} Z_1, V\rangle = 0 \tag{12}$$

Similarly,

$$\begin{aligned}
K(V, Z_2) &= \langle \nabla_V Z_1, Z_2\rangle\langle \nabla_{Z_2} Z_1 + \nabla_{Z_1} Z_2, V\rangle - \langle \nabla_{Z_2} Z_2, Z_1\rangle\langle \nabla_V V, Z_1\rangle \\
&- \langle \nabla_{Z_2} Z_1, V\rangle\langle \nabla_{Z_1} Z_2, V\rangle \\
K(X, Z_2) &= -\langle \nabla_{Z_2} Z_2, Z_1\rangle\langle \nabla_X X, Z_1\rangle.
\end{aligned}$$

and since

$$\begin{aligned} K(V, Z_2) &= c + \lambda\langle A_\xi Z_2, Z_2\rangle - \langle A_\xi Z_2, V\rangle^2 \\ K(X, Z_2) &= c + \lambda\langle A_\xi Z_2, Z_2\rangle, \end{aligned}$$

we have

$$K(V, Z_2) = K(X, Z_2) - \langle A_\xi Z_2, V\rangle^2.$$

Substituting above we obtain

$$\langle\nabla_V Z_1, Z_2\rangle\langle\nabla_{Z_2} Z_1 + \nabla_{Z_1} Z_2, V\rangle - \langle\nabla_{Z_2} Z_1, V\rangle\langle\nabla_{Z_1} Z_2, V\rangle = -\langle A_\xi Z_2, V\rangle^2.$$

This equation together with (11) and (12) implies

$$2\langle\nabla_{Z_2} Z_1, V\rangle\langle\nabla_{Z_1} Z_2, V\rangle = \langle A_\xi Z_2, V\rangle^2.$$

Using again (11) we have

$$\langle\nabla_{Z_2} Z_1, V\rangle = 2\langle\nabla^\perp_W \eta, \xi\rangle^2\langle\nabla_{Z_1} Z_2, V\rangle,$$

which substituted in (12) gives

$$\langle\nabla_V Z_1, Z_2\rangle\Big[1 + 2\langle\nabla^\perp_W \eta, \xi\rangle^2\Big] + 2\langle\nabla^\perp_W \eta, \xi\rangle^2\langle\nabla_{Z_1} Z_2, V\rangle = 0.$$

Finallly, since $\langle\nabla^\perp_W \eta, \xi\rangle\langle\nabla_V Z_1, Z_2\rangle = \langle A_\xi V, Z_2\rangle$, by (4), using (11) once more we conclude that

$$\langle A_\xi Z_2, V\rangle(1 + \langle\nabla^\perp_W \eta, \xi\rangle^2) = 0,$$

which contradicts the initial assumption, namely that $\langle A_\xi Z_2, V\rangle \neq 0$. Therefore $A_\xi Z_2$ is orthogonal to $E_{\bar\lambda}$. $\boxed{QED}$

## 3 Submanifolds of the Sphere

With the assumptions of Section 3, we consider here the case that the ambient space is the sphere of constant curvature 1, denoted by $\mathbf{S}^{n+2}$. We point out first, that these assumptions immediately imply that $\xi$ is not an umbilical direction. In fact, if $A_\xi = \lambda I$, then the substantial codimension of $f$ in $\mathbf{R}^{n+3}$ is 2, that is, $f(M)$ lies in a totally geodesic sphere $S^{n+1} \subset S^{n+2}$. The type number of $f : M \to S^{n+1}$ is 2. Moreover, our assumption on the index of relative nullity implies that $M$ is at least 5-dimensional. Since the scalar curvature is constant, we apply a theorem of Harle (see [10]) which states that with such conditions, the immersion of $M$ in $S^{n+1}$ is rigid. From the homogeneity of $M$ we conclude that the eigenvalues of $A_\eta$ are constant. Therefore, $M = S^2 \times S^k$, by Proposition 6.4(b) of [4]. But this contradicts that $A_\xi = \lambda I$, for $\lambda \neq 0$.

We now have that $g : N^{n-2} \to S^{n-1} \subset S^{n+2}$. It is well known that each homogeneous (isoparametric) hypersurface of the sphere is an orbit of the isotropy representation of a Riemannian symmetric pair of rank 2 and thus contained in the list given in [21], Table II. We will show however that under our hypotheses, the Weingarten operator of $\bar{A}_\zeta$ has one eigenvalue of multiplicity at least $n-3$. For that, we consider the distribution $D$ given by

$$D = \{X \in \operatorname{Ker} A_\eta \mid A_\xi X \in \operatorname{Ker} A_\eta\}.$$

Since $A_\xi Z_1 \in \operatorname{Ker} A_\eta^\perp$ we have that $\dim(D) \geq n-3$, and $D = \operatorname{Ker} A_\eta$ if and only if $A_\xi Z_2 \in \operatorname{Im} A_\eta$. Notice that since $A_\xi$ is constant and $\operatorname{Ker} A_\eta$ is invariant by isometries, the distribution $D$ is also invariant by isometries.

**12 Lemma.** *Let $X_1$ and $X_2$ be eigenvectors of $A_\eta$ corresponding to non-zero eigenvalues $\delta_1$ and $\delta_2$ respectively. Then we have:*

*(a)* $(\delta_1 - \delta_2)\langle \nabla_X X_1, X_2\rangle = \delta_2 \langle \nabla_{X_1} X_2, X\rangle = \delta_1 \langle \nabla_{X_2} X_1, X\rangle$, $\forall\ X \in D$.

*(b)* $X(\delta_1) = \delta_1 \langle \nabla_{X_1} X_1, X\rangle$, $X(\delta_2) = \delta_2 \langle \nabla_{X_2} X_2, X\rangle$, $\forall\ X \in D$.

Proof. Consider the Codazzi equation

$$\nabla_X A_\eta X_i - A_\eta \nabla_X X_i - \nabla^\perp_X \eta, \xi\rangle A_\xi X_i = \nabla_{X_i} A_\eta X - A_\eta - A_\eta \nabla_{X_i} X - \langle \nabla^\perp_{X_i} \eta, \xi\rangle A_\xi X,$$

where $X \in D$ and $i = 1, 2$. Taking inner product with $X_j$, $j = 1, 2$, we obtain $(a)$ and with $X \in D$ we obtain (b). $\boxed{\mathit{QED}}$

**13 Lemma.** *If the leaves $N$ are not totally geodesic in $M$ then $[X, Z_i] \in \operatorname{Ker} A_\eta$, for $i = 1, 2$, $\forall\ X \in D$.*

Proof. Since $\nabla_X Z_1 \in \operatorname{Ker} A_\eta$ and $\nabla_X Z_2 \in \operatorname{Ker} A_\eta$ $\forall\ X \in D$, it suffices to show that $\langle \nabla_{Z_i} Z_j, X\rangle = 0$, $\forall\ X \in D$ and $i, j = 1, 2$. We divide this proof into the following steps.

*Step 1* We show that there exists a point $x_0$ such that $\nabla_{X_i} X_j(x_0)$ is orthogonal to $D(x_0)$.

In fact, if there exists $x_0$ such that $\delta_1(x_0) = \delta_2(x_0)$, Lemma 12 (a) implies that $\nabla_{X_1} X_2(x_0)$ and $\nabla_{X_2} X_1(x_0)$ are both orthogonal to $D(x_0)$.

If not, we suppose that $\delta_1(x) < \delta_2(x)$, $\forall\ x \in M^n$, since $\delta_1$ and $\delta_2$ are continuous functions defined on a connected manifold. This implies in particular that $X_1$ and $X_2$ determine globally defined distributions in $M^n$. Moreover, if $N$ is not totally geodesic in $M$ then the vector field $Z_1$ is also globally defined on $M^n$, for if $X \in \operatorname{Ker} A_\eta$ and $\langle A_\xi X, X\rangle \neq 0$, $\nabla_X X$ defines a unique direction for $Z_1$ (observe that $\nabla_{(-X)}(-X) = \nabla_X X$) by 3.

If for every $x \in N$, the vector $Z_1(x)$ is not an eigenvector of $A_\eta(x)$, the function $h_1 : N \to \mathbf{R}$, given by $h_1(x) = \langle X_1(x), Z_1(x)\rangle$ is well defined and $h_1(x) \neq 0$, $\forall\, x \in N$. Moreover $X_1$ can be chosen so that $h_1(x) > 0$, for every $x \in N$. Since we are supposing that $Z_1$ we have $h_1(x) < 1$.

Since $h_1$ is continuous and $N$ is compact, let $x_0 \in N$ be a point where $h_1$ achieves its minimum. Then $X(h_1)(x_0) = 0$. Writing

$$X_1 = h_1 Z_1 + h_2 Z_2,$$

we get

$$\nabla_X X_1 = X(h_1) Z_1 + X(h_2) Z_2 + h_1 \nabla_X Z_1 + h_2 \nabla_X Z_2.$$

We know that the vector fields $\nabla_X Z_1$ and $\nabla_X Z_2$ are in $\operatorname{Ker} A_\eta$ for $X \in D$. $X(g_1)(x_0) = 0$. Further, $h_1 X(h_1) + h_2 X(h_2) = 0$, since $X_1$ is a unit vector. The fact that $h_1(x) < 1$ implies that $h_2(x_0) \neq 0$ and therefore $X(h_2)(x_0) = 0$. It follows that $\nabla_X X_1(x_0) \in \operatorname{Ker} A_\eta$. Now, Lemma 12(a) implies that $\nabla_{X_1} X_2(x_0)$ and $\nabla_{X_2} X_1(x_0)$ are orthogonal to $D(x_0)$. If there exists a point $x_0$ such that $Z_1(x_0) = X_1(x_0)$ but $Z_1(x) \neq X_2(x), \forall x \neq x_0$, we consider the function $h_2 = \langle Z_1, X_2\rangle$, and the same type of arguments apply to this case.

Now we consider the remaining case, that is, there exist points $x, y \in N$ such that $Z_1(x) = X_1(x)$ and $Z_1(y) = X_2(y)$. Using the Codazzi equation

$$\nabla_{Z_1} A_\eta Z_2 - A_\eta \nabla_{Z_1} Z_2 - \langle \nabla^\perp_{Z_1}, \xi\rangle A_\xi Z_2 = \nabla_{Z_2} A_\eta Z_1 - A_\eta \nabla_{Z_2} Z_1 - \langle \nabla^\perp_{Z_2}, \xi\rangle A_\xi Z_1,$$

and taking inner product with $X$ we get

$$\langle A_\eta Z_2, \nabla_{Z_1} X\rangle = \langle A_\eta Z_1, \nabla_{Z_2} X\rangle.$$

For the particular points $x$ and $y$, we have

$$\delta_2(x)\langle \nabla_{Z_1} Z_2, X\rangle(x) = \delta_1(x)\langle \nabla_{Z_2} Z_1, X\rangle(x)$$

$$\delta_1(y)\langle \nabla_{Z_1} Z_2, X\rangle(y) = \delta_2(y)\langle \nabla_{Z_2} Z_1, X\rangle(y).$$

The distributions $\nabla_{Z_1} Z_2$, $\nabla_{Z_2} Z_1$ and $D$ are invariant by isometries and then solving for $\langle \nabla_{Z_1} Z_2, X\rangle$ in the second equation and substituting into the first we obtain

$$\delta_2(x)\delta_2(y)\langle \nabla_{Z_2} Z_1, X\rangle = \delta_1(x)\delta_1(y)\langle \nabla_{Z_2} Z_1, X\rangle.$$

Since $\delta_2(x)\delta_2(y) - \delta_1(x)\delta_1(y) \neq 0$, for $\delta_1 < \delta_2$ we conclude that $\nabla_{Z_1} Z_2$ and $\nabla_{Z_2} Z_1$ are orthogonal to $D$. Therefore $\langle [Z_1, Z_2], X\rangle = 0$, which implies $\langle [X_1, X_2], X\rangle = 0$. Now the last two equalities in Lemma 12(a) imply

$$\langle \nabla_{X_1} X_2, X\rangle = \langle \nabla_{X_2} X_1, X\rangle = 0, \ \ \forall\, X \in D,$$

since we are supposing $\delta_1 \neq \delta_2$.

*Step 2* The vector field $\nabla_{X_i} X_j(x)$ is orthogonal to $D$, for every $x \in N$.

Since the distributions $\nabla_{Z_1} Z_2$, $\nabla_{Z_2} Z_1$ and $D$ are invariant by isometries, we have that $\langle [Z_1, Z_2], X\rangle$ is constant on $M$. *Step 1* implies that $[\langle [X_1, X_2], X\rangle = \langle [Z_1, Z_2], X\rangle$ is zero at $x_0$ and hence $[\langle [X_1, X_2], X\rangle = 0$ for all points of $N$. This and the two last equalities of Lemma 12(a)imply that if $\delta_1(x) \neq \delta_2(x)$ then $\langle \nabla_{X_1} X_2, X\rangle(x) = \langle \nabla_{X_2} X_1, X\rangle(x) = 0$. For points such that $\delta_1 = \delta_2$, the first equality of Lemma 12(a) implies that $\nabla_{X_1} X_2$ and $\nabla_{X_1} X_2$ are orthogonal to $D$.

*Step 3* There exists a point $p$ such that vector field $\nabla_{X_i} X_i(x)(p)$ is orthogonal to $D(p)$.

Here we use again the compactness of $N$. Let $t : N \to \mathbf{R}$ denote the trace of the Weingarten operator $A_\eta$ restricted to $N$. Since $t$ is continuous, let $p_1$ and $p_2$ denote points where $t$ achieves its minimum and maximum respectively. We then have $X(\delta_1)(p_i) = -X(\delta_2)(p_i)$. Since $\delta_1\delta_2$ is constant, we also have $\delta_1 X(\delta_2) + \delta_2 X(\delta_1) = 0$. These two equations imply

$$(\delta_1(p_i) - \delta_2(p_i))X(\delta_j) = 0, \ \ \forall\, i, j = 1, 2.$$

Let us suppose that $\delta_1(p_1) = \delta_2(p_1) = \delta$. Since $\delta_1\delta_2$ is constant, this constant is $\delta^2$. If $\delta_1(p_2) = \delta_2(p_2)$, then $\delta_1(p_2) = \delta_2(p_2) = \delta$ (notice that if not, there would be a point $p$ such that $t(p) = 0$ and then $\delta_1\delta_2(p)$ would be negative). We then conclude that $t$ is constant which in turn implies that $\delta_1$ and $\delta_2$ are constants. The results then follows from Lemma 12(b). If $\delta_1(p_2) \neq \delta_2(p_2)$ then $X(\delta_i)(p_2) = 0$ and then Lemma 12(b) implies $\nabla_{X_i} X_i(p_2)$ is orthogonal to $D(p_2)$.

Now we finish the proof of the lemma by observing that $\langle \nabla_{Z_1} Z_2, X\rangle$ is constant and therefore we use the point $p$ of *Step 3*. We write $Z_i$, $i = 1, 2$ as linear combinations of $X_1$ and $X_2$ and by the previous steps we conclude that $\langle \nabla_{Z_1} Z_2, X\rangle(p) = 0$. $\boxed{QED}$

**14 Proposition.** *If $[X, Z_i] \in \operatorname{Ker} A_\eta$, for $i = 1, 2$, $\forall\ X \in D$ then the Weingarten operator of $\bar{A}_\zeta$ has one eigenvalue of multiplicity at least $n-3$.*

Proof. Consider the Codazzi equation

$$\nabla_X A_\xi Z_1 - A_\xi \nabla_X Z_1 = \nabla_{Z_1} A_\xi X - A_\xi \nabla_{Z_1} X \ , \ \ X \in \operatorname{Ker} A_\eta.$$

If $X \in D$, by taking inner product with $Z_1$ we obtain

$$\langle A_\xi X, \nabla_{Z_1} Z_1\rangle = 0, \ \ \forall\ X \ \in D,$$

which in turn implies either $A_\xi(D) \subset D$ or $\nabla_{Z_1} Z_1 \in \operatorname{Ker} A_\eta^\perp$. The latter case implies that $\langle \nabla_{Z_2} Z_2, X\rangle = 0$ para $\forall\ X \in \operatorname{Ker} A_\eta$, by Lemma 12. We will see that both cases imply that $A_\xi X = \lambda X$, $\forall\ X \in D$.

Let us suppose first that $\nabla_{Z_2} Z_2$ is orthogonal to $\operatorname{Ker} A_\eta$. We compute the expression $\langle R(X, Z_2)Z_2, Y\rangle$ for $X \in D$ and $Y \in \operatorname{Ker} A_\eta$.

$$\begin{aligned}
\langle R(X, Z_2)Z_2, Y\rangle &= \langle \nabla_X \nabla_{Z_2} Z_2, Y\rangle - \langle \nabla_{Z_2}\nabla_X Z_2, Y\rangle - \langle \nabla_{[X,Z_2]} Z_2, Y\rangle \\
&= -\langle \nabla_{Z_2} Z_2, \nabla_X Y\rangle \\
&= -\langle \nabla_{Z_2} Z_2, Z_1\rangle\langle Z_1, \nabla_X Y\rangle \\
&= -a\langle \nabla_{Z_2} Z_2, Z_1\rangle\langle A_\xi X, Y\rangle
\end{aligned}$$

for $\langle \nabla_{Z_2} Z_2, Y\rangle = 0$, $\nabla_X Z_2 = 0$, and $[X, Z_2] \in \operatorname{Ker} A_\eta$, by Lemma 13. Since $\langle A_\xi Z_2, X\rangle = 0$, from the Gauss equation we get

$$\langle R(X, Z_2)Z_2, Y\rangle = \langle X, Y\rangle + \langle A_\xi X, Y\rangle\langle A_\xi Z_2, Z_2\rangle.$$

Comparing the two equations above we obtain

$$\langle X, Y\rangle + (\langle A_\xi Z_2, Z_2\rangle + a\langle \nabla_{Z_2} Z_2, Z_1\rangle)\langle A_\xi X, Y\rangle = 0, \ \ \forall X \in D, \ \forall Y \in \operatorname{Ker} A_\eta.$$

This equation implies $A_\xi X = \lambda X, \forall\, X \in D$, where

$$\lambda = \frac{-1}{\langle A_\xi Z_2, Z_2\rangle + a\langle \nabla_{Z_2} Z_2, Z_1\rangle}.$$

Now we suppose that $A_\xi(D) \subset D$. Let $X_i \in D$ be an eigenvector of $A_\xi$ with eigenvalue $\lambda_i$. Considering again the Codazzi equation

$$\nabla_{X_i} A_\xi Z_1 - A_\xi \nabla_{X_i} Z_1 = \nabla_{Z_1} A_\xi X_i - A_\xi \nabla_{Z_1} X_i\ , \ \ X \in \operatorname{Ker} A_\eta,$$

and taking inner product with $X_i$ we conclude that $\lambda_i = \langle A_\xi Z_1, Z_1\rangle, \forall i$. $\boxed{QED}$

**15 Lemma.** *The leaves $N$ are totally geodesic in $M$.*

PROOF. If $N$ is not totally geodesic then Lemmas 13 and 14 imply that $\bar{A}_\zeta$ has an eigenvalue $\lambda$ of multiplicity $m \geq n-3$. Therefore $N$ is either a sphere or a product a circle with a sphere. In either case, Lemma 11 implies that $D$ is the tangent space of a sphere, denoted by $S$, of constant curvature $k$. The Gauss equation for $S \to S^{n+2}$ implies

$$k = 1 + \lambda^2 + \langle \nabla_Y Y, Z_1\rangle^2, \tag{13}$$

since each vector $Y$ in $D$ is also an eigenvector of $A_\xi$ corresponding to the same eigenvalue $\lambda$.

Moreover, it follows from Lemma 6.2(a) of [4] that if $X, Y \in E_\lambda$ then $\nabla_X Y$ is also an eigenvector of $A_\xi$ corresponding to $\lambda$. From this and the fact that the eigenspaces of $\bar{A}_\zeta$ are auto-parallel distributions, we get that if $\langle \nabla_X X, Z_1\rangle \neq 0$, for $X \in D$ then $Z_1$ is an eigenvector of $A_\xi$ with eigenvalue $\lambda$. If the orthogonal projection $(\nabla_{Z_1}Z_1)'$ of $\nabla_{Z_1}Z_1$ onto $D$ is not zero then let us consider a unit vector field $Y \in E_\lambda$ in the direction of $(\nabla_{Z_1}Z_1)'$. Now we compute the curvature $K(Y, Z_1)$ and we have

$$\langle R(Y,Z_1)Z_1,Y\rangle = Y\langle \nabla_{Z_1}Z_1, Y\rangle - \langle \nabla_{Z_1}Z_1, \nabla_Y Y\rangle - Z_1\langle \nabla_Y Z_1, Y\rangle + \\ +\langle \nabla_Y Z_1, \nabla_{Z_1} Y\rangle - \langle \nabla_{[Y,Z_1]}Z_1, Y\rangle.$$

Our choice of $Y$ implies $\langle \nabla_{Z_1}Z_1, \nabla_Y Y\rangle = 0$. Moreover, we have

$$\langle \nabla_X Y, Z_1\rangle = 0 \quad \forall\, X \perp Y,\ X, Y \in E_\lambda \qquad \text{and} \quad \langle \nabla_Y Z_2, Z_1\rangle = 0,$$

where the last equality comes from (4) and the fact that $Y$ is an eigenvector of $A_\xi$. We then obtain that

$$\langle R(Y,Z_1)Z_1,Y\rangle = -\langle \nabla_Y Y, Z_1\rangle^2 - \langle \nabla_{Z_1}Z_1, Y\rangle^2. \tag{14}$$

Computing the same curvature through the Gauss equation we get

$$\langle R(Y,Z_1)Z_1,Y\rangle = 1 + \lambda^2. \tag{15}$$

This and (13) above would imply that $S$ would have curvature $-\langle \nabla_{Z_1}Z_1, Y\rangle^2$, which is clearly a contradiction. $\boxed{QED}$

**16 Theorem.** *Let $f : M^n \to S^{n+2}$ be an isometric immersion of a homogeneous Riemannian manifold such that for each $x \in M$ there exists an orthonormal frame $\{\xi, \eta\}$ of the normal space with $A_\xi$ constant, rank$A_\eta \equiv 2$ and $\bar\nu \le n-5$. Then one of the following occurs:*

*(a) $f(M^n)$ is a Riemannian product $\Sigma^2 \times S^{n-2}$, where $\Sigma^2$ is a surface of constant curvature contained in a 3- sphere.*

*(b) $f(M^n)$ is a Riemannian product $\Sigma^3 \times S^{n-3}$, where $\Sigma^3$ is a homogeneous hypersurface of a 4-sphere.*

Proof. Let $\{X_1, \ldots, X_n\}$ be an ortonormal basis of eigenvectors of the operator $A_\xi$ with the corresponding eigenvalues $\lambda_i$. Since Lemmas 8 and 15 imply that the normal bundle of the immersion $f$ is flat, we can suppose that $X_i \in \operatorname{Ker} A_\eta$ for $i \ge 3$.

We then consider the Codazzi equation for $X_1, X_2, \eta$ is

$$\nabla_{X_1}A_\eta X_2 - A_\eta(\nabla_{X_1}X_2) - A_{\nabla^\perp_{X_1}\eta}(X_2) = \nabla_{X_2}A_\eta X_1 - A_\eta(\nabla_{X_2}X_1) - A_{\nabla^\perp_{X_2}\eta}(X_1).$$

and taking inner product with $X \in \operatorname{Ker} A_\eta$ we get

$$\langle \nabla_{X_1}X_2, X\rangle\, \delta_2 = \langle \nabla_{X_2}X_1, X\rangle\, \delta_1. \tag{16}$$

Since $\delta_i \neq 0$ we conclude that the orthogonal projections of $\nabla_{X_1}X_2$ and $\nabla_{X_2}X_1$ onto $\operatorname{Ker} A_\eta$ are lineraly dependent. In addition, the eigenvalues of $A_\xi$ are constant and a standard application of the Codazzi equation gives that

$$\langle \nabla_{X_i}X_i, X_j\rangle = 0 \quad \text{whenever} \quad \lambda_i \neq \lambda_j. \tag{17}$$

We claim first that $\lambda_1 = \lambda_2$. Suppose that they are distinct. Then for some $i = 1, 2$, there exists $j \geq 3$ such that $\lambda_i \neq \lambda_j$. Then (17) implies that $\langle \nabla_{X_i}X_i, X_j\rangle = 0$. Recall that Lemma 12(c) gives

$$\langle \nabla_{X_1}X_1, X\rangle = -\langle \nabla_{X_2}X_2, X\rangle, \forall X \in \operatorname{Ker} A_\eta,$$

and hence we conclude that $\nabla_{X_i}X_i \in \operatorname{Ker} A_\eta^\perp$, for $i = 1, 2$. Further, the last two equalities of Lemma 12(a) imply the orthogonal projections of $\nabla_{X_1}X_1$ and $\nabla_{X_2}X_2$ onto $\operatorname{Ker} A_\eta$ are collinear. We then consider $X \in \operatorname{Ker} A_\eta$ orthogonal to both of them. Computing the sectional curvature $K(X_i, X)$ for $i = 1, 2$ we have

$$\langle R(X, X_i)X_i, X\rangle = \langle \nabla_X \nabla_{X_i}X_i, X\rangle - \langle \nabla_{X_i}\nabla_X X_i, X\rangle - \langle \nabla_{[X,X_i]}X_i, X\rangle.$$

Since the leaves of $\operatorname{Ker} A_\eta$ are totally geodesic we have that $\nabla_X X_i \perp \operatorname{Ker} A_\eta$ and then our choice of $X$ implies that $\langle \nabla_{X_i}\nabla_X X_i, X\rangle = 0$. Similarly we obtain $\langle \nabla_{[X,X_i]}X_i, X\rangle = 0$. Now we use again that the leaves of $\operatorname{Ker} A_\eta$ are totally geodesic and the fact that $\nabla_{X_i}X_i \in \operatorname{Ker} A_\eta^\perp$ to conclude that $K(X_i, X) = 0$. On the other hand the Gauss equation implies that

$$0 = K(X_i, X) = 1 + \lambda_i \, \langle A_\xi X, X\rangle,$$

yielding $\lambda_1 = \lambda_2$ and this contradicts our assumption.

Therefore we have $\lambda_1 = \lambda_2 = \mu$ and we denote $D_\mu$ the eigenspace corresponding $\mu$. Since $\xi$ is not an umbilical direction, there exists $\lambda_i \neq \mu$ and then $\langle \nabla_X Y, X_i\rangle = \langle \nabla_X Y, X_i\rangle = 0$, for all $X, Y \in D_\mu$. It follows that $D_\mu$ is an auto-parallel distribution and its leaf $N$ is a homogeneous submanifold of $\mathbf{S}^{n+2}$. Observe that its codimension in the sphere can be reduced to 1 and its normal space has one direction, $\eta$ such that $rank A_\eta = 2$. Therefore, if $\dim D_\mu = k \geq 4$, $N$ would split in a Riemannian product $S^2 \times S^{k-2}$ contradicting that for $X \in \operatorname{Ker} A_\eta$ and $Y \in \operatorname{Im} A_\eta$, the sectional curvature $K(X, Y) = 1 + \lambda^2$.

Then $\dim D_\mu \leq 3$. We will show that $A_\xi$ has only two distinct eigenvalues. Let $X_i \in D_\lambda^\perp$ be an eigenvector of $A_\xi$. We compute, using the Riemannan tensor, the sectional curvature of the plane $span\{X_j, X_i\}, j \leq 2$. Since $X_i$ is orthogonal to $\nabla_{X_k}X_j, k, j, = 1, 2$ we obtain (as before)

$$K(X_i, X_j) = 0 = 1 + \lambda\lambda_i,$$

and hence $\lambda_i = \lambda_l$, $\forall\, j, l \geq 3$.

Now we have that $D_\lambda$ and $D_\lambda^\perp$ are both parallel and involutive and thus from the de Rham Theorem we get that the universal cover $\tilde{M}$ is a Riemannian product.

If $\dim D_\mu = 2$ we have that $\tilde{M} = \Sigma^2 \times N^{n-2}$. Since we have $\alpha(X, Y) = 0$ for $X \in T\Sigma^2$ and $Y \in TN^{n-2}$, the immersion $f$ is a product of immersions. Then $N^{n-2}$ is umbilical in $\mathbf{R}^{n+3}$ and hence a sphere of constant curvature $\bar{c}$ immersed in an umbilical sphere $S_c^{n-1}$, while $\Sigma^2$, since its homogeneous, is a surface of constant curvature contained in a 3-sphere. If $\dim D_\mu = 3$ then $\tilde{M} = \Sigma^3 \times N^{n-3}$. Again we have, product of immersions and then that $N^{n-3}$ is an umbilical sphere $S_c^{n-2}$ and $\Sigma^3$ is a homogeneous hypersurface of 4-dimensional sphere. $\boxed{QED}$

## 4 Submanifolds of Hyperbolic Space

In this section we suppose that the ambient space is the Hyperbolic space of curvature $-1$. We still assume the hypotheses of Section 3. Homogeneous hypesurfaces of the hyperbolic space have been classified by Tsunero Takahashi in [19] and [20]. He proves that there are only three possibilities for the type number of a codimension 1 isometric immersion of a homogeneous space into the hyperbolic space, namely, $1, 2$ or $n$. Moreover, the case equal to 2 occurs only for 3-dimensional manifolds. This immediately implies, in our case, that $\xi$ is not an umbilical direction. In fact, suppose it is. Then, from the fundamental theorem for submanifols, we conclude that $M$ (or its universal covering) is immersed in an umbilical hypersurface $\mathbf{Q}^{n+1}$ of $\mathbf{H}^{n+2}$ and, since $\eta$ is the normal direction, with typer number 2. The results of Takahashi imply that $\mathbf{Q}$ is not the hyperbolic space, since $n \geq 5$. It is clear that $\mathbf{Q}$ is not the Euclidean space either. Therefore $\mathbf{Q}$ would have to be a sphere, and in this case, we get the same contradiction obtained in the previous section. Let $\tau$ denote the type number of the immersion $g : N_p^{n-2} \to \mathbf{H}^{n-1}$. The results of Takahashi imply:

(i) $\tau \leq 1$.

(ii) $\tau = n - 2$.

(iii) $\tau = 2$ and $n = 5$.

Further, if $\tau = n - 2$ then we have the following:

(a) The immersion $g$ is umbilical and each $N_p$ is isometric to a sphere or to the hyperbolic space or to the Euclidean space.

(b) The immersion is not umbilical and each $N_p$ is isometric to the Riemannian product of the sphere $S^m$ with the hyperbolic space $H^{n-2-m}$, $m \geq 1$.

Case (i) cannot occur under our assumption on the relative nullity. In fact, since the orthogonal projection of $A_\xi(\operatorname{Im} A_\eta)$ onto $\operatorname{Ker} A_\eta$ is at most one dimensional, $n - 4$ linearly independent directions of relative nullity of $g$ are in the relative nullity space of $f$, implying that $\bar{\nu} \geq n-4$, which contradicts that $\bar{\nu} \leq n-5$.

**17 Lemma.** *If $g$ is umbilical and $N$ is not a Euclidean space then $N$ is totally geodesic in $M$.*

Proof. From Lemma 11 we get that $A_\xi Z_2$ is orthogonal to $\operatorname{Ker} A_\eta$. This implies that if $X \in \operatorname{Ker} A_\eta$ then $X$ is an eigenvector of $A_\xi$; it also implies that $A_\xi$ has an eigenvalue, denoted by $\lambda$, of multiplicity at least $n-2$. As in the proof of Lemma 9, we conclude that if $\langle \nabla_X X, Z_1\rangle \neq 0$, then $Z_1$ is an eigenvector of $A_\xi$ with eigenvalue $\lambda$. It follows that $Z_2$ is also an eigenvector of $A_\xi$, with corresponding eigenvalue $\lambda_1 \neq \lambda$, since $A_\xi$ is not umbilical. Let us consider the Codazzi equation

$$\nabla_X A_\xi Z_2 - A_\xi \nabla_X Z_2 - \langle \nabla^\perp_X \xi, \eta\rangle A_\eta Z_2 = \nabla_{Z_2} A_\xi X - A_\xi \nabla_{Z_2} X - \langle \nabla^\perp_{Z_2}\xi, \eta\rangle A_\eta X,$$

$X \in \operatorname{Ker} A_\eta$. Taking inner product with $Z_2$ we obtain $(\lambda_1 - \lambda)\langle \nabla_{Z_2} Z_2, X\rangle = 0$, giving that $\langle \nabla_{Z_2} Z_2, X\rangle = 0$. ¿From Lemma 10(b) we get $\langle \nabla_{Z_1} Z_1, X\rangle = 0$. The same Codazzi equation for $X$ and $Z_1$ implies $[Z_1, X]$ is in the eigenspace of $\lambda$. Now we compute the curvature $K(Y, Z_1)$ and we have

$$\langle R(X,Z_1)Z_1, X\rangle = X\langle \nabla_{Z_1} Z_1, X\rangle - \langle \nabla_{Z_1} Z_1, \nabla_X X\rangle - Z_1\langle \nabla_X Z_1, Y\rangle +$$
$$+\langle \nabla_X Z_1, \nabla_{Z_1} X\rangle - \langle \nabla_{[X,Z_1]} Z_1, X\rangle.$$

Moreover, we have

$$\langle \nabla_X Y, Z_1\rangle = 0 \quad \forall\, X \perp Y,\ X, Y \in E_\lambda \qquad \text{and} \quad \langle \nabla_X Z_2, Z_1\rangle = 0,$$

where the last equality comes from (4) and the fact that $X$ is an eigenvector of $A_\xi$. We then obtain that

$$\langle R(X,Z_1)Z_1, X\rangle = -\langle \nabla_X X, Z_1\rangle^2. \tag{18}$$

Computing the same curvature through the Gauss equation we get

$$\langle R(X,Z_1)Z_1, X\rangle = -1 + \lambda^2. \tag{19}$$

On the other hand, applying the Gauss equation to the immersion $N \to \mathbf{H}^{n+2}$, we obtain

$$K_N = -1 + \lambda^2 + \langle \nabla_X X, Z_1\rangle^2,$$

and therefore (18) and(19) imply $K_N = 0$, contradicting that $N$ has non-zero curvature. $\boxed{QED}$

**18 Lemma.** *If $\tau = n-2$ and $g$ is not umbilical then $N$ is totally geodesic in $M$.*

Proof. In this case we have that $N$ is Riemannian product $S^m \times \mathbf{H}^{k=n-2-m}$. If $m, k \geq 2$ then $\bar{A}_\zeta$ has two eigenvalues and each has muliplicity at least two. It follows from Lemma 11 that each vector tangent to $S^m$ and each vector tangent to $\mathbf{H}^k$ are eigenvectors of $A_\xi$. The assumption on the relative nullity implies that the corresponding eigenvalues are non-zero. Now from Lemma 9 we get that $N$ is totally geodesic in $M$.

If $m = 1$, then $k \geq 2$ and Lemma 11 implies that $\langle A_\xi Z_2, X\rangle = 0$, for all $X$ tangent to $\mathbf{H}^k$ and then all vectors tangent to $\mathbf{H}^k$ are eigenvectors of $A_\xi$ corresponding to the same eigenvalue that we denote by $\lambda$. As before we conclude that if $\langle \nabla_X X, Z_1\rangle \neq 0$ then $Z_1$ is eigenvector of $A_\xi$ correponding to $\lambda$.

Let $\lambda_i, i = 1, 2$, denote the other two eigenvalues of $A_\xi$. Let $Y$ denote a unit vector tangent to $S^1$ and $E_i$ eigenvectors of corresponding to $\lambda_i$. If $\lambda_1 = \lambda_2$, then $Y$ is also an eigenvector of $A_\xi$ and from Lemma 9 we obtain that $N$ is totally geodesic.

If $\lambda_1 \neq \lambda_2$, then $\lambda_i \neq \lambda$ for some $i = 1, 2$, say $\lambda_1 \neq \lambda$.

If $\lambda_2 \neq \lambda$, a standard application of the Codazzi equation implies that

$$\langle \nabla_{E_1} E_1, X\rangle = 0 \quad \text{and} \quad \langle \nabla_{E_2} E_2, X\rangle = 0,$$

for $X$ tangent to $\mathbf{H}^k$. We then write

$$E_1 = aZ_2 + bY, \qquad E_2 = -bZ_2 + aY,$$

and obtain

$$\langle \nabla_{E_1} E_1, X\rangle = a^2\langle \nabla_{Z_2} Z_2, X\rangle + ba\langle \nabla_{Z_2} Y, X\rangle = 0$$
$$\langle \nabla_{E_2} E_2, X\rangle = b^2\langle \nabla_{Z_2} Z_2, X\rangle - ba\langle \nabla_{Z_2} Y, X\rangle = 0.$$

Notice that if $ba \neq 0$, the homogeneous system above has only the trivial solution and thus $\langle \nabla_{Z_2} Z_2, X\rangle = 0$, which in turn implies $\langle \nabla_{Z_1} Z_1, X\rangle = 0$. Now the same arguments used at the end of the proof of Lemma 17 gives that $K_N(X, X') =$ contradicting that $X$ and $X'$ are tangent to the Hyperbolic space. If $ba = 0$, then $Y$ is an eigenvector of $A_\xi$ and we apply Lemma 9.

If $\lambda_2 = \lambda$, we write the Codazzi equation

$$\nabla_{Z_1} A_\xi E_2 - A_\xi \nabla_{Z_1} E_2 - \langle \nabla^\perp_{Z_1}\xi, \eta\rangle A_\eta E_2 = \nabla_{E_2} A_\xi Z_1 - A_\xi \nabla_{E_2} Z_1 - \langle \nabla^\perp_{E_2}\xi, \eta\rangle A_\eta Z_1.$$

Taking inner product with $E_2$ we have

$$b^2\langle \nabla^\perp_{Z_1}\xi, \eta\rangle\langle A_\eta Z_2, Z_2\rangle = b^2\langle \nabla^\perp_{Z_2}\xi, \eta\rangle\langle A_\eta Z_1, Z_2\rangle. \tag{20}$$

Let $U$ denote

$$U = \langle \nabla^\perp_{Z_1}\xi, \eta\rangle Z_2 - \langle \nabla^\perp_{Z_2}\xi, \eta\rangle Z_1.$$

Observe that $\langle \nabla^{\perp}_{U}\xi, \eta\rangle = 0$ and hence $A_\eta U$ is in the direction of $Z_2$. On the other hand, (20) implies $b^2\langle A_\eta U, Z_2\rangle = 0$. Since $rank A_\eta = 2$, $U \notin \operatorname{Ker} A_\eta$ and we conclude that $b = 0$. This implies that $Y = E_2$, which is a contradiction for $\langle A_\eta Y, Y\rangle \neq \langle A_\eta X, X\rangle = \lambda$. $\boxed{QED}$

The last case to be considered is case (iii), which cannot occur under our assumption on the relative nullity. In fact, let $X_i$ be orthonormal eigenvectors of $\bar{A}_\zeta$ with $X_1$ corresponding to the zero eigenvalue. We basically repeat the arguments (and the notation) used in the proof of Lemma 11, considering the unit vector field $V$ obtained by the orthogonal projection of $A_\xi Z_2$ onto $\operatorname{Ker} A_\eta$. The Ricci equation for the immersion $N \to M$ gives

$$\langle \bar{R}^{\perp}(X, V)Z_1, Z_2\rangle = \langle \bar{\nabla}^{\perp}_{[X,V]}Z_1, Z_2\rangle = 0,$$

$$\langle \bar{R}^{\perp}(X, Y)Z_1, Z_2\rangle = \langle \bar{\nabla}^{\perp}_{[X,Y]}Z_1, Z_2\rangle = 0.$$

The first equation gives that $[X, V]$ is orthogonal to $V$, while the second implies that $\langle [X, Y], V\rangle = 0$. In particular, $\langle V, [X_i, X_j]\rangle = 0$.

If $\lambda_2 = \lambda_3$, Lemma 11 implies that $X_2$ and $X_3$ are also eigenvectors of $A_\xi$. Proceeding as in the proof of Lemma 18, we would conclude that $N$ is totally geodesic which in turn implies that $\xi$ and $\eta$ are parallel sections and hence $\operatorname{Im} A_\eta$ is invariant by $A_\xi$. Then we would conclude that $X_1$ is also an eigenvector of $A_\xi$ with eigenvalue 0. Therefore $\bar{\nu} \geq 1$, contradicting our assumption that $\bar{\nu} \leq n-5$, since in this case $n = 5$.

If $\lambda_2 \neq \lambda_3$, since they are non-null, $\bar{A}_\zeta$ has three distinct eigenvalues. The eigenspaces of $\bar{A}_\zeta$ form auto-parallel distributions and hence

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = a_k X_k.$$

Suppose $[X_1, X_j] \neq 0$, then $V$ is in the direction of $X_i$, $i \neq j, i, j = 2, 3$ which implies that $X_1$ is an eigenvector of $A_\xi$ with eigenvalue 0, contradicting our assumption on $\bar{\nu}$. If $[X_1, X_j] = 0, j = 1, 2$ and $[X_2, X_3] \neq 0$ then $V$ is in the direction of $X_1$ and $X_2, X_3$ are eigenvectors of $A_\xi$. Since $\lambda_2 \neq \lambda_3$, Lemma 9 implies that $N$ is totally geodesic in $M$. Now, if $[X_2, X_3] = 0$, then $N$ has three parallel orthonormal vector fields and hence its a flat space. But using the Gauss equation we obtain that $K'(X_1, X_i) = -1$, and we have a contradiction.

**19 Theorem.** *Let $f : M^n \to \boldsymbol{H}^{n+2}$ be an isometric immersion of a homogeneous Riemannian manifold such that for each $x \in M$ there exists an orthonormal frame $\{\xi, \eta\}$ of the normal space with $A_\xi$ constant, $rank A_\eta \equiv 2$ and $\bar{\nu} \leq n - 5$. Then one of the following occurs:*

*(a) $\tilde{M}$, the universal covering of $M$, is a Riemannian product $\Sigma^2 \times N^{n-2}$, where $\Sigma^2$ is a surface of constant curvature isometrically immersed in a 3-dimensional space form and $N^{n-2}$ is isometric to one of the following:*

*(i) a sphere* $S_c^{n-2}$.

*(ii) the hyperbolic space* $\boldsymbol{H}_{c_1}^{n-2}$, $-1 < c_1 < 0$.

*(iii) the Euclidean space.*

*(b)* $\tilde{M}$ *is a Riemannian product* $\Sigma^3 \times \mathbf{H}_{c_1}^{n-3}$, $-1 < c_1 < 0$, *where* $\Sigma^3$ *is a homogeneous hypersurface of a* 4*-dimensional sphere.*

*(c)* $M$ *is a cohomogeneity one manifold such that all orbits are flat spaces.*

Proof. We start by supposing that $N$ is totally geodesic in $M$. Then the same arguments used in the beginning of the proof of Theorem 16 can be repeated to conclude that there exists a parallel distribution $D_\mu$ containing $\operatorname{Im} A_\eta$. We claim that $\dim D_\mu \leq 3$. In fact, if not, since $D_\mu$ is auto-parallel, its leaf would live in an umbilical hypersurface of $\mathbf{H}^{n+2}$. Since its dimension would be at least 4, we would have the same contradiction obtained when we supposed that $\xi$ was an umbilical direction.

If $\dim D_\mu = 2$, then $\tilde{M}$ is a Riemannian product of $\Sigma^2 \times N^{n-2}$. The immersion $\tilde{f} : \tilde{M} \to \mathbf{H}^{n+2}$ reduces codimension, that is, $\tilde{f}(\Sigma^2)$ lies in the hyperbolic space $\mathbf{H}^4$, which is totally geodesic in $\mathbf{H}^{n+2}$. Further, $N$ is a space form of curvature $-1 + \lambda^2$ and, since $-1 + \lambda\mu = 0$, $\tilde{f}(\Sigma^2)$ lies in a umbilical hypersurface of $\mathbf{H}^4$ of curvature $-1 + \mu^2 = (1 - \lambda^2)\lambda^{-2}$. This gives (a)

If $\dim D_\mu = 3$, then $A_\xi$ restricted to $\operatorname{Ker} A_\eta$ has an eigenvalue $\lambda$ of mutiplicity $n-3$ and $N$ is $S^1 \times \mathbf{H}_c^{n-3}$, where $c = -1+\lambda^2$. In this case $\tilde{M}$ splits in Riemannian product $\Sigma^3 \times \mathbf{H}_c^{n-3}$ and $\tilde{f}(\Sigma^3)$ lies in the hyperbolic space $\mathbf{H}^5$, which is totally geodesic in $\mathbf{H}^{n+2}$. Moreover, $\tilde{f}(\Sigma^3)$ is contained in an umbilical hypersurface of $\mathbf{H}^5$. Notice that the eigenvalue corresponding to the direction tangent to $S^1$ is equal to $\mu$. Therefore $-1 + \mu\lambda = 0$, by the Gauss equation, which in turn gives $-1 + \mu^2 = (1 - \lambda^2)\lambda^{-2}$. It follows then that $\Sigma^3$ lives in a 4-dimensional sphere, and this is (b).

Now we consider the case that $N$ is not totally geodesic in $M$. It follows from Lemmas 17 and 18 that $N$ is the Euclidean space. We then consider the vector fields $Z_1$ and $Z_2$ defined previously in (2). Recall that Lemma 11 implies that $\operatorname{Ker} A_\eta$ is invariant by $A_\xi$. The fact that $N$ is not totally geodesic gives that $Z_1$ is an eigenvector of $A_\xi$ with corresponding eigenvalue $\lambda$ and hence $Z_2$ is also an eigenvector with eigenvalue that we will denote by $\lambda_1$. Standard applications of the Codazzi equation give

$$\langle \nabla_{Z_2} Z_2, X \rangle = 0, \quad \langle \nabla_X Z_1, Z_2 \rangle = 0, \quad \text{and} \quad \langle \nabla_{Z_1} X, Z_2 \rangle = 0.$$

Now, Lemma 10(b) implies $\langle \nabla_{Z_1} Z_1, X \rangle = 0$. We will show that $\langle \nabla_{Z_2} Z_1, X \rangle = 0$. For that, consider the Codazzi equation

$$\nabla_{Z_1} A_\eta X - A_\eta \nabla_{Z_1} X - \langle \nabla^\perp_{Z_1} \eta, \xi \rangle A_\xi X = \nabla_X A_\eta Z_1 - A_\eta \nabla_X Z_1.$$

Taking inner product with $Z_i$, $i = 1, 2$, we obtain

$$-\langle \nabla_{Z_1} X, A_\eta Z_i \rangle = X(\langle A_\eta Z_1, Z_i \rangle) - \langle \nabla_X Z_1, A\eta Z_i \rangle.$$

Since $\nabla_X Z_1$ and $\nabla_{Z_1} X$ are both in $\operatorname{Ker} A_\eta$, we conclude that $X(\langle A_\eta Z_1, Z_i \rangle) = 0$. The homogeneity of $M$ and the fact that $A_\xi$ is constant implies $X(\langle A_\eta Z_2, Z_2 \rangle) = 0$, and hence $X(\delta_i) = 0$, where $\delta_i$, $i = 1, 2$ denote the non-null eigenvalues of $A_\eta$ corresponding to eigenvectors denoted by $X_i$. From Lemma 12 we get that $\langle \nabla_{X_i} X_i, X \rangle = 0$. Then we write $X_i$ as linear combination of $Z_1$ and $Z_2$ and we get $\langle \nabla_{Z_2} Z_1, X \rangle = 0$.

Now we consider the distribution $L = \mathit{span}\{Z_2, X_1, \ldots, X_{n-2}\}$, where the vectors $X_1, \ldots, X_{n-2}$ is basis of $\operatorname{Ker} A_\eta$. This distribution is invariant by isometries, involutive and whose leaves are homogeneous submanifolds. We will show that $\nabla_{Z_1} Z_1 = 0$. We already know that $\langle \nabla_{Z_1} Z_1, X \rangle = 0$. Now, using the Codazzi equation

$$\nabla_{Z_1} A_\xi Z_2 - A_\xi \nabla_{Z_1} Z_2 - \langle \nabla^\perp_{Z_1} \eta, \xi \rangle A_\eta Z_2 = \nabla_{Z_2} A_\xi Z_1 - A_\xi \nabla_{Z_2} Z_1 - \langle \nabla^\perp_{Z_2} \eta, \xi \rangle A_\eta Z_1,$$

and taking product with $Z_1$ yields

$$(\lambda_1 - \lambda) \langle \nabla_{Z_1} Z_1, Z_2 \rangle = \langle A_\eta U, Z_1 \rangle,$$

where $U$ is a vector given

$$U = \langle \nabla^\perp_{Z_2} \eta, \xi \rangle Z_1 - \langle \nabla^\perp_{Z_1} \eta, \xi \rangle Z_2.$$

Since $\langle \nabla^\perp_U \eta, \xi \rangle = 0$ we have that $A_\eta U$ is in the direction of $Z_2$, which gives us that $\langle \nabla_{Z_1} Z_1, Z_2 \rangle = 0$, an hence $\nabla_{Z_1} Z_1 = 0$. It follows that the integral curve $\gamma$ of $Z_1$ is a geodesic that is orthogonal to the leaves of $L$. Let $S$ denote the maximal leaf at $p$ and

$$K = \{g \in I(M) \,|\, g(S) \subset S\}.$$

We then have that $M$ is a Riemannian $K$-cohomogeneity one manifold and $S$ is principal orbit. Proposition 4.1 of [1], states that $\gamma$ crosses each orbit of $K$ orthogonally and this implies that the leaves of $L$ are the orbits of $K$.

Since $N$ is totally geodesic in $S$, we have that $K_S(X, X') = 0$, for $X$ and $X'$ in $\operatorname{Ker} A_\eta$. We will show that $K_S(Z_2, X) = 0$. First, from the Gauss equation for the immersion $f_{|S} : S \to \mathbf{H}^{n+2}$ we have

$$K_S(Z_2, X) = -1 + \lambda_1 \lambda + \langle \nabla_X X, Z_1 \rangle \langle \nabla_{Z_2} Z_2, Z_1 \rangle.$$

Now we compute the curvature of plane $\mathit{span}\{Z_2, X\}$ in $M$. First, we use the curvature tensor and from the properties of the vector fields $Z_2$ and $X$ we get

$$\begin{aligned} \langle R(X, Z_2) Z_2, X \rangle &= \langle \nabla_X \nabla_{Z_2} Z_2, X \rangle - \langle \nabla_{Z_2} \nabla_X Z_2, X \rangle - \langle \nabla_{[X, Z_2]} Z_2, X \rangle \\ &= -\langle \nabla_{Z_2} Z_2, Z_1 \rangle \langle \nabla_X X, Z_1 \rangle. \end{aligned}$$

Using the Gauss equation, we obtain $\langle R(X, Z_2)Z_2, X\rangle = -1 + \lambda_1\lambda$. It follows then that $K_S(Z_2, X) = 0$. $\boxed{QED}$

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