Note di Matematica **21,** n. 1, 2002, 127–135.

Real parallelisms

Norman L. Johnson
Mathematics Department, University of Iowa
Iowa City, IA. 52242
njohnson@math.uiowa.edu

Rolando Pomareda
Mathematics Dept., University of Chile
Casilla 653 Santiago, Chile

Received: 20 April 2002; accepted: 20 April 2002.

Abstract. A garden of parallelisms in $PG(3, R)$ are constructed, where R is the field of real numbers.

Keywords: parallelisms, infinite parallelisms, real parallelisms.

MSC 2000 classification: 51E23; 51A40.

Introduction

The first author (Johnson [3]) constructs a class of parallelisms in $PG(3, K)$, where K is an arbitrary field which admits a quadratic extension. The construction involves the use of a central collineation group G of a Pappian spread lying in the parallelism so that G also acts as a collineation group of the parallelism. The authors have recently enumerated isomorphism classes for the finite examples. Furthermore, there is the following classification.

Theorem 1. *(see Johnson and Pomareda [2]) Let K be a skewfield, Σ a spread in $PG(3, K)$ and $\mathcal{P}$ a partial parallelism of $PG(3, K)$ containing Σ.*

If $\mathcal{P}$ admits as a collineation group the full central collineation group G of Σ with a given axis ℓ that acts two-transitive on the remaining spread lines then

(1) Σ is Pappian,

(2) $\mathcal{P}$ is a parallelism,

(3) the spreads of $\mathcal{P} - \{\Sigma\}$ are Hall, and

(4) G acts transitively on the spreads of $\mathcal{P} - \{\Sigma\}$.

(5) Moreover, $\mathcal{P}$ is one of the parallelisms of the construction of Johnson.

Furthermore, in Johnson [1], this general idea was used to construct a variety of parallelisms admitting a subgroup G^- of the full central collineation group of a finite Desarguesian spread Σ where G^- acts transitively on the remaining spreads of the parallelism.

Theorem 2. *(Johnson [1]) Let q be odd equal to $p^{2^b z}$ where z is an odd integer > 1. Assume that $2^a \parallel (q-1)$ then there exists a nonidentity automorphism σ of $GF(q)$ such that $2^a \mid (\sigma - 1)$.*

Let γ_2 and γ_1 be nonsquares of $GF(q)$ such that the equation $\gamma_2 t^\sigma = \gamma_1 t$ implies that $t = 0$.

(1) Then, there exists a parallelism $\mathcal{P}_{\gamma_2,\sigma}$ of derived Knuth type with $q^2 + q$ derived Knuth planes and one Desarguesian plane.

(2) The collineation group of this parallelism contains the central collineation group of the Desarguesian plane with fixed axis ℓ of order $q^2 2^a (q+1)$.

Actually, all of these examples stem from a general construction process, we will list as follows:

Let Σ be any Pappian spread in $PG(3,K)$ and let Σ' any spread which shares a regulus R with Σ such that Σ' is derivable with respect to R. Assume that there exists a subgroup G^- of the central collineation group G with fixed axis L with the following properties:

(0) : Σ and Σ' share exactly R,

(i) : Every line skew to L and not in Σ is in $\Sigma' G^-$,

(ii) : G^- is transitive on the reguli that share L and

(iii) : a collineation g of G^- such that for $L' \in \Sigma'$ then $L'g \in \Sigma'$ implies that g is a collineation of Σ'.

Let $(Rg)^*$ denote the opposite regulus to Rg.

Theorem 3. *Under the above assumptions, $\Sigma \cup \{(\Sigma' g - Rg) \cup (Rg)^*$ for all $g \in G^-$ is a parallelism of $PG(3,K)$ consisting of one Pappian spread Σ and the remaining spreads derived Σ'-spreads.*

Moreover, there are some related parallelisms, called the 'derived parallelisms'.

Theorem 4. *(see Johnson [1]) Assume that $\Sigma \cup \{(\Sigma' g - Rg) \cup (Rg)^*$for all $g \in G^-\}$ is a parallelism.Then $\{\Sigma - R\} \cup R^* \cup \Sigma' \cup \{(\Sigma' g - Rg) \cup (Rg)^*$ for all $g \in G^- - \{1\}\}$ is a parallelism. In this case, the spreads are Hall, Σ' (undetermined) and derived Σ' type spreads.*

As mentioned above, the application of this construction technique has been applied most successfully when the spreads other than the Pappian spread are derived conical flock spreads and when the group contains a large normal subgroup that is a central collineation group. (By 'conical flock spreads', we intend to mean those spreads that correspond to flocks of quadratic cones.)

Actually, there is a classification of sorts of such parallelisms.

Theorem 5. *(see Johnson [1]) Let $\mathcal{P}$ be a parallelism in $PG(3, K)$, for K a field, that admits a Pappian spread Σ and a collineation group G^- fixing a line ℓ of Σ that acts transitively on the remaining spreads of $\mathcal{P}$.*

(1) If K is finite and if G^- contains the full elation group with axis ℓ then the spreads of $\mathcal{P} - \{\Sigma\}$ are derived conical flock spreads.

(2) If G^- contains the full elation group with axis ℓ and for ρ a spread of $\mathcal{P} - \{\Sigma\}$, G_ρ^- contains a non-trivial homology (i.e. homology in Σ) then the spreads of $\mathcal{P} - \{\Sigma\}$ are derived conical flock spreads.

So, we see that good candidates for the initial spreads involved in the construction of such parallelisms are (derived) conical flock spreads.

In this article, we ask if the above construction can be considered over infinite fields and we isolate on the question when the field is the field of real numbers. Here we are able to show that there are a vast number of parallelisms depending on the class of strictly increasing functions f on the reals that define a class of conical flock spreads. We point out that our construction process constructs not only parallelisms but (proper) maximal partial parallelisms, and such form the first known classes of such objects.

1 Constructions over the Reals

We work over the field of real numbers $K = \mathcal{R}$.

We consider a Pappian spread Σ_1 defined as follows:

$$x = 0, y = x \begin{bmatrix} u & -t \\ t & u \end{bmatrix} \forall u, t \in \mathcal{R}.$$

We let Σ_2 be a spread in $PG(3, \mathcal{R})$, defined by a function f:

$$x = 0, y = x \begin{bmatrix} u & -f(t) \\ t & u \end{bmatrix} \forall u, t \in \mathcal{R}$$

where f is a function such that $f(t) = t$ implies that $t = 0$ and $f(0) = 0$.

Thus, if a spread exists then the two spreads Σ_1 and Σ_2 share exactly the regulus $\mathcal{D}$ with partial spread:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in \mathcal{R}.$$

Lemma 1. *Let f be any continuous strictly increasing function on the field of real numbers such that $\lim_{x \longmapsto \pm\infty} f(t) = \pm\infty$.*

(1) Then Σ_2 is a spread.

(2) Let $G^{-} = EH^{-}$ where H^{-} denotes the homology group of Σ_1 (or rather the associated affine plane) whose elements are given by

$$\left\langle \begin{bmatrix} u & -t & 0 & 0 \\ t & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; u^2 + t^2 = 1 \right\rangle .$$

and where E denotes the full elation group with axis $x = 0$.

(2) Then G^{-} is transitive on the set of reguli of Σ_1 that share $x = 0$.

PROOF. The proof of (1) is in Johnson and Liu [4].

We consider part (2). Since E is transitive on the components of Σ_1 not equal to $x = 0$, then first assume that there is a regulus $\mathcal{D}_1$ which shares exactly $x = 0$ with $\mathcal{D}$. Then there is an elation subgroup $E_{\mathcal{D}_1}$ which acts transitively on the components not equal $x = 0$. It follows easily that $E_{\mathcal{D}_1} = E_{\mathcal{D}}$ and this group induces a partition of the components of Σ_1 into a unique set of reguli that mutually share $x = 0$. (In this context, the set of 'elation-base' reguli determine a flock of a quadratic cone in $PG(3, \mathcal{R})$.)

Since $E_{\mathcal{D}}$ is a normal subgroup of E, these elation-base reguli are permuted transitively by E.

Now assume that a regulus $\mathcal{D}_2$ shares two components with $\mathcal{D}$ which we may take without loss of generality to be $x = 0$ and $y = 0$. Now there is a unique set of reguli sharing $x = 0$ and $y = 0$ which cover the components of Σ_1. These reguli have the property that there is a collineation group H_1 of the full central collineation group with axis $x = 0$ and coaxis $y = 0$ with the property that H_1 acts transitively on the non-fixed components of each regulus. (Here the set of 'homology-base' reguli determine a flock of a hyperbolic quadric in $PG(3, \mathcal{R})$.)

The group H_1 has the following form:

$$\left\langle \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; u \in \mathcal{R} - \{0\} \right\rangle .$$

We note that this is the form for the group due to the form of $\mathcal{D}$.

Hence, $\mathcal{D}_2$ has the following basic form:

$$x = 0, y = x \begin{bmatrix} w & -s \\ s & w \end{bmatrix} vI_2 \ \forall \ \textit{nonzero} \ v \in \mathcal{R}$$

where w and s are fixed elements of $\mathcal{R}$.

It remains to show that a determinant 1 homology maps $\mathcal{D}$ onto $\mathcal{D}_2$. We note that, since we are dealing with reguli, if $y = x$ of $\mathcal{D}$ maps to some component of $\mathcal{D}_2$ then $\mathcal{D}$ must map to $\mathcal{D}_2$. Hence, equivalently, for a given w, s does there exist elements u and t such that $u^2+t^2=1$ and a nonzero element v of $\mathcal{R}$ such that

$$\begin{bmatrix} w & -s \\ s & w \end{bmatrix} v^{-1} I_2 = \begin{bmatrix} u & -t \\ t & u \end{bmatrix}.$$

The determinant 1 group determines a circle of radius 1 and center $(0,0)$ in the real 2-dimensional plane. Since $(uv)^2+(tv)^2=v^2$ determine a circle of radius v in the real 2-dimensional plane, then any affine point (w,s) lines on one of these circles. Considering that the mapping $(x,y)\longmapsto(xv,yv)$ for v fixed and nonzero is a bijective mapping, it follows that if (w,s) is on the circle of radius v and center $(0,0)$ then $w^2+s^2=v^2$ if and only if $(w/v,s/v)$ is a point on the circle of radius 1 and equal to some (u,t) such that $u^2+t^2=1$.

Hence, we must have $\begin{bmatrix} w & -s \\ s & w \end{bmatrix} v^{-1} I_2 = \begin{bmatrix} u & -t \\ t & u \end{bmatrix}$.

This completes the proof of part (2). $\boxed{QED}$

Theorem 6. *Under the above assumptions, assume also that f is symmetric with respect to the origin in the real Euclidean 2-space and $f(t_o+r)=f(t_o)+r$ for some t_o and r in the reals implies that $r=0$.*

Then $\Sigma_1\cup\Sigma_2^*g$ for all $g\in G^-$ and where Σ_2^* denotes the derived spread of Σ_2 by derivation of $\mathcal{D}$, is a partial parallelism $\mathcal{P}_f$ in $PG(3,\mathcal{R})$.

Proof. By previous arguments, it suffices to show that the set of spreads $\cup\Sigma_2 g$ for all $g\in G^-$ covers uniquely a line of $PG(3,\mathcal{R})$ that does not lie in Σ_1 and which is disjoint from $x=0$ provided it covers it.

Assume that $\Sigma_2 g$ and $\Sigma_2 h$ share a component. Then so do $\Sigma_2 gh^{-1}$ and Σ_2 share a component M. Let $gh^{-1}=k$ and represent k as follows:

$$k=\begin{bmatrix} 1 & 0 & m & -r \\ 0 & 1 & r & m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} w & -s & 0 & 0 \\ s & w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

such that $w^2+s^2=1$.

Let M be

$$y=x\begin{bmatrix} u^* & -f(t^*) \\ t^* & u^* \end{bmatrix}$$

and let the preimage of k be

$$y=x\begin{bmatrix} u & -f(t) \\ t & u \end{bmatrix}.$$

Hence, we must have

$$\begin{bmatrix} w & -s \\ s & w \end{bmatrix} \begin{bmatrix} u^* & -f(t^*) \\ t^* & u^* \end{bmatrix} = \begin{bmatrix} u & -f(t) \\ t & u \end{bmatrix} + \begin{bmatrix} m & -r \\ r & m \end{bmatrix}.$$

Equating the $(1,1)$ and $(2,2)$ entries, we must have:

$$wu^* - st = wu^* - sf(t^*).$$

However, by our conditions on f, we must have that $s = 0$. Since $w^2+s^2 = 1$, this implies that $w = \pm 1$. Note that since f is symmetric with respect to the origin then $-f(-t) = f(t)$ for all t in the reals. This means that the homology $(x, y) \longmapsto (-x, -y)$ is a collineation of Σ_2. Hence, we may assume that $w = 1$.

Equating the $(1,2)$ and $(2,1)$ entries above, we obtain:

$$\begin{aligned} -f(t^*) &= -f(t) - r = f(-t) - r \text{ and} \\ t^* &= t + r. \end{aligned}$$

Hence, we obtain:

$$f(t + r) = -(f(-t) - r) = f(t) + r.$$

By our condition, this implies that $r = 0$.

In this case, we see that k is a collineation of Σ_2. Hence, we have the proof to our theorem. $\boxed{QED}$

Example 1. For examples of continuous, strictly increasing functions f such that $f(t) = t$ implies that $t = 0$ and $f(t_o + r) = f(t_o) + r$ for some t_o implies $r = 0$ which are also onto functions, we consider the following set of examples:

Let $f(t) = t + a^t - 1$ for $a > 1$ if $t \geq 0$ and let
$f(t) = t - a^{-t} + 1$ if $t < 0$.

Note that we are basically 'defining' the function so that $-f(-t) = f(t)$.

PROOF. Let $f(t) = t = t+a^t-1$ if and only if $a^t = 1$ if and only if $t = 0$. Now assume that $f(t_o+r) = f(t_o)+r$. Without loss of generality, we may assume that t_o is positive. If r is non-negative then we obtain $t_o+r+a^{t_o+r}-1 = t_o+a^{t_o}-1+r$ if and only if $a^{t_o+r} = a^{t_o}$ so that $r = 0$. Hence, assume that r is non-positive but $t_o + r$ is positive, the above proof applies. Hence, assume that t_o is positive, r is negative and $t_o + r$ is negative.

Hence, this implies that $t_o + r - a^{-(t_o+r)} + 1 = t_o + a^{t_o} - 1 + r$. This equation is valid if and only if $a^{-(t_o+r)} - 2 + a^{t_o} = 0$. Hence, we must have:

$$a^{2t_o} - 2a^{t_o} + a^{-r} = 0.$$

Hence, we must have

$$a^{t_o} = (2 \pm \sqrt{4 - 4a^{-r}})/2 = 1 \pm \sqrt{1 - a^{-r}}.$$

So, we must then have that $1 - a^{-r} \geq 0$ if and only if

$$1 \geq a^{-r}.$$

However, since $-r$ is positive or zero and $a > 1$, this implies that $r = 0$.

Clearly, $f(t)$ is strictly increasing since $f'(t) = 1 + a^t \ln a > 0$ for $t > 0$ and $f'(t) = 1 + a^{-t}\ln(-t) > 0$ for t negative. Moreover, $\lim_{t \longmapsto \infty} f(t) = \lim_{t \longmapsto \infty} t + a^t - 1 = \infty$. $\boxed{QED}$

Remark 1. Note that the same proof works for $f(t) = t + a^{g(t)} - 1$ for t positive and $f(t) = t - a^{-g(t)} + 1$ for t negative provided we have the following conditions:

$g(0) = 0$, $g(t)$ is differentiable and $g'(t) > 0$,

Proof. It suffices to show that

$$a^{-g(t_o+r)} - 2 + a^{g(t_o)} = 0$$

for $t_o > 0$ and $t_o + r < 0$ cannot have a real solution.

If $r < 0$, since g is strictly increasing, we have $g(t_o + r) < g(t_o)$ if $t_o > 0$. Let $g(t_o + r) + s_o = g(t_o)$ for $s_o > 0$ if $r < 0$. Then

$$a^{-g(t_o+r)} - 2 + a^{g(t_o)} = 0$$

is

$$a^{-g(t_o)+s_o} - 2 + a^{g(t_o)} = 0$$

has a solution only if $1 - a^{s_o} \geq 0$ and since s_o is positive this forces $s_o = 0$ which is a contradiction. $\boxed{QED}$

Theorem 7. *The above construction produces a parallelism if and only if $f(t) - t$ is surjective.*

Proof. We have the group E as a collineation group of the partial parallelism. Any line disjoint from $x = 0$ has the form $y = x \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. A typical element of E has the following form:

$$\tau_{m,r} = \begin{bmatrix} 1 & 0 & m & -r \\ 0 & 1 & r & m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, the given line is covered if and only if an image is covered. If we let $r = b$ and $m = -d$, we see that it suffices to consider lines with $b = d = 0$.

Hence, we consider $y = x \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$.

First assume that $a = 0$. Since $t - f(t)$ is onto, there is an element t_o such that $c = t_o - f(t_o)$. Apply $\tau_{0,-f(t_o)}$ to $y = x \begin{bmatrix} 0 & -f(t_o) \\ t_o & 0 \end{bmatrix}$ to obtain $y = x \begin{bmatrix} 0 & 0 \\ t_o - f(t_o) = c & 0 \end{bmatrix}$. Conversely, if the element $y = x \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ is covered then $t - f(t)$ is forced to be onto.

Now assume that $a \neq 0$.

Consider the element

$$\sigma_{w,s,m,r} = \begin{bmatrix} w & -s & m & -r \\ s & w & r & m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $w^2 + s^2 = 1$. We consider the image of $y = x \begin{bmatrix} 0 & -f(t) \\ t & 0 \end{bmatrix}$ under $\sigma_{w,s,m,r}$. Such an image will cover $y = x \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$ if and only if

$$\begin{bmatrix} w & -s \\ s & w \end{bmatrix} \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} wa - sc & 0 \\ sa + wc & 0 \end{bmatrix}$$
$$\begin{bmatrix} m & -f(t) - r \\ t - r & m \end{bmatrix}.$$

Hence, we must have:

$$f(t) = r, m = 0, wa = sc, sa + wc = t - f(t).$$

Hence,

$$t - f(t) = s(a^2 + c^2)/a$$

and since $w^2 + s^2 = 1$, we have $s^2(c^2 + a^2)/a^2$ so that

$$s = \pm a\sqrt{1/(c^2 + a^2)}.$$

Thus, the requirement is that

$$t - f(t) = \pm\sqrt{a^2 + c^2}.$$

This is so if and only if $t - f(t)$ is surjective. $\boxed{QED}$

Remark 2. To see examples of functions f such that $f(t)-t$ is not surjective, we note that the projection of $y=-\tan t$ onto the lines $y=x$ or $y=-x$ is surjective. Thus, rotate $y=-\tan t$ thru $\pi/4$ to find a continuous function on $(0,\infty)$, which is bounded between $y=x$ and $y=x+\pi/\sqrt{2}$. The function is continuous and strictly increasing and is bijective. Furthermore, since $y=x+\pi/\sqrt{2}$ is an asymptote and the function is concave down when x is positive, it follows that $f(t+r)=f(t)+r$ if and only if $r=0$.

In this case, $|f(t)-t|\leq\pi/\sqrt{2}$. Hence, a partial parallelism which is not a parallelism is constructed which has the property that for each regulus R of Σ_1 (the Pappian spread) containing a fixed line ℓ, the opposite regulus R^* is in a unique spread of the parallelism.

We assert that when the function f produces a partial parallelism, it must be a maximal partial parallelism.

To see this, suppose there is an another spread ρ which is not in the constructed partial parallelism $\mathcal{P}$. We have noted that none of the lines of ρ can intersect $x=0$, the axis of the central collineation group G^-, since we have a covering of such lines by $\mathcal{P}$. However, this means that we have a spread ρ which covers the points of our unique Pappian spread Σ without intersecting the axis $x=0$, a contradiction. Hence, the partial parallelism is a maximal partial parallelism.

Theorem 8. *When the function f produces a partial parallelism $\mathcal{P}$ and $f(t)-t$ is not an onto function then $\mathcal{P}$ is a proper maximal partial parallelism.*

Corollary 1. *If $\mathcal{P}$ is a proper maximal partial parallelism then so is any derived partial parallelism $\mathcal{P}^*$.*

Proof. All lines which nontrivially intersect the Baer subplane π_o corresponding to the axis of the central collineation group must be covered. So, any spread extra to the parallelism $\mathcal{P}^*$ must have its lines such that they are all disjoint from π_o, a contradiction. $\boxed{QED}$

References

[1] N. L. Johnson: *Some New Classes of Finite Parallelisms*, Note di Mat. (to appear).

[2] N. L. Johnson, R. Pomareda: *Transitive Deficiency One Parallelisms*, European J. Combin. (to appear).

[3] N. L. Johnson: Subplane Covered Nets, Monographs/Textbooks in Pure and Applied Mathematics, vol. 222, Marcel Dekker, New York 2000.

[4] N. L. Johnson, X. Liu: *Flocks of quadratic and semi-elliptic cones*, Mostly finite geometries, Lecture Notes in Pure and Applied Math., Vol. 190, Marcel Dekker. New York-Basil-Hong Kong, (l997), 275–304.