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Selectively strictly A-function spaces

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Abstract. We consider a selective version of the notion of countable set tightness and characterize $C_p(X)$-spaces with this property via a corresponding covering property of the space X.

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A space X is said to have *countable set tightness* if for each $A \subseteq X$ and each point $x \in \overline{A} \setminus A$ there is a sequence $\langle A_n \mid n < \infty \rangle$ of subsets of A such that $x \in \overline{\bigcup_{n<\infty} A_n} \setminus \bigcup_{n<\infty} \overline{A_n}$. This notion was first introduced in [1], where a bit different terminology was used, while the term set tightness was suggested later in [2]. The selective version of this notion is given by the next definition.

1 Definition. A space X is said to be a *selectively strictly A-space* if for each sequence $\langle A_n \mid n < \infty \rangle$ of subsets of X and each point $x \in X$ with $x \in (\bigcap_{n<\infty} \overline{A_n}) \setminus (\bigcup_{n<\infty} A_n)$, there is a sequence $\langle B_n \mid n < \infty \rangle$, where for each n $B_n \subseteq A_n$, such that $x \in \overline{\bigcup_{n<\infty} B_n} \setminus \bigcup_{n<\infty} \overline{B_n}$.

This notion can also be viewed as a modification of the notion of *strictly A-spaces*, which are defined in exactly the same way with the addition that the sequence $\langle A_n \mid n < \infty \rangle$ must be decreasing (see [3] and [4]). So this should account for the term selectively strictly A-space.

As it has become customary that closure properties of the function space $C_p(X)$ are characterized by covering properties of X, it is natural to ask if a similar characterization could be established for selectively strictly A-function spaces. The analogue of this question for function spaces with countable set tightness has been answered in [5]. It is from there that the following definition actually originates.

For a set H we put $\operatorname{dom} H = \{ x \mid \exists y \ (\ (x,y) \in H \ ) \}$, $\operatorname{ran} H = \{ x \mid \exists y \ (\ (y,x) \in H \ ) \}$, and call $\operatorname{dom} H$ and $\operatorname{ran} H$, respectively, the *domain* and the *range* of the set H.

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2 Definition. Call a set $\mathcal{R}$ a *t-relation* on a space X if:

(i) the domain of $\mathcal{R}$ is a family of cozero sets of the space X not containing the whole space X,

(ii) if $(U, Z) \in \mathcal{R}$ then $Z \subseteq U$ and Z is a zero set of the space X,

(iii) the range of $\mathcal{R}$ is an ω-cover of X.

Let us point out here that by an *ω-cover* of a set X we mean any collection $\mathcal{L}$ of its subsets different from X such that every finite subset of X is contained in a member of $\mathcal{L}$.

We shall make use of these t-relations to formulate the appropriate dual property for selectively strictly A-spaces.

3 Definition. We say that a space X has *property* $S(X)$ if for each sequence $\langle \mathcal{R}_n \mid n < \infty \rangle$ of t-relations of the space X there is a sequence $\langle \mathcal{V}_n \mid n < \infty \rangle$ such that:

- $\mathcal{V}_n \subseteq \mathcal{R}_n$ for all n,
- no $\operatorname{ran} \mathcal{V}_n$ is an ω-cover of X, but
- $\bigcup_{n<\infty} \operatorname{dom} \mathcal{V}_n$ is an ω-cover of X.

Now the mentioned result can be stated in the following theorem, the proof of which is based on the Sakai's technique used in [5].

4 Theorem. *A $T_{3\frac{1}{2}}$ space X has the property $S(X)$ if and only if $C_p(X)$ is a selectively strictly A-space.*

Proof. Let $C_p(X)$ be a selectively strictly A-space and let $\langle \mathcal{R}_n \mid n < \infty \rangle$ be a sequence of t-relations on X. For each $n \in \{1, 2, \dots\}$ and $(U, Z) \in \mathcal{R}_n$ choose a function $f^n_{U,Z} \in C_p(X)$ such that $f^n_{U,Z} | (X \setminus U) \equiv 1$, $f^n_{U,Z} | Z \equiv 0$. Put $A_n = \{ f^n_{U,Z} \mid (U, Z) \in \mathcal{R}_n \}$.

Since for each n $\mathcal{Z}_n$ is an ω-cover, it follows that $\mathbf{o} \in \overline{A_n}$ for each n, where $\mathbf{o} \in C_p(X)$ stands for the constant 0 function. Also, we have that $\mathbf{o} \notin \bigcup_{n<\infty} A_n$, because the contrary would mean that $\mathbf{o} = f^n_{U,Z}$, for some $n \in \{1, 2, \dots\}$ and some $(U, Z) \in \mathcal{R}_n$, so $f^n_{U,Z} | (X \setminus U) \equiv 1$ would imply $U = X$, which is impossible.

As $C_p(X)$ is a selectively strictly A-space, there is a sequence $\langle B_n \mid n < \infty \rangle$ such that: $B_n \subseteq A_n$ for all n and $\mathbf{o} \in \overline{\bigcup_{n<\infty} B_n} \setminus \bigcup_{n<\infty} \overline{B_n}$. For each n, let $\mathcal{V}_n$ denote the set $\{ (U, Z) \in \mathcal{R}_n \mid f^n_{U,Z} \in B_n \}$. Then $\mathcal{V}_n \subseteq \mathcal{R}_n$. Also, no $\operatorname{ran} \mathcal{V}_n$ is an ω-cover of X; for otherwise $\mathbf{o} \in \overline{B_n}$.

Finally, assume an arbitrary $F \subseteq X$ is given. As $\mathbf{o} \in \overline{\bigcup_{n<\infty} B_n}$ there is a n_0 and a $(U, Z) \in \mathcal{R}_{n_0}$ such that $f^{n_0}_{U,Z} \in B_{n_0}$ and $F \subseteq f^{n_0\leftarrow}_{U,Z}(R \setminus \{1\})$, where R is the set of reals. But then $(U, Z) \in \mathcal{V}_n$ and $f^{n_0}_{U,Z}|(X \setminus U) \equiv 1$, so $F \subseteq U \in \operatorname{dom} \mathcal{V}_{n_0} \subseteq \bigcup_{n<\infty} \operatorname{dom} \mathcal{V}_n$. Therefore $\bigcup_{n<\infty} \operatorname{dom} \mathcal{V}_n$ is an ω-cover of X.

Thus we have shown that X has the property $S(X)$.

Conversely, let X has the property $S(X)$ and let a sequence $\langle A_n \mid n < \infty \rangle$ be given, where for each n $A_n \subseteq C_p(X)$ and $\mathbf{o} \in \bigcap_{n<\infty} \overline{A_n}$. Consider a bijection $i : \omega^2 \to \omega$, and denote $A_{n,m} = A_{i(n,m)}$. For each m, n and $f \in C_p(X)$ put $U^m_f = f^{\leftarrow}\left(-\frac{1}{m}, \frac{1}{m}\right)$, $Z^m_f = f^{\leftarrow}\left[-\frac{1}{m+1}, \frac{1}{m+1}\right]$, $\mathcal{R}_{n,m} = \{ (U^m_f, Z^m_f) \mid f \in A_{n,m} \}$.
Case 1: The set $M = \{ m \in \omega \mid \exists\, n\ \exists\, f \in A_{n,m}\ (U^m_f = X) \}$ is infinite. Then there are sequences $m_1 < m_2 < \cdots < m_k < \cdots$, $(n_k)_{k\in\omega}$ and $(f_k)_{k\in\omega}$ such that $f_k \in A_{n_k,m_k}$, $U^{m_k}_{f_k} = X$. It is now clear that $f_k \underset{k\to\infty}{\overset{X}{\rightrightarrows}} \mathbf{o}$, so that, if we put $B_{n_k,m_k} = \{ f_k \}$ and $B_{n,m} = \emptyset$ if $\forall k\ ((n,m) \neq (n_k, m_k))$, we get the required sequence of subsets of the sets $A_{n,m}$.
Case 2: The set M is finite. Let $B_{n,m} = \emptyset$ for each $n \in \omega$ and each $m \in \{ 0, 1, \ldots, \max M \}$. For each $m > \max M$, $\langle \mathcal{R}_{n,m} \mid n < \infty \rangle$ is a sequence of t-relations on the space X so, as X has the property $S(X)$, there is a sequence $\langle \mathcal{V}_{n,m} \mid n < \infty \rangle$, such that $\mathcal{V}_{n,m} \subseteq \mathcal{R}_{n,m}$, $\bigcup_{n<\infty} \operatorname{dom} \mathcal{V}_{n,m}$ is an ω-cover but none of the sets $\operatorname{ran} \mathcal{V}_{n,m}$ is an ω-cover.

For each $m > \max M$ put $B_{n,m} = \{ f \in A_{n,m} \mid (U^m_f, Z^m_f) \in \mathcal{V}_{n,m} \}$. The fact that $\mathbf{o} \in \overline{B_{n,m_0}}$, for a $m_0 > \max M$, would imply that $\operatorname{ran} \mathcal{V}_{n,m_0}$ is an ω-cover, which is impossible. Hence, $\mathbf{o} \notin \overline{B_{n,m}}$ whenever $m > \max M$.

Let $F \subseteq X$ be an arbitrary finite set and $\epsilon > 0$ any real number. Choose an integer $m' > \max\left(\frac{1}{\epsilon}, \max M\right)$. As $\bigcup_{n<\infty} \operatorname{dom} \mathcal{V}_{n,m'}$ is an ω-cover there exist a $n' \in \omega$ and a $f \in B_{n',m'}$ such that $F \subseteq U^{m'}_f = f^{\leftarrow}\left(-\frac{1}{m'}, \frac{1}{m'}\right)$. In other words, $f_{\rightarrow} F \subseteq \left(-\frac{1}{m'}, \frac{1}{m'}\right) \subseteq (-\epsilon, \epsilon)$.

These last few lines actually say that $\mathbf{o} \in \overline{\bigcup_{n,m\in\omega} B_{n,m}}$, which ends the proof. *QED*

As the definition of strictly A-spaces differs from the definition of selectively strictly A-spaces only in the part where the given family of sets is required to be decreasing, it is no surprise that the next slight modification of the definition of the property $S(X)$ gives the dual property for strictly A-function spaces.

5 Definition. We say that a space X has *property* $S'(X)$ if for each decreasing sequence $\langle \mathcal{R}_n \mid n < \infty \rangle$ of t-relations of the space X there is a sequence $\langle \mathcal{V}_n \mid n < \infty \rangle$ such that:

- $\mathcal{V}_n \subseteq \mathcal{R}_n$ for all n,
- no $\operatorname{ran} \mathcal{V}_n$ is an ω-cover of X, but
- $\bigcup_{n<\infty} \operatorname{dom} \mathcal{V}_n$ is an ω-cover of X.

Practically the same proof as the proof of the previous theorem shows that:

6 Theorem. *A* $T_{3\frac{1}{2}}$ *space* X *has property* $S'(X)$ *if and only if the function space* $C_p(X)$ *is a strictly A-space.*

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