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Limit of nonlinear Signorini problems for incompressible materials

Edoardo Mainini
Università degli Studi di Genova, Dipartimento di Ingegneria Meccanica, Energetica, Gestionale e dei Trasporti (DIME), Via Opera Pia 15, 16129 Genova, Italy
edoardo.mainini@unige.it

Danilo Percivale
Università degli Studi di Genova, Dipartimento di Ingegneria Meccanica, Energetica, Gestionale e dei Trasporti (DIME), Via Opera Pia 15, 16129 Genova, Italy
danilo.percivale@unige.it

Robertus Van der Putten
Università degli Studi di Genova, Dipartimento di Ingegneria Meccanica, Energetica, Gestionale e dei Trasporti (DIME), Via Opera Pia 15, 16129 Genova, Italy
robertus.van.der.putten@unige.it

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Abstract. The paper is devoted to the linearization of the non linear Signorini functional in the incompressible case. The limit functional, in the sense of Γ-convergence, may coincide with the expected one only in some particular cases.

Keywords: Calculus of Variations; Signorini problem; Linear elasticity; Finite elasticity; Gamma-convergence; Unilateral constraint

MSC 2020 classification: 49J45 · 74K30 · 74K35 · 74R10

1 Introduction

In linear elastostatics, the classical Signorini problem [23] requires to find the equilibrium of an elastic body subject to external forces and resting on a rigid support $E \subset \partial\Omega$ in its reference configuration Ω. Precisely, if Ω is subject to suitable volume and surface forces $\mathbf{f} : \Omega \to \mathbb{R}^3$ and $\mathbf{g} : \partial\Omega \setminus E \to \mathbb{R}^3$ such that

$$\mathcal{L}(\mathbf{u}) := \int_\Omega \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x} + \int_{\partial\Omega\setminus E} \mathbf{g} \cdot \mathbf{u} \, d\mathcal{H}^2 \tag{1.1}$$

is the load potential and $\mathbf{u} : \Omega \to \mathbb{R}^3$ is the displacement field, then, by assuming that $\mathcal{H}^2(E) > 0$, the variational formulation of the Signorini problem consists in

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finding a minimizer of the functional

$$\mathcal{E}(\mathbf{u}) := \int_\Omega \mathcal{Q}\big(\mathbf{x}, \mathbb{E}(\mathbf{u})\big)\, d\mathbf{x} - \mathcal{L}(\mathbf{u}) \tag{1.2}$$

among all $\mathbf{u}$ in the Sobolev space $H^1(\Omega;\mathbb{R}^3)$ such that $\mathbf{u}\cdot\mathbf{n} \geq 0$ $\mathcal{H}^2$- a.e. on E where $\mathbf{n}$ is the inward unit vector normal to $\partial\Omega$ and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. As usually happens, here $\mathbb{E}$ denotes the linear strain tensor, $\mathbb{C}$ represents the classical linear elasticity tensor and

$$\mathcal{Q}(\mathbf{x}, \mathbb{E}) := \frac{1}{2}\, \mathbb{E}^T \mathbb{C}(\mathbf{x})\, \mathbb{E}$$

is the corresponding strain energy density (see [8]). A classical result (see [6]) states that a minimizer exists if $\mathcal{L}(\mathbf{v}) \leq 0$ for every infinitesimal rigid displacement $\mathbf{v}$ such that $\mathbf{v}\cdot\mathbf{n} \geq 0$ $\mathcal{H}^2$- a.e. on E and $\mathcal{L}(\mathbf{v}) = 0$ if and only if $\mathbf{v}\cdot\mathbf{n} \equiv 0$ $\mathcal{H}^2$- a.e. on E. More recently a proper generalization of the latter formulation has been given by assuming that the set E has positive Sobolev capacity and accordingly modifying the obstacle condition by requiring $\mathbf{v}\cdot\mathbf{n} \geq 0$ on E up to a set of null capacity (shortly, q.e. on E): the existence of minimizers for this general setting was proved in [2, Theorem 4.5]. Although the original formulation given in [23] may look different from the generalized notion exploited in [2], it can be shown (see Remark 2.1) that if the set $E \subset \partial\Omega$ is regular in an appropriate sense then the two frameworks coincide.

In the recent paper [16] it has been shown that, under sharp conditions on $\mathcal{L}$, there exists a sequence of (suitably rescaled) functionals $\mathcal{G}_h$ of finite elasticity such that $\inf \mathcal{G}_h$ converges to the minimum of $\mathcal{E}$ and that in addition there are examples in which this convergence fails. The aim of this paper is to show that at least in the planar obstacle case, namely when $E \subset \partial\Omega \cap \{\mathbf{x}\cdot\mathbf{e}_3 = 0\} \not\equiv \emptyset$ (being $(\mathbf{e}_i)$, $i = 1,2,3$, the canonical basis of $\mathbb{R}^3$) a similar result holds also for incompressible energy densities. More in detail, denoting by $\mathbf{y} : \Omega \to \mathbb{R}^3$ the deformation field and by $h > 0$ an adimensional parameter, we introduce a strain energy density $\mathcal{W} : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty]$, which is frame indifferent and minimized at the identity, and the family of energy functionals

$$\mathcal{F}_h^I(\mathbf{y}) := h^{-2} \int_\Omega \mathcal{W}^I\big(\mathbf{x}, \nabla\mathbf{y}(\mathbf{x})\big)\, d\mathbf{x} - h^{-1}\mathcal{L}(\mathbf{y} - \mathbf{x})$$

where $\mathcal{L}$ is defined as in (1.1) and $\mathcal{W}^I$ is the incompressible strain energy density (coinciding with $\mathcal{W}$ if $\det \nabla\mathbf{y} \equiv 1$ and set equal to $+\infty$ otherwise). We define

$$\mathcal{G}_h^I(\mathbf{y}) = \begin{cases} \mathcal{F}_h^I(\mathbf{y}) & \text{if } \mathbf{y} \in \mathcal{A} \\ +\infty & \text{otherwise in } H^1(\Omega;\mathbb{R}^3), \end{cases}$$

where E, the portion of the elastic body that is sensitive to the obstacle, is such that $\operatorname{cap} E > 0$, and where $\mathcal{A} = \mathcal{A}(E)$ denotes the class of admissible deformations (i.e., those $\mathbf{y} \in H^1(\Omega;\mathbb{R}^3)$ such that $\mathbf{y}\cdot\mathbf{e}_3 \geq 0$ quasi-everywhere on E, i.e., up to sets of null capacity). Moreover we have to assume that $\mathcal{L}(\mathbf{y}-\mathbf{x}) \leq 0$ for every rigid deformation $\mathbf{y} \in \mathcal{A}$. Indeed, if there exists a rigid deformation $\overline{\mathbf{y}} \in \mathcal{A}$ such that $\mathcal{L}(\overline{\mathbf{y}} - \mathbf{x}) > 0$, then

$$\mathcal{G}_h^I(\overline{\mathbf{y}}) = -h^{-1}\mathcal{L}(\overline{\mathbf{y}} - \mathbf{x}) \to -\infty \text{ as } h \to 0^+.$$

Therefore, since a rigid deformation $\mathbf{y}(\mathbf{x}) = \mathbf{R}\mathbf{x} + \mathbf{c}$ with $\mathbf{R} \in SO(3)$, $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3$, belongs to $\mathcal{A}$ whenever $c_3 \geq -(\mathbf{R}\mathbf{x})_3$ on E, we have to assume

$$\mathcal{L}((\mathbf{R}-\mathbf{I})\mathbf{x} + \mathbf{c}) \leq 0 \quad \text{for every } \mathbf{R} \in SO(3) \text{ and every } \mathbf{c} \in \mathbb{R}^3 \quad \text{s.t. } c_3 \geq -(\mathbf{R}\mathbf{x})_3 \text{ on } E. \tag{1.3}$$

In the main result of this paper (Theorem 2.8 below) we show that if $\mathcal{L}$ satisfies the necessary condition (1.3) together with $\mathcal{L}(\mathbf{e}_3) < 0$ and

$$\mathcal{L}\left((\mathbf{R}\mathbf{x} - \mathbf{x})_1\, \mathbf{e}_1 + (\mathbf{R}\mathbf{x} - \mathbf{x})_2\, \mathbf{e}_2 \right) \leq 0 \quad \forall\, \mathbf{R} \in SO(3), \tag{1.4}$$

and if $\mathcal{W}$ satisfies some standard assumptions to be detailed in the next section, then

$$\lim_{h\to 0} \inf_{H^1(\Omega;\mathbb{R}^3)} \mathcal{G}_h^I = \min_{H^1(\Omega;\mathbb{R}^3)} \mathcal{G}^I,$$

where, with the notation D^2 for Hessian of $\mathcal{W}(\mathbf{x},\cdot)$, the limit functional is characterized as

$$\mathcal{G}^I(\mathbf{u}) := \int_\Omega \mathcal{Q}^I(\mathbf{x}, \mathbb{E}(\mathbf{u}))\, d\mathbf{x} - \max_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{R}\mathbf{u})$$

if $\mathbf{u}\cdot\mathbf{e}_3 \geq 0$ quasi-everywhere on E, and

$$\mathcal{G}^I(\mathbf{u}) := +\infty \qquad \text{otherwise in } H^1(\Omega;\mathbb{R}^3),$$

where

$$\mathcal{Q}^I(\mathbf{x},\mathbf{F}) := \begin{cases} \dfrac{1}{2}\mathbf{F}^T D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathbf{F} & \text{if } \operatorname{Tr}\mathbf{F} = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{S}_{\mathcal{L},E} = \left\{ \mathbf{R} \in SO(3) : \ \mathcal{L}\left((\mathbf{R}-\mathbf{I})\,\mathbf{x}\right) - \inf_{\mathbf{x}\in E}\big((\mathbf{R}\mathbf{x})_3\big)\mathcal{L}(\mathbf{e}_3) = 0 \right\}.$$

In general, $\mathcal{G}^I$ differs from the expected limit functional $\mathcal{E}^I$, defined by replacing $\mathcal{Q}$ with $\mathcal{Q}^I$ in (1.2). However, under the hypotheses on E and $\mathcal{L}$ that were previously detailed, it has been proven in [16] (see Lemma 2.6 below) that either $\mathcal{S}_{\mathcal{L},E} \equiv \{\mathbf{I}\}$ or $\mathcal{S}_{\mathcal{L},E} = \{\mathbf{R} \in SO(3) : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\}$. If $\mathcal{S}_{\mathcal{L},E} \equiv \{\mathbf{I}\}$ then clearly $\mathcal{G}^I \equiv \mathcal{E}^I$, hence in this case we recover the minimum of the Signorini problem in linearized elasticity as the limit of the nonlinear energies $\inf \mathcal{G}_h^I$. In particular, if Ω is contained in the upper half-space, $E = \partial\Omega \cap \{x_3 = 0\}$, if $\mathbf{f} = f\mathbf{e}_3$, $\mathbf{g} = g\mathbf{e}_3$ and $\mathcal{L}$ satisfies conditions (1.3)-(1.4) and $\mathcal{L}(\mathbf{e}_3) < 0$, then since $\mathcal{L}(\mathbf{v}) = \mathcal{L}(\mathbf{R}\mathbf{v})$ for every $\mathbf{R} \in \mathcal{S}_{\mathcal{L},E}$ we get $\min \mathcal{G}^I = \min \mathcal{E}^I$ hence $\inf \mathcal{G}_h^I \to \min \mathcal{E}^I$ in this case.

Plan of the paper. In section 2 we rigorously provide assumptions on the obstacle, on the external forces and the global energy functionals. Then, we state the main variational convergence result. In Section 3 we give the proof of the main result after having proved suitable technical lemmas. With respect to the analysis from [16] about the compressible case, some care is needed in constructing the suitable upper and lower bounds while simultaneously taking care of the obstacle condition and the incompressibility constraint.

2 Notation and main results

In the following, Ω will denote the reference configuration of an elastic body. Ω is always assumed to be a nonempty, bounded, connected, Lipschitz open set in $\mathbb{R}^3$.

2.1 The obstacle

Notations $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ will be used to represent generic points in $\mathbb{R}^3$, with components referred to the canonical basis $(\mathbf{e}_i)$, $i = 1, 2, 3$. In the Signorini problem, the elastic body rests on a frictionless rigid support $E \subset \partial\Omega$. The set E will be assumed to be planar, i.e., contained in $\{x_3 = 0\}$, and of positive capacity, according to the next definition.

For every compact set $K \subset \mathbb{R}^N$ we define the Sobolev capacity of K by setting, see [1, Section 2.2],

$$\operatorname{cap} K = \inf\left\{ \|w\|^2_{H^1(\mathbb{R}^N)} : w \in C_0^\infty(\mathbb{R}^N),\ w \geq 1 \text{ on } K \right\}.$$

If $G \subset \mathbb{R}^N$ is open we define

$$\operatorname{cap} G := \sup\{\operatorname{cap} K : K \text{ compact},\ K \subset G\},$$

and for a generic set $F \subset \mathbb{R}^3$

$$\operatorname{cap} F := \inf\{\operatorname{cap} G : G \text{ open},\ F \subset G\}.$$

A property is said to hold quasi-everywhere (q.e. for short) if it holds true outside a set of zero capacity. We introduce (see [2]) a canonical representative of a set F, called the *essential part* of F and denoted by F_{ess}, which coincides with F whenever F is a smooth closed manifold or the closure of an open subset of $\mathbb{R}^N$: for every set $F \subset \mathbb{R}^3$ we define

$$F_{ess} \ := \ \bigcap \{ \, C : \ C \text{ is closed and } \ \mathrm{cap}(F \backslash C) = 0 \, \}.$$

As shown in [2], we have

$$F_{ess} \ \text{ is a closed subset of } \overline{F}, \qquad \mathrm{cap}(F \backslash F_{ess}) = 0,$$
$$\mathrm{cap}\, F = 0 \ \text{ if and only if } F_{ess} = \emptyset.$$

Moreover, if $\mathrm{cap}\, F > 0$ and $\mathbf{x} \in F_{ess}$, then $\mathrm{cap}(F \cap B_r(\mathbf{x})) > 0$ for every $r > 0$.

Throughout the paper we will assume that the set E giving the obstacle condition satisfies

$$E \subset \partial\Omega \cap \{x_3 = 0\} \not\equiv \emptyset \qquad \text{and} \qquad \mathrm{cap}\, E > 0. \tag{2.5}$$

We recall that every function in $H^1(\Omega;\mathbb{R}^3)$ has a precise representative defined quasi-everywhere on the whole $\overline{\Omega}$. Indeed, if $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$ and $\mathbf{v} \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ is a Sobolev extension of $\mathbf{u}$, it is well known (see [1, Proposition 6.1.3]) that the limit

$$\mathbf{v}^*(\mathbf{x}) \ := \ \lim_{r \downarrow 0} \ \frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} \mathbf{v}(\xi)\, d\xi$$

exists for q.e. $\mathbf{x} \in \mathbb{R}^3$. The function $\mathbf{v}^*$ is called the precise representative of $\mathbf{v}$ and is a quasicontinuous function in $\mathbb{R}^3$, meaning that for every $\varepsilon > 0$ there exists an open set $V \subset \mathbb{R}^3$ such that $\mathrm{cap}\, V < \varepsilon$ and $\mathbf{v}^*$ is continuous in $\mathbb{R}^3 \backslash V$. It has been proven in [16] that if $\mathbf{v}_1, \ \mathbf{v}_2$ are two distinct Sobolev extensions of $\mathbf{u}$ then $\mathbf{v}_1^*(\mathbf{x}) \ = \ \mathbf{v}_2^*(\mathbf{x})$ for q.e. $\mathbf{x} \in \overline{\Omega}$. Therefore if $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$ we may define its precise representative for quasi-every $\mathbf{x} \in \overline{\Omega}$ by

$$\mathbf{u}^*(\mathbf{x}) = \lim_{r \downarrow 0} \ \frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} \mathbf{v}(\xi)\, d\xi, \qquad \text{q.e. } \mathbf{x} \in \overline{\Omega}, \tag{2.6}$$

where $\mathbf{v}$ is any Sobolev extension of $\mathbf{u}$. The function $\mathbf{u}^*$ is pointwise quasi-everywhere defined by (2.6) and is *quasicontinuous* on $\overline{\Omega}$ i.e., for every $\varepsilon > 0$ there exists a relatively open set $V \subset \overline{\Omega}$ such that $\mathrm{cap}\, V < \varepsilon$ and $\mathbf{u}^*$ is continuous in $\overline{\Omega} \setminus V$.

Remark 2.1. If $w \in H^1(\Omega)$ then its negative part w^- is in $H^1(\Omega)$ too. Moreover, both $(w^-)^*$ and $(w^*)^-$ are quasicontinuous in $\overline{\Omega}$ and $(w^-)^* = (w^*)^- = w^-$ a.e. in Ω. Then, by [11], $(w^-)^* = (w^*)^-$ q.e. in $\overline{\Omega}$. Therefore the condition $(w^-)^* = 0$ q.e. in E_{ess} is equivalent to $w^* \geq 0$ q.e. in E_{ess}. In particular, as it was pointed out in in [16], Theorem 2.1 of [5] entails that if $E_{ess} \subset \partial\Omega$ is Ahlfors 2-regular and $\mathcal{H}^2(E_{ess}) > 0$, then the condition $w \geq 0$ q.e. on E_{ess} is equivalent to $w \geq 0$ $\mathcal{H}^2$ a.e. on E_{ess} so the classical framework of [2, 23, 12] is equivalent to ours in this case.

2.2 The elastic energy density

Let $\mathbb{R}^{3\times 3}$ denote the set of 3×3 real matrices, endowed with the Euclidean norm $|\mathbf{F}| = \sqrt{\mathrm{Tr}(\mathbf{F}^T\mathbf{F})}$. We let $\mathrm{sym}\mathbf{F} := \frac{1}{2}(\mathbf{F}^T + \mathbf{F})$. $SO(3)$ will denote the special orthogonal group. Let $\mathcal{L}^3$ and $\mathcal{B}^3$ denote respectively the σ-algebras of Lebesgue measurable and Borel measurable subsets of $\mathbb{R}^3$ and let $\mathcal{W} : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty]$ be $\mathcal{L}^3 \times \mathcal{B}^9$- measurable satisfying the following standard assumptions, see also [19, 20]:

$$\mathcal{W}(\mathbf{x}, \mathbf{R}\mathbf{F}) = \mathcal{W}(\mathbf{x}, \mathbf{F}) \quad \forall\, \mathbf{R} \in SO(3) \ \ \forall\, \mathbf{F} \in \mathbb{R}^{3\times 3}, \qquad \text{for a.e. } \mathbf{x} \in \Omega, \tag{2.7}$$

$$\min_{\mathbf{F}} \mathcal{W}(\mathbf{x}, \mathbf{F}) = \mathcal{W}(\mathbf{x}, \mathbf{I}) = 0 \quad \text{for a.e. } \mathbf{x} \in \Omega \tag{2.8}$$

and as far as it concerns the regularity of $\mathcal{W}$, we assume that there exist an open neighborhood $\mathcal{U}$ of $SO(3)$ in $\mathbb{R}^{3\times 3}$, an increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\lim_{t\to 0^+} \omega(t) = 0$ and a constant $K > 0$ such that for a.e. $\mathbf{x} \in \Omega$

$$\begin{aligned} &\mathcal{W}(\mathbf{x}, \cdot) \in C^2(\mathcal{U}), \quad |D^2\mathcal{W}(\mathbf{x}, \mathbf{I})| \leq K \ \text{ and} \\ &|D^2\mathcal{W}(\mathbf{x}, \mathbf{F}) - D^2\mathcal{W}(\mathbf{x}, \mathbf{G})| \leq \omega(|\mathbf{F} - \mathbf{G}|), \quad \forall\, \mathbf{F}, \mathbf{G} \in \mathcal{U}. \end{aligned} \tag{2.9}$$

Moreover we assume that there exists $C > 0$ such that for a.e. $\mathbf{x} \in \Omega$

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) \geq C(d(\mathbf{F}, SO(3)))^2 \qquad \forall\, \mathbf{F} \in \mathbb{R}^{3\times 3}, \tag{2.10}$$

where $d(\,\cdot\,, SO(3))$ denotes the Euclidean distance function from the set of rotations. In order to consider incompressible elasticity models, starting from a function $\mathcal{W}$ as above we introduce the incompressible strain energy density by letting, for a.e. $\mathbf{x} \in \Omega$,

$$\mathcal{W}^I(\mathbf{x}, \mathbf{F}) := \begin{cases} \mathcal{W}(\mathbf{x}, \mathbf{F}) & \text{if } \det \mathbf{F} = 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{2.11}$$

We recall some basic properties that follow from the above assumptions, see [19, 20]. For a.e. $\mathbf{x} \in \Omega$ there holds

$$\lim_{h\to 0} h^{-2}\mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{F}) = \frac{1}{2}\,\mathbf{F}^T D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\,\mathbf{F} = \\ = \frac{1}{2}\,\mathrm{sym}\mathbf{F}\, D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\ \mathrm{sym}\mathbf{F} \qquad \forall\ \mathbf{F}\in\mathbb{R}^{3\times 3},$$

hinting to the linear elastic energy density as limit of the nonlinear energies, with $\mathbb{C}(\mathbf{x}) = D^2\mathcal{W}(\mathbf{x}, \mathbf{I})$, and

$$\frac{1}{2}\,\mathbf{F}^T D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\,\mathbf{F} \geq C|\mathrm{sym}\mathbf{F}|^2 \qquad \forall\ \mathbf{F}\in\mathbb{R}^{3\times 3},$$

where C is the constant in (2.10). Taking (2.9) into account we have, for a.e. $\mathbf{x} \in \Omega$,

$$\left|\mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{F}) - \frac{h^2}{2}\,\mathrm{sym}\mathbf{F}\, D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\ \mathrm{sym}\mathbf{F}\right| \leq h^2\omega(h|\mathbf{F}|)|\mathbf{F}|^2 \qquad (2.12)$$

for every $\mathbf{F} \in \mathbb{R}^{3\times 3}$ and every $h > 0$ such that $\mathbf{I} + h\mathbf{F} \in \mathcal{U}$.

We mention a class of energy densities $\mathcal{W}$ (the so called Yeoh materials [22, 19]) fulfilling the assumptions above (2.7)–(2.10) and for which the main result of the present paper (see Theorem 2.8 below) applies (for simplicity, we consider the homogeneous case):

$$\mathcal{W}(\mathbf{F}) := \sum_{k=1}^{3} c_k(|\mathbf{F}|^2 - 3)^k$$

with $c_k > 0$. It is easy to check that with this choice the energy density satisfies all assumptions from (2.7) to (2.9) while inequality (2.10) has been proven in [17]. It is worth noticing that when material constants are suitably chosen then also the classical Ogden-type energies may fulfill the assumptions from (2.7) to (2.10) and we refer to [17] for all details.

2.3 External forces

We introduce a body force field $\mathbf{f} \in L^{6/5}(\Omega, \mathbb{R}^3)$ and a surface force field $\mathbf{g} \in L^{4/3}(\partial\Omega, \mathbb{R}^3)$. From now on, $\mathbf{f}$ and $\mathbf{g}$ will always be understood to satisfy these summability assumptions. The load functional is the following linear functional

$$\mathcal{L}(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} + \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v}\,d\mathcal{H}^2(\mathbf{x}), \qquad \mathbf{v} \in H^1(\Omega, \mathbb{R}^3).$$

We note that since Ω is a bounded Lipschitz domain, the Sobolev embedding $H^1(\Omega,\mathbb{R}^3) \hookrightarrow L^6(\Omega,\mathbb{R}^3)$ and the Sobolev trace embedding $H^1(\Omega,\mathbb{R}^3) \hookrightarrow L^4(\partial\Omega,\mathbb{R}^3)$ imply that $\mathcal{L}$ is a bounded functional over $H^1(\Omega,\mathbb{R}^3)$ and throughout the paper we denote its norm with $\|\mathcal{L}\|_*$.

For every $\mathbf{R}\in SO(3)$ we set

$$\mathcal{C}_{\mathbf{R}} := \{\mathbf{c} : c_3 \geq -\min_{\mathbf{x}\in E_{ess}} (\mathbf{R}\mathbf{x})_3\}$$

and we assume the following *geometrical compatibility between load and obstacle*

$$\mathcal{L}((\mathbf{R}-\mathbf{I})\mathbf{x}+\mathbf{c}) \leq 0 \qquad \forall \mathbf{R} \in SO(3),\ \forall \mathbf{c} \in \mathcal{C}_{\mathbf{R}} \tag{2.13}$$

together with

$$\mathcal{L}\left((\mathbf{R}\mathbf{x}-\mathbf{x})_\alpha \, \mathbf{e}_\alpha\right) \leq 0 \quad \forall\, \mathbf{R} \in SO(3), \tag{2.14}$$

the summation convention over the repeated index $\alpha = 1, 2$ being understood all along this paper. It can be shown that condition (2.13) is equivalent to (see [16, Remark 3.4])

$$\mathcal{L}(\mathbf{e}_3) \leq 0 = \mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2), \qquad \Phi(\mathbf{R}, E, \mathcal{L}) \leq 0 \quad \forall\, \mathbf{R}\in SO(3), \tag{2.15}$$

where we have set

$$\Phi(\mathbf{R}, E, \mathcal{L}) := \mathcal{L}((\mathbf{R}-\mathbf{I})\mathbf{x}) \;-\; \mathcal{L}(\mathbf{e}_3) \min_{\mathbf{x}\in E_{ess}} (\mathbf{R}\mathbf{x})_3$$

and from now on we will use (2.15) in place of (2.13). It has been shown in [16] that conditions (2.14) and (2.15) are compatible and that they do not imply each other, see also Remark 3.3 below. We next state three auxiliary lemmas, in order to better clarify the role of conditions (2.14) and (2.15). Proofs may be found in [16]. In the following, $\wedge$ denotes the cross product.

Lemma 2.2. *Assume that* (2.14) *holds. Then,*

$$\mathcal{L}((\mathbf{a}\wedge\mathbf{x})_\alpha \, \mathbf{e}_\alpha) = 0 \quad \textit{and} \quad \mathcal{L}((\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{x}))_\alpha \, \mathbf{e}_\alpha) \leq 0 \quad \forall\, \mathbf{a} \in \mathbb{R}^3. \tag{2.16}$$

Remark 2.3. It is worth noticing that, by inserting $\mathbf{a} = \mathbf{e}_1$ or $\mathbf{a} = \mathbf{e}_2$, the condition (2.16) entails $\mathcal{L}(x_3\mathbf{e}_2) = 0$ and $\mathcal{L}(x_3\mathbf{e}_1) = 0$ respectively.

Lemma 2.4. *Assume* (2.5) *and* (2.13). *Then*

(1) $\mathcal{L}(\mathbf{e}_1) = \mathcal{L}(\mathbf{e}_2) = 0$ *and* $\mathcal{L}(\mathbf{e}_3) \leq 0$;

(2) $\mathcal{L}(\mathbf{e}_3\wedge\mathbf{x}) = 0$;

(3) $\mathcal{L}\big(\mathbf{e}_3\wedge(\mathbf{e}_3\wedge\mathbf{x})\big) \leq 0$;

(4) there exists $\mathbf{x}_{\mathcal{L}}$ *in the relative interior of the convex envelope of* E_{ess} *such that*
$\mathcal{L}\big(\mathbf{a} \wedge (\mathbf{x} - \mathbf{x}_{\mathcal{L}})\big) = 0 \quad \forall \mathbf{a} \in \mathbb{R}^3.$

Remark 2.5. We emphasize that conditions (1) and (4) in Lemma 2.4 together with (2.5) and $\mathcal{L}(\mathbf{e}_3) < 0$ coincide with conditions (4.9)–(4.11) of [2, Theorem 4.5], which provides the solution to Signorini problem in linear elasticity. We also emphasize that the whole set of conditions (1)–(4) appearing in the claim of Lemma 2.4 together with condition (2.5) on the set E is not equivalent to admissibility of the loads as expressed by (2.15). This phenomenon is made explicit in [16, Example 3.6].

Finally, by setting

$$\mathcal{S}_{\mathcal{L},E} = \{\, \mathbf{R} \in SO(3) : \Phi(\mathbf{R}, E, \mathcal{L}) = 0 \,\}\,, \tag{2.17}$$

we have

Lemma 2.6. *Assume that* (2.5), (2.15) *hold and that* $\mathcal{L}(\mathbf{e}_3) < 0$. *Then*

$$\textit{either} \quad \mathcal{S}_{\mathcal{L},E} = \{\, \mathbf{I} \,\} \qquad \textit{or} \quad \mathcal{S}_{\mathcal{L},E} = \{\, \mathbf{R} \in SO(3) : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3 \,\}.$$

2.4 Energy functionals

If E fulfils (2.5) the incompressible Signorini problem in linear elasticity can be described as the minimization of the functional $\mathcal{E}^I : H^1(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{E}^I(\mathbf{u}) := \begin{cases} \displaystyle\int_\Omega \mathcal{Q}^I(\mathbf{x}, \mathbb{E}(\mathbf{u}))\, d\mathbf{x} - \mathcal{L}(\mathbf{u}) & \text{if } \mathbf{u} \in \mathcal{A} \\ +\infty & \text{otherwise in } H^1(\Omega, \mathbb{R}^3), \end{cases}$$

where $\mathbb{E}(\mathbf{u}) := \operatorname{sym} \nabla \mathbf{u}, \quad \mathcal{Q}(\mathbf{x}, \mathbf{F}) := \frac{1}{2}\, \mathbf{F}^T \mathbb{C}(\mathbf{x})\, \mathbf{F}, \quad \mathbb{C}(\mathbf{x}) := D^2 \mathcal{W}(\mathbf{x}, \mathbf{I}),$

$$\mathcal{Q}^I(\mathbf{x}, \mathbf{F}) := \begin{cases} \mathcal{Q}(\mathbf{x}, \mathbf{F}) & \text{if } \operatorname{Tr} \mathbf{F} = 0 \\ +\infty \text{ otherwise} \end{cases}$$

and where $\mathcal{A}$ is defined by

$$\mathcal{A} \ := \ \big\{\, \mathbf{u} \in H^1(\Omega; \mathbb{R}^3) : \ \ u_3^*(\mathbf{x}) \geq 0 \ \text{ q.e. } \mathbf{x} \in E \,\big\}\,.$$

The meaning of such constraint is that the deformed configuration of E, namely $\{\mathbf{y}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x}),\ \mathbf{x} \in E\}$, is constrained to remain in $\{y_3 \geq 0\}$.

For every $\mathbf{y} \in H^1(\Omega, \mathbb{R}^3)$ we introduce the set

$$\mathcal{M}(\mathbf{y}) := \operatorname{argmin}\left\{ \int_\Omega |\nabla \mathbf{y} - \mathbf{R}|^2 \, d\mathbf{x} : \ \mathbf{R} \in SO(3) \right\}. \tag{2.18}$$

Thus, due to the rigidity inequality of [7], there exists a constant $C = C(\Omega) > 0$ such that for every $\mathbf{y} \in H^1(\Omega, \mathbb{R}^3)$ and every $\mathbf{R} \in \mathcal{M}(\mathbf{y})$

$$\int_\Omega \Big(d\big(\nabla \mathbf{y}, SO(3)\big)\Big)^2 \, d\mathbf{x} \ \geq \ C \int_\Omega |\nabla \mathbf{y} - \mathbf{R}|^2 \, d\mathbf{x}, \tag{2.19}$$

where $d\big(\mathbf{F}, SO(3)\big) := \min\{|\mathbf{F} - \mathbf{R}| : \mathbf{R} \in SO(3)\}$. The rigidity inequality is a crucial tool for obtaining linear elasticity as limit of finite elasticity via Γ-convergence, as seen in [4] and subsequent works [9, 10, 15, 14, 16, 17, 18, 19, 20, 21, 22].

Let us introduce nonlinear elastic energies. If $(h_j)_{j \in \mathbb{N}} \subset (0,1)$ is a vanishing sequence, the rescaled finite elasticity functionals $\mathcal{G}_j : H^1(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ are defined by

$$\mathcal{G}_j(\mathbf{y}) = \begin{cases} h_j^{-2} \displaystyle\int_\Omega \mathcal{W}(\mathbf{x}, \nabla \mathbf{y}) \, d\mathbf{x} - h_j^{-1} \mathcal{L}(\mathbf{y} - \mathbf{x}) & \text{if } \mathbf{y} \in \mathcal{A} \\ \\ +\infty & \text{otherwise.} \end{cases}$$

The related rescaled incompressible finite elasticity functionals $\mathcal{G}_j^I : H^1(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ are defined by

$$\mathcal{G}_j^I(\mathbf{y}) = \begin{cases} h_j^{-2} \displaystyle\int_\Omega \mathcal{W}^I(\mathbf{x}, \nabla \mathbf{y}) \, d\mathbf{x} - h_j^{-1} \mathcal{L}(\mathbf{y} - \mathbf{x}) & \text{if } \mathbf{y} \in \mathcal{A}, \\ \\ +\infty & \text{otherwise.} \end{cases}$$

It will be shown (see Lemma 3.1 below) that $\inf_{H^1(\Omega;\mathbb{R}^3)} \mathcal{G}_j^I > -\infty$ for every $j \in \mathbb{N}$. Moreover, $(\mathbf{y}_j)_{j \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)$ is said a *minimizing sequence of the sequence of functionals* $\mathcal{G}_j^I$ if

$$\lim_{j \to +\infty} \left(\mathcal{G}_j^I(\mathbf{y}_j) - \inf_{H^1(\Omega, \mathbb{R}^3)} \mathcal{G}_j^I \right) = 0. \tag{2.20}$$

The main focus of the paper is to investigate if $\mathcal{G}_j^I(\mathbf{y}_j)$ converges to a minimum of (2.20) whenever $(\mathbf{y}_j)_{j \in \mathbb{N}}$ is a minimizing sequence of $\mathcal{G}_j^I$. To this end we introduce the functionals $\mathcal{I}^I, \ \widetilde{\mathcal{G}}^I, \ \mathcal{G}^I : H^1(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{I}^I(\mathbf{u}) := \min_{\mathbf{b} \in \mathbb{R}^2} \int_\Omega \mathcal{Q}^I(\mathbf{x}, \mathbb{E}(\mathbf{u}) + \tfrac{1}{2} b_\alpha (\mathbf{e}_\alpha \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\alpha)) \, d\mathbf{x},$$

$$\widetilde{\mathcal{G}}^I(\mathbf{u}) := \begin{cases} \mathcal{I}^I(\mathbf{u}) - \max\limits_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{R}\mathbf{u}) & \text{if } \mathbf{u}\in\mathcal{A} \\ +\infty & \text{otherwise in } H^1(\Omega,\mathbb{R}^3), \end{cases}$$

and

$$\mathcal{G}^I(\mathbf{u}) := \begin{cases} \displaystyle\int_\Omega \mathcal{Q}^I(\mathbf{x},\mathbb{E}(\mathbf{u}))\,d\mathbf{x} - \max\limits_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{R}\mathbf{u}) & \text{if } \mathbf{u}\in\mathcal{A} \\ +\infty & \text{otherwise in } H^1(\Omega,\mathbb{R}^3), \end{cases}$$

where $\mathcal{S}_{\mathcal{L},E}$ is defined by (2.17).

Remark 2.7. It is worth noticing that $\widetilde{\mathcal{G}}^I \le \mathcal{G}^I \le \mathcal{E}^I$, since $\mathbf{I}\in\mathcal{S}_{\mathcal{L},E}$, and it is straightforward to check that $\mathcal{I}^I$, $\widetilde{\mathcal{G}}^I$, $\mathcal{G}^I$ are all continuous with respect to the strong convergence in $H^1(\Omega;\mathbb{R}^3)$.

2.5 The variational convergence result

The main result is stated in the next theorem, referring to (2.20) for the notion of minimizing sequence. The technical assumption that $\partial\Omega$ has a finite number of connected components will be needed for applying an extension theorem about divergence-free vector fields from [10].

Theorem 2.8. *Assume that $\partial\Omega$ has a finite number of connected components, that* (2.5), (2.7), (2.8), (2.9), (2.10), (2.11), (2.14), (2.15) *hold true and that* $\mathcal{L}(\mathbf{e}_3) < 0$. *Let $(h_j)_{j\in\mathbb{N}}\subset(0,1)$ be a vanishing sequence. Let $(\overline{\mathbf{y}}_j)_{j\in\mathbb{N}}\subset H^1(\Omega,\mathbb{R}^3)$ be a minimizing sequence of $\mathcal{G}^I_j$. If $\mathbf{R}_j\in\mathcal{M}(\overline{\mathbf{y}}_j)$ for every $j\in\mathbb{N}$, then there are $\overline{\mathbf{c}}_j\in\mathbb{R}^3$, $j\in\mathbb{N}$, such that the sequence*

$$\overline{\mathbf{u}}_j(\mathbf{x}) := {h_j}^{-1}\mathbf{R}_j^T\big\{\big(\overline{\mathbf{y}}_j - \overline{\mathbf{c}}_j - \mathbf{R}_j\mathbf{x}\big)_\alpha\,\mathbf{e}_\alpha + (\overline{y}_{j,3} - x_3)\mathbf{e}_3\,\big\}$$

is weakly compact in $H^1(\Omega,\mathbb{R}^3)$. Therefore up to subsequences, $\overline{\mathbf{u}}_j \rightharpoonup \overline{\mathbf{u}}$ in $H^1(\Omega,\mathbb{R}^3)$ and also

$$\mathcal{G}^I_j(\mathbf{y}_j) \to \widetilde{\mathcal{G}}^I(\overline{\mathbf{u}}) = \min_{H^1(\Omega,\mathbb{R}^3)} \widetilde{\mathcal{G}}^I = \min_{H^1(\Omega,\mathbb{R}^3)} \mathcal{G}^I, \qquad \text{as } j\to+\infty.$$

Remark 2.9. Since $\widetilde{\mathcal{G}}^I \le \mathcal{G}^I$ then equality $\min\widetilde{\mathcal{G}}^I = \min\mathcal{G}^I$ is equivalent to $\operatorname{argmin}\mathcal{G}^I \subset \operatorname{argmin}\widetilde{\mathcal{G}}^I$ with possible strict inclusion.

3 Proof of the variational convergence result

This section contains the proof of our main result. We start by showing that sequences of deformations with equibounded energy correspond (up to suitably tuned rotations and translations of the horizontal components) to displacements that are equibounded in H^1.

Lemma 3.1. *(**compactness**) Assume that E, $\mathcal{L}$ and $\mathcal{W}$ fulfil* (2.5), (2.7), (2.8), (2.9), (2.10), (2.11), (2.15) *and $\mathcal{L}(\mathbf{e}_3) < 0$. Let $(h_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. Let $(\mathbf{y}_j)_{j\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^3)$ satisfy $\sup_{j\in\mathbb{N}} \mathcal{G}^I_j(\mathbf{y}_j) < +\infty$. If $\mathbf{R}_j \in \mathcal{M}(\mathbf{y}_j)$ for every $j \in \mathbb{N}$, then every limit point of the sequence $(\mathbf{R}_j)_{j\in\mathbb{N}}$ belongs to $\mathcal{S}_{\mathcal{L},E}$, and by setting*

$$c_{j,\alpha} = |\Omega|^{-1}\int_\Omega (\mathbf{y}_j(\mathbf{x}) - \mathbf{R}_j\mathbf{x})_\alpha \, d\mathbf{x}, \qquad \alpha = 1, 2, \tag{3.21}$$

$$c_{j,3} = -\min_{\mathbf{x}\in E_{ess}} (\mathbf{R}_j\mathbf{x})_3, \tag{3.22}$$

the sequence

$$h_j^{-1}\big(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\big)_\alpha \mathbf{e}_\alpha + h_j^{-1}(y_{j,3} - x_3)\mathbf{e}_3$$

is bounded in $H^1(\Omega;\mathbb{R}^3)$. Moreover, $\inf_{j\in\mathbb{N}} \inf_{H^1(\Omega;\mathbb{R}^3)} \mathcal{G}^I_j > -\infty$.

Proof. Since $\mathcal{G}^I_j \geq \mathcal{G}_j$ the proof follows from the analogous [16, Lemma 4.1] and we report here that proof for the sake of completeness and also making some steps more precise.

Step 1. We start by providing some estimates using the finiteness of $M := \sup_{j\in\mathbb{N}} \mathcal{G}^I_j(\mathbf{y}_j)$. Referring to (2.18), let $\mathbf{R}_j \in \mathcal{M}(\mathbf{y}_j)$ for every $j \in \mathbb{N}$. Up to subsequences, $\mathbf{R}_j \to \mathbf{R}$ for some $\mathbf{R} \in SO(3)$. Then we define $\mathbf{c}_j = (c_{j,1}, c_{j,2}, c_{j,3})$ by (3.21) and (3.22). By the rigidity inequality (2.19) there exists a constant $C = C(\Omega) > 0$ such that

$$\begin{aligned} M \ &\geq\ \mathcal{G}^I_j(\mathbf{y}_j) \ \geq\ C\, h_j^{-2}\int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - h_j^{-1}\mathcal{L}(\mathbf{y}_j - \mathbf{x}) \\ &= C\, h_j^{-2}\int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - h_j^{-1}\mathcal{L}(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j) - h_j^{-1}\mathcal{L}(\mathbf{R}_j\mathbf{x} - \mathbf{x} + \mathbf{c}_j). \end{aligned} \tag{3.23}$$

Thus, by (2.15) and the definition of $c_{j,3}$ we get

$$M \ \geq\ C\, h_j^{-2} \int_\Omega |\nabla\mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - h_j^{-1}\mathcal{L}(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j) \tag{3.24}$$

and Poincaré inequality (with constant denoted by C_P) along with Young inequality entail, for every $\varepsilon > 0$,

$$\begin{aligned} h_j^{-1}\mathcal{L}\big((\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_\alpha\, \mathbf{e}_\alpha\big) &\le h_j^{-1} C_P \|\mathcal{L}\|_* \Big(\sum_{\alpha=1}^{2} \int_\Omega |\, (\nabla \mathbf{y}_j - \mathbf{R}_j)_\alpha\,|^2\, d\mathbf{x}\Big)^{1/2} \\ &\le \frac{C_P\|\mathcal{L}\|_*^2}{2\varepsilon} + \frac{\varepsilon\, h_j^{-2}\, C_P}{2} \sum_{\alpha=1}^{2} \int_\Omega |(\nabla \mathbf{y}_j - \mathbf{R}_j)_\alpha|^2\, d\mathbf{x}. \end{aligned} \tag{3.25}$$

Estimates (3.24) and (3.25) together with Young inequality provide

$$\begin{aligned} M &\ge h_j^{-2}\Big(C - \frac{\varepsilon\, C_P}{2}\Big)\int_\Omega |\nabla \mathbf{y}_j - \mathbf{R}_j|^2 d\mathbf{x} - \frac{C_P\,\|\mathcal{L}\|_*^2}{2\,\varepsilon} - h_j^{-1}\mathcal{L}\big((\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3\, \mathbf{e}_3\big) \\ &\ge h_j^{-2}\Big(C - \frac{\varepsilon\, C_P}{2}\Big)\int_\Omega |\nabla \mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - \frac{C_P\|\mathcal{L}\|_*^2}{2\,\varepsilon} \\ &\qquad - h_j^{-1}\, \|\mathcal{L}\|_*\, \big(\, \|(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3\|_{L^2(\Omega)} + \|\nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_3\|_{L^2(\Omega)}\,\big) \\ &\ge h_j^{-2}\Big(C - \frac{\varepsilon\, C_P}{2} - \frac{\varepsilon}{2}\Big)\int_\Omega |\nabla \mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} - \Big(\frac{C_P}{2\,\varepsilon} + \frac{1}{2\,\varepsilon}\Big)\, \|\mathcal{L}\|_*^2 \\ &\qquad - h_j^{-1}\, \|\mathcal{L}\|_*\, \big(\, \|(\mathbf{y}_j - \mathbf{R}_h\mathbf{x} - \mathbf{c}_j)_3\|_{L^2(\Omega)}. \end{aligned}$$

By choosing $\varepsilon = C/(C_P + 1)$, we get

$$\begin{aligned} & h_j^{-2}\, \frac{C}{2} \int_\Omega |\nabla \mathbf{y}_j - \mathbf{R}_j|^2\, d\mathbf{x} \\ & \qquad \le M + \frac{(C_P+1)^2}{2C}\|\mathcal{L}\|_*^2 + h_j^{-1}\, \|\mathcal{L}\|_*\, \|(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3\|_{L^2(\Omega;\mathbb{R}^3)}. \end{aligned} \tag{3.26}$$

Thus, if we show that $h_j^{-1}\|(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3\|_{L^2(\Omega)}$ is uniformly bounded, then, due to estimate (3.26), $\|h_j^{-1}(\nabla \mathbf{y}_j - \mathbf{R}_j)\|_{L^2(\Omega)}$ is uniformly bounded too and Poincarè inequality entails uniform boundedness of $h_j^{-1}\|\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\|_{H^1(\Omega;\mathbb{R}^3)}$.

Let us define for every j

$$t_j \;:=\; h_j^{-1}\, \|\, (\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3\, \|_{L^2(\Omega)} \qquad \text{and} \qquad w_j \;:= t_j^{-1} h_j^{-1}\, (\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_3. \tag{3.27}$$

Then

$$\|w_j\|_{L^2(\Omega)} = 1, \qquad |\nabla \mathbf{y}_j - \mathbf{R}_j|^2 \;=\; \sum_{\alpha=1}^{2} |\, \nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha\,|^2 \;+\; h_j^2 t_j^2\, |\nabla w_j|^2, \tag{3.28}$$

so that (3.23), (3.25) and (2.15) imply that for every $\varepsilon > 0$

$$\begin{aligned} &Ct_j^2 \int_\Omega |\nabla w_j|^2 \, d\mathbf{x} \,-\, t_j\, \mathcal{L}(w_j\, \mathbf{e}_3) - h_j^{-1} \mathcal{L}(\mathbf{R}_j\mathbf{x} - \mathbf{x} + c_{j,3}\mathbf{e}_3) \\ &\quad \le M - C\, h_j^{-2} \sum_{\alpha=1}^{2} \int_\Omega |\, \nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha\, |^2 \, d\mathbf{x} \,+\, h_j^{-1}\, \mathcal{L}\big(\, (\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j)_\alpha\, \mathbf{e}_\alpha\, \big) \\ &\quad \le M - C\, h_j^{-2} \sum_{\alpha=1}^{2} \int_\Omega |\, \nabla(\mathbf{y}_h - \mathbf{R}_j\mathbf{x})_\alpha\, |^2 \, d\mathbf{x} \,+\, \frac{C_P\, \|\mathcal{L}\|_*^2}{2\,\varepsilon} \\ &\qquad + \frac{h_j^{-2}\, \varepsilon\, C_P}{2} \sum_{\alpha=1}^{2} \int_\Omega |\nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha\, |^2 \, d\mathbf{x}, \end{aligned} \tag{3.29}$$

and by choosing $\varepsilon = 2C/C_P$ in (3.29) we get

$$C\, t_j^2 \int_\Omega |\nabla w_j|^2 \, d\mathbf{x} \,-\, t_j\, \mathcal{L}(w_j\, \mathbf{e}_3) - h_j^{-1} \mathcal{L}(\mathbf{R}_j\mathbf{x} - \mathbf{x} + c_{j,3}\mathbf{e}_3) \;\le\; \frac{C_P^2\, \|\mathcal{L}\|_*^2}{4\,C} + M \tag{3.30}$$

while, by choosing $\varepsilon = C/C_P$, (3.29) yields

$$\begin{aligned} &\frac{1}{2}\, C\, h_j^{-2} \sum_{\alpha=1}^{2} \int_\Omega \nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha\, |^2 \, d\mathbf{x} \,+\, C\, t_j^2 \int_\Omega |\nabla w_j|^2 \,-\, t_j\, \mathcal{L}(w_j\, \mathbf{e}_3) \;\le \\ &\quad \le\; \frac{C_P^2}{2C}\, \|\mathcal{L}\|_*^2 + M + h_j^{-1} \mathcal{L}(\mathbf{R}_j\mathbf{x} - \mathbf{x} + c_{j,3}\mathbf{e}_3). \end{aligned}$$

Thus, by taking account of the fact that (2.15) entails $\mathcal{L}(\mathbf{R}_j\mathbf{x} - \mathbf{x} + c_{j,3}\mathbf{e}_3) \le 0$, we get

$$\frac{1}{2}\, C\, \frac{h_j^{-2}}{t_j^2} \sum_{\alpha=1}^{2} \int_\Omega |\, \nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha\, |^2 \, d\mathbf{x} \;\le\; \frac{1}{t_j}\, \mathcal{L}(w_j\, \mathbf{e}_3) \,+\, \frac{1}{{t_j}^2}\, \frac{C_P^2}{2C}\, \|\mathcal{L}\|_*^2 + \frac{M}{t_j^2}. \tag{3.31}$$

Moreover, (3.30) and (2.15) yield

$$-h_j \frac{C_P^2\, \|\mathcal{L}\|_*^2}{4\,C} - M h_j - \, h_j t_j\, \mathcal{L}(w_j\, \mathbf{e}_3) \le \Phi(\mathbf{R}_j, E, \mathcal{L}) = \mathcal{L}(\mathbf{R}_j\mathbf{x} - \mathbf{x} + c_{j,3}\mathbf{e}_3) \;\le 0. \tag{3.32}$$

Step 2. Here we prove that t_j from (3.27) is uniformly bounded, thus yielding as shown through the previous step the uniform boundedness of $h_j^{-1}\|\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\|_{H^1(\Omega;\mathbb{R}^3)}$. In order to check that the sequence $(t_j)_{j\in\mathbb{N}}$ is uniformly bounded we assume by contradiction that, up to subsequences, $\lim_{j\to+\infty} t_j =$

$+\infty$. Normalization $\|w_j\|_{L^2(\Omega)} = 1$ entails, for every $\varepsilon > 0$,

$$\begin{aligned}\mathcal{L}\big(w_j\,\mathbf{e}_3\big) \;&\le\; \|\mathcal{L}\|_*\big(\,\|w_j\|_{L^2} + \|\nabla w_j\|_{L^2}\,\big) \;=\; \|\mathcal{L}\|_*\big(\,1 + \|\nabla w_j\|_{L^2}\,\big)\\ &\le\; \|\mathcal{L}\|_* \;+\; \frac{\|\mathcal{L}\|_*^2}{2\,\varepsilon} \;+\; \frac{\varepsilon}{2}\,\|\nabla w_j\|_{L^2}^2,\end{aligned}$$

and choosing $\varepsilon = C\,t_j^2$ we get, by (3.30),

$$\frac{C}{2}t_j^2\int_\Omega |\nabla w_j|^2\,d\mathbf{x} \;\le\; t_j\,\|\mathcal{L}\|_* \;+\; \frac{\|\mathcal{L}\|_*^2}{2C\,t_j} + M, \tag{3.33}$$

thus $\int_\Omega |\nabla w_j|^2\,d\mathbf{x} \to 0$, so by (3.28) $w_j \to w$ in $H^1(\Omega;\mathbb{R}^3)$ with $\nabla w = 0$ a.e. in Ω, that is, w is a constant function since Ω is a connected open set. Combining estimates (3.31) and (3.33) we get

$$\begin{aligned}&\frac{1}{2}C\,\frac{h_j^{-2}}{t_j^2}\sum_{\alpha=1}^{2}\int_\Omega |\,\nabla(\mathbf{y}_j - \mathbf{R}_j\mathbf{x})_\alpha\,|^2\,d\mathbf{x}\\ &\quad\le\; \frac{1}{t_j}\left(\|\mathcal{L}\|_*^2 \;+\; \frac{\|\mathcal{L}\|_*^2}{2} \;+\; \frac{1}{2}\,\|\nabla w_j\|_{L^p}^2\right) \;+\; \frac{1}{t_j{}^2}\,\frac{C_P^2}{2\,C_R\,C}\,\|\mathcal{L}\|_*^2 + \frac{M}{t_j^2},\end{aligned}$$

hence

$$\frac{1}{h_j\,t_j}\,\nabla\big(\mathbf{y}_j - \mathbf{R}_j\mathbf{x}\big)_\alpha \;\to\; 0 \qquad \text{in } L^2(\Omega;\mathbb{R}^{3\times 3}) \qquad \text{if } \alpha = 1,2,$$

and by definition of w_j we have

$$h_j^{-1}t_j^{-1}\,\left(\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\right)_3 \;\to\; w \qquad \text{q.e. } \mathbf{x} \in E\,.$$

Moreover, by (3.30) and (2.15) we get

$$\mathcal{L}(w_j\,\mathbf{e}_3) \;\ge\; -\,\frac{C_P^2}{4\,C\,t_j}\,\|\mathcal{L}\|_*^2 - \frac{M}{t_j}.$$

Hence, due to $\mathcal{L}(w_j\,\mathbf{e}_3) \to \mathcal{L}(w\,\mathbf{e}_3) = w\,\mathcal{L}(\mathbf{e}_3)$, we have $w\,\mathcal{L}(\mathbf{e}_3) \ge 0$, thus, by taking account of $\mathcal{L}(\mathbf{e}_3) < 0$, we get $w \le 0$ and eventually, by $\|w_j\|_{L^2} = 1$, we obtain $w \;<\; 0$.

We notice now that if we set $L := \liminf_{j\to+\infty} h_j t_j$, then either $L \in (0,+\infty]$ or $L = 0$. Assume first that $L \in (0,+\infty)$. Since on a subsequence $\mathbf{R}_j \to \mathbf{R} \in SO(3)$, we have for q.e. $\mathbf{x}\in E_{ess}$

$$\begin{aligned}\frac{y_{j,3}^*}{h_j\,t_j} = \frac{y_{j,3}^* - (\mathbf{R}_j\mathbf{x})_3 - \mathbf{c}_{j,3}}{h_j\,t_j} \;+\; \frac{(\mathbf{R}_j\mathbf{x})_3 + \mathbf{c}_{j,3}}{h_j\,t_j} \to&\\ w + L^{-1}\{(\mathbf{R}\mathbf{x})_3 - \min_{E_{ess}}(\mathbf{R}\mathbf{x})_3\}&\end{aligned} \tag{3.34}$$

as $j \to +\infty$ (along a suitable subsequence). Since

$$\min_{E_{ess}} \left\{ (\mathbf{Rx})_3 - \min_{E_{ess}}((\mathbf{Rx})_3) \right\} = 0$$

then there exists $E' \subset E_{ess}$ with $\operatorname{cap} E' > 0$ such that

$$(\mathbf{Rx})_3 - \min_{E_{ess}}(\mathbf{R}_x x)_3 < -\frac{wL}{2}$$

on E', so that by (3.34)

$$\frac{y^*_{j,3}}{h_j\, t_j} \to w + L^{-1}\{(\mathbf{Rx})_3 - \min_{E_{ess}}(\mathbf{Rx})_3\} < \frac{w}{2} < 0$$

for q.e. $\mathbf{x} \in E'$, a contradiction since the assumption $\sup_{j\in\mathbb{N}} \mathcal{G}^I_j(\mathbf{y}_j) < +\infty$ implies $\mathbf{y}_j \in \mathcal{A}$, i.e., $y^*_{j,3} \geq 0$ q.e. in E. If $L = +\infty$ then by arguing as in estimate (3.34) we easily get $h_j^{-1} t_j^{-1} y^*_{j,3} \to w < 0$ for q.e. $\mathbf{x} \in E'$ which is again a contradiction. Therefore we are left to assume that, up to subsequences, $h_j t_j \to 0$.

In order to complete this step we notice that either $\mathbf{R}_j\mathbf{e}_3 \neq \mathbf{e}_3$ for j large enough or $\mathbf{R}_j\mathbf{e}_3 = \mathbf{e}_3$ for infinitely many j, and we separately treat these two cases. In the first case, by taking account that $\mathcal{L}(\mathbf{e}_3) < 0$, [16, Lemma 3.10] entails

$$\limsup_{j\to+\infty} \frac{\Phi(\mathbf{R}_j, E, \mathcal{L})}{|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|} \; < \; 0. \tag{3.35}$$

By (3.32) we get therefore

$$\gamma := \liminf_{j\to+\infty} \frac{h_j t_j}{|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|} > 0 \tag{3.36}$$

and for large enough j (3.36) yields

$$|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3| \; \leq \; \frac{2h_j t_j}{\gamma}\,.$$

Hence, if $\mathbf{x}_j \in \operatorname{argmin}_{E_{ess}}\{(\mathbf{R}_j\mathbf{x})_3\}$ for every j, so we may assume that, up to subsequences, $\mathbf{x}_j \to \overline{\mathbf{x}} \in E_{ess}$, then for every $\mathbf{x} \in E_{ess}$

$$\begin{aligned}\theta_j(\mathbf{x}) :&= (\mathbf{R}_j\mathbf{x})_3 - \min_{\mathbf{x}\in E_{ess}}((\mathbf{R}_j\mathbf{x})_3) = (\mathbf{R}_j(\mathbf{x}-\mathbf{x}_j))_3 = (\mathbf{R}_j(\mathbf{x}-\mathbf{x}_j))_3 - (\mathbf{x}-\mathbf{x}_j)_3 \\ &= (\mathbf{x}-\mathbf{x}_j)\cdot(\mathbf{R}_j^T - \mathbf{I})\mathbf{e}_3 \leq |\mathbf{x}-\mathbf{x}_j||\mathbf{R}_j^T\mathbf{e}_3 - \mathbf{e}_3| \leq \frac{2h_j t_j}{\gamma}|\mathbf{x}-\mathbf{x}_j|\end{aligned}$$

and then we get, for j large enough,

$$\theta_j(\mathbf{x}) \leq \frac{2h_j t_j}{\gamma}|\mathbf{x}-\overline{\mathbf{x}}| \leq \frac{h_j t_j}{4}|w| \qquad \text{for every } \mathbf{x} \in E'' := B_{\frac{\gamma|w|}{16}}(\overline{\mathbf{x}}) \cap E_{ess}. \tag{3.37}$$

Moreover, by taking account of $w_j \to w$ in $H^1(\Omega;\mathbb{R}^3)$ we get

$$\mathbf{y}^*_{j,3} = h_j t_j w_j + \theta_j = h_j t_j w + \theta_j + \delta_j \qquad \text{where } \delta_j := h_j t_j (w_j - w),$$

with $h_j^{-1} t_j^{-1} \delta_j \to 0$ in $H^1(\Omega;\mathbb{R}^3)$ and thus, up to subsequences, cap quasi uniformly in Ω. Since $\operatorname{cap} E'' > 0$ we may assume that there exists $E''' \subset E''$ with $\operatorname{cap} E''' > 0$ such that $h_j^{-1} t_j^{-1}\delta_j \to 0$ uniformly in E''' hence for j large enough

$$w + h_j^{-1} t_j^{-1} \delta_j(\mathbf{x}) < \frac{w}{2} \quad \text{q.e. } \mathbf{x} \in E'''.$$

From the latter estimate and (3.37) we get for j large enough

$$h_j^{-1} y^*_{j,3}(\mathbf{x}) \leq \frac{t_j}{4} w < \frac{w}{4} < 0 \qquad \text{q.e. } \mathbf{x} \in E''',$$

a contradiction since $y^*_{j,3}(\mathbf{x}) \geq 0$ for q.e. $\mathbf{x} \in E$. In the second case we may assume that $\mathbf{R}_j \mathbf{e}_3 = \mathbf{e}_3$ for every j so $c_{j,3} = 0$ and $x_3 = 0 \Rightarrow (\mathbf{R}_j\mathbf{x})_3 = 0$ for every j. In particular $w_j \to w < 0$ in $H^1(\Omega;\mathbb{R}^3)$ implies

$$t_j^{-1} h_j^{-1} y_{j,3}(\mathbf{x}) = t_j^{-1} h_j^{-1}(y_{j,3}(\mathbf{x}) - (\mathbf{R}_j\mathbf{x})_3 - c_{j,3}) = w_j(\mathbf{x}) \to w < 0 \qquad \text{q.e. } \mathbf{x} \in E.$$

again contradicting $y^*_{j,3}(\mathbf{x}) \geq 0$ for q.e. $\mathbf{x} \in E$. This proves that $(t_j)_{j\in\mathbb{N}}$ is a bounded sequence.

Step 3. We have shown that $h_j^{-1}\|\mathbf{y}_j - \mathbf{R}_j\mathbf{x} - \mathbf{c}_j\|_{H^1(\Omega;\mathbb{R}^3)}$ is uniformly bounded, yielding in particular that $(t_j w_j)_{j\in\mathbb{N}}$ is a bounded sequence in $H^1(\Omega;\mathbb{R}^3)$. But then (3.32) yields

$$-M' h_j \leq \Phi(\mathbf{R}_j, E, \mathcal{L}) = \mathcal{L}(\mathbf{R}_j\mathbf{x} - \mathbf{x} + c_{j,3}\mathbf{e}_3) \ \leq 0$$

for some suitable constant $M' > 0$ independent of j. If, up to subsequences, $\mathbf{R}_j \to \mathbf{R}$, we get $\Phi(\mathbf{R}_j, E, \mathcal{L}) \to \Phi(\mathbf{R}, E, \mathcal{L}) = 0$ thus proving that $\mathbf{R} \in \mathcal{S}_{\mathcal{L},E}$.

Therefore, in order to end the proof, we have to show that the sequence $(h_j^{-1}(y_{j,3} - x_3))_{j\in\mathbb{N}}$ is equibounded in $H^1(\Omega;\mathbb{R}^3)$. We claim that the sequence $(h_j^{-1}|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|)_{j\in\mathbb{N}}$ is bounded. Indeed, if we assume that $\mathbf{R}_j\mathbf{e}_3 \neq \mathbf{e}_3$ for every j large enough, (3.35) entails

$$\gamma' := \liminf_{j\to+\infty} \frac{h_j}{|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|} > 0,$$

so that $|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3| \le 2h_j/\gamma'$ for j large enough, thus proving the claim. As a consequence, since

$$\begin{aligned}\left|h_j^{-1}(y_{j,3} - x_3)\right| &\le \left|h_j^{-1}(y_{j,3} - (\mathbf{R}_j\mathbf{x})_3 - c_{j,3})\right| + h_j^{-1}|(\mathbf{R}_j\mathbf{x})_3 - x_3| \\ &\le \left|h_j^{-1}(y_{j,3} - (\mathbf{R}_j\mathbf{x})_3 - c_{j,3})\right| + h_j^{-1}|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|\,|\mathbf{x}|\end{aligned}$$

and

$$\begin{aligned}\left|h_j^{-1}\nabla(y_{j,3} - x_3)\right| &\le h_j^{-1}\,|\nabla(y_{j,3} - (\mathbf{R}_j\mathbf{x})_3)| + h_j^{-1}|\nabla((\mathbf{R}_j\mathbf{x})_3 - x_3)| \\ &\le h_j^{-1}\,|\nabla(y_{j,3} - (\mathbf{R}_j\mathbf{x})_3)| + h_j^{-1}|\mathbf{R}_j\mathbf{e}_3 - \mathbf{e}_3|,\end{aligned}$$

we obtain that the sequence $(h_j^{-1}(y_{j,3} - x_3))_{j\in\mathbb{N}}$ is equibounded in $H^1(\Omega;\mathbb{R}^3)$.

Step 4. Eventually, we prove the last statement by contradiction, assuming that there exists a sequence $(\overline{\mathbf{y}}_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $\mathcal{G}_j^I(\overline{\mathbf{y}}_j) \to -\infty$. In such case, $\mathcal{G}_j^I(\overline{\mathbf{y}}_j) \le 0$ for j large enough so, as we have just proved, there exist sequences $(\mathbf{R}_j)_{j\in\mathbb{N}} \subset SO(3)$ and $(\overline{\mathbf{c}}_j)_{j\in\mathbb{N}} \subset \mathbb{R}^3$ and there exists $\mathbf{v} \in H^1(\Omega;\mathbb{R}^3)$ such that, up to subsequences,

$$\mathbf{v}_j := h_j^{-1}(\overline{\mathbf{y}}_j - \mathbf{R}_j\mathbf{x} - \overline{\mathbf{c}}_j) \rightharpoonup \mathbf{v}$$

weakly in $H^1(\Omega;\mathbb{R}^3)$. Therefore by (2.15)

$$\liminf_{j\to+\infty} \mathcal{G}_j^I(\overline{\mathbf{y}}_j) \ge -\limsup_{j\to+\infty} \mathcal{L}(\mathbf{v}_j) = -\mathcal{L}(\mathbf{v}) > -\infty,$$

a contradiction. QED

Lemma 3.2. ***(Lower bound)*** *Assume that E, $\mathcal{L}$, $\mathcal{W}$ fulfill the conditions* (2.5), (2.7), (2.8), (2.9), (2.10), (2.11), (2.14), (2.15) *and* $\mathcal{L}(\mathbf{e}_3) < 0$. *Let $(h_j)_{j\in\mathbb{N}} \subset (0,1)$ be a vanishing sequence. Let $(\mathbf{y}_j)_{j\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^3)$ satisfy $\sup_{j\in\mathbb{N}} \mathcal{G}_j^I(\mathbf{y}_j) < +\infty$. Let $\mathbf{R}_j \in \mathcal{M}(\mathbf{y}_j)$ for every $j \in \mathbb{N}$. If*

$$\mathbf{u}_j(\mathbf{x}) := h_j^{-1}\mathbf{R}_j^T\left\{\left(\mathbf{y}_j - \mathbf{c}_j - \mathbf{R}_j\mathbf{x}\right)_\alpha \mathbf{e}_\alpha + (y_{j,3} - x_3)\mathbf{e}_3\right\}, \quad j \in \mathbb{N}, \tag{3.38}$$

where $\mathbf{c}_j$ are defined by (3.21)-(3.22), *then, up to passing to a not relabeled subsequence, we have $\mathbf{u}_j \rightharpoonup \mathbf{u}$ weakly in $H^1(\Omega;\mathbb{R}^3)$, $\mathbf{u} \in \mathcal{A}$ and*

$$\liminf_{j\to+\infty} \mathcal{G}_j^I(\mathbf{y}_j) \ge \widetilde{\mathcal{G}}^I(\mathbf{u}).$$

PROOF. Finiteness of $\mathcal{G}_j^I(\mathbf{y}_j)$ imply $\mathbf{y}_j \in \mathcal{A}$ for every $j \in \mathbb{N}$. Due to Lemma 3.1, the sequence defined in (3.38) is equibounded in $H^1(\Omega;\mathbb{R}^3)$ hence there exists $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$ such that up to subsequences $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $H^1(\Omega;\mathbb{R}^3)$. Moreover since

$$\begin{aligned} 1 &= \det \nabla \mathbf{y}_j = \det(\mathbf{R}_j(\mathbf{I} + h_j \nabla \mathbf{u}_j)) = \det(\mathbf{I} + h_j \nabla \mathbf{u}_j) = \\ &= 1 + h_j \operatorname{div} \mathbf{u}_j - \frac{1}{2} h_j^2 (\operatorname{Tr}(\nabla \mathbf{u}_j)^2 - (\operatorname{Tr} \nabla \mathbf{u}_j)^2) + h_j^3 \det \nabla \mathbf{u}_j \end{aligned}$$

a.e. in Ω we get

$$\operatorname{div} \mathbf{u}_j = \frac{1}{2} h_j (\operatorname{Tr}(\nabla \mathbf{u}_j)^2 - (\operatorname{Tr} \nabla \mathbf{u}_j)^2) - h_j^2 \det \nabla \mathbf{u}_j.$$

By taking into account that $\nabla \mathbf{u}_j$ are uniformly bounded in L^2 we get $h_j^\alpha |\nabla \mathbf{u}_j| \to 0$ a.e. in Ω for every $\alpha > 0$ hence $\operatorname{div} \mathbf{u}_j = \frac{1}{2} h_j (\operatorname{Tr}(\nabla \mathbf{u}_j)^2 - (\operatorname{Tr} \nabla \mathbf{u}_j)^2) - h_j^2 \det \nabla \mathbf{u}_j \to 0$ a.e. in Ω. Since the weak convergence of $\nabla \mathbf{u}_j$ implies $\operatorname{div} \mathbf{u}_j \rightharpoonup \operatorname{div} \mathbf{u}$ weakly in $L^2(\Omega)$ we get $\operatorname{div} \mathbf{u} = 0$ a.e. in Ω. By recalling [2, Lemma A1] we get, again up to subsequences, $\mathbf{u}_j^*(\mathbf{x}) \to \mathbf{u}^*(\mathbf{x})$ for q.e. $\mathbf{x} \in E$ hence by taking account of

$$u_{j,3}^*(\mathbf{x}) = {h_j}^{-1} y_{j,3}^*(\mathbf{x}) \geq 0$$

for q.e. $\mathbf{x} \in E$ we get $u_3^*(\mathbf{x}) \geq 0$ for q.e. $\mathbf{x} \in E$ that is $\mathbf{u} \in \mathcal{A}$. By taking account of $\mathcal{G}_j^I \geq \mathcal{G}_j$, of $\operatorname{div} \mathbf{u} = 0$, and of $\mathbf{R}_j \to \mathbf{R} \in \mathcal{S}_{\mathcal{L},E}$ up to subsequences as shown in Lemma 3.1, we may invoke [16, Lemma 4.3], which entails that

$$\begin{aligned} \liminf_{j\to+\infty} \mathcal{G}_j^I(\mathbf{y}_j) &\geq \liminf_{j\to+\infty} \mathcal{G}_j(\mathbf{y}_j) \\ &\geq \min_{\mathbf{b}\in\mathbb{R}^2} \int_\Omega \mathcal{Q}(\mathbf{x}, \mathbb{E}(\mathbf{u}) + \tfrac{1}{2} b_\alpha (\mathbf{e}_\alpha \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\alpha))\, d\mathbf{x} - \mathcal{L}(\mathbf{R}\mathbf{u}) \\ &\geq \widetilde{\mathcal{G}}^I(\mathbf{u}) \end{aligned}$$

thus proving the claim. *QED*

Remark 3.3. If condition (2.14) is not satisfied then the claim of Lemma 3.2 may fail. Indeed, the example [16, Remark 4.5] applies also to the incompressible case.

In the next auxiliary lemma we provide some estimates for the Lagrangian flow

$$\begin{cases} \dfrac{\partial \mathbf{z}}{\partial t}(t,\mathbf{x}) = \mathbf{v}(\mathbf{z}(t,\mathbf{x})) & t > 0 \\ \\ \mathbf{z}(0,\mathbf{x}) = \mathbf{x} \end{cases} \tag{3.39}$$

associated to a vector field $\mathbf{v} \in C_c^1(\mathbb{R}^3,\mathbb{R}^3)$.

Lemma 3.4. *Let* $\mathbf{v} \in C^1_c(\mathbb{R}^3, \mathbb{R}^3)$ *be satisfying* $\operatorname{div} \mathbf{v} = 0$ *in an open neighborhood* Ω_* *of* $\overline{\Omega}$. *Let* $\mathbf{z} \in C^1([0, +\infty) \times \mathbb{R}^3; \mathbb{R}^3)$ *be the unique global solution to* (3.39). *Then there exists* $T > 0$ *such that*

$$\mathbf{z}(t, \mathbf{x}) \in \Omega_* \quad \textit{and} \quad \det \nabla \mathbf{z}(t, \mathbf{x}) = 1 \qquad \forall\, t \in [0, T] \quad \forall \mathbf{x} \in \overline{\Omega}. \tag{3.40}$$

Moreover

$$\sup_{\mathbf{x} \in \overline{\Omega}} |\mathbf{z}(t, \mathbf{x}) - \mathbf{x}| \leq t \|\mathbf{v}\|_\infty \, \exp(t \|\nabla \mathbf{v}\|_\infty) \qquad \forall\, t \geq 0, \tag{3.41}$$

$$\sup_{\mathbf{x} \in \overline{\Omega}} |t^{-1}(\mathbf{z}(t, \mathbf{x}) - \mathbf{x}) - \mathbf{v}(\mathbf{x})| \leq \|\mathbf{v}\|_{L^\infty} (\exp(t \|\nabla \mathbf{v}\|_{L^\infty}) - 1) \qquad \forall\, t > 0, \tag{3.42}$$

$$\sup_{\mathbf{x} \in \overline{\Omega}} |\nabla \mathbf{z}(t, \mathbf{x})| \leq 3 \exp(t \|\nabla v\|_\infty) \qquad \forall\, t \geq 0, \tag{3.43}$$

$$\sup_{\mathbf{x} \in \overline{\Omega}} |(\nabla \mathbf{z}(t, \mathbf{x}) - \mathbf{I})| \leq 3(\exp(t \|\nabla \mathbf{v}\|_{L^\infty}) - 1) \qquad \forall\, t \geq 0, \tag{3.44}$$

where $\nabla \mathbf{z}$ *denotes the gradient of* $\mathbf{z}$ *with respect to the* $\mathbf{x}$ *variable.*

Proof. Since $\mathbf{v} \in C^1_c(\mathbb{R}^3, \mathbb{R}^3)$, (3.39) has a unique global solution $\mathbf{z} \in C^1([0, +\infty) \times \mathbb{R}^3; \mathbb{R}^3)$. By integrating (3.39) we have

$$(\mathbf{z}(t, \mathbf{x}) - \mathbf{x}) - t\mathbf{v}(\mathbf{x}) = \int_0^t (\mathbf{v}(\mathbf{z}(s, \mathbf{x})) - \mathbf{v}(\mathbf{x}))\, ds \tag{3.45}$$

for any $\mathbf{x} \in \overline{\Omega}$ and for every $t \geq 0$, whence

$$|\mathbf{z}(t, \mathbf{x}) - \mathbf{x}| \leq t|\mathbf{v}(\mathbf{x})| + \|\nabla \mathbf{v}\|_\infty \int_0^t |\mathbf{z}(s, \mathbf{x}) - \mathbf{x}|\, ds$$

and Gronwall lemma entails

$$|\mathbf{z}(t, \mathbf{x}) - \mathbf{x}| \leq t|\mathbf{v}(\mathbf{x})| \exp(t \|\nabla \mathbf{v}\|_\infty) \leq t \|\mathbf{v}\|_\infty \exp(t \|\nabla \mathbf{v}\|_\infty). \tag{3.46}$$

Therefore there exists $T > 0$ such that $\mathbf{z}(t, \mathbf{x}) \in \Omega_*$ for every $(t, \mathbf{x}) \in [0, T] \times \overline{\Omega}$. Moreover, (3.40) follows by a standard argument, since $\mathbf{v}$ is divergence free in Ω_*, see also [18, Lemma 4.1]. By exploiting (3.45) and (3.46) we get

$$\begin{aligned}\left|t^{-1}(\mathbf{z}(t, \mathbf{x}) - \mathbf{x}) - \mathbf{v}(\mathbf{x})\right| &\leq \|\mathbf{v}\|_\infty \int_0^t s^{-1}\ |\mathbf{z}(s, \mathbf{x}) - \mathbf{x}|\ ds \\ &\leq \|\mathbf{v}\|_\infty \int_0^t \exp(s \|\nabla \mathbf{v}\|_\infty)\, ds \leq \|\mathbf{v}\|_\infty (\exp(t \|\nabla \mathbf{v}\|_\infty) - 1)\end{aligned}$$

for any $\mathbf{x} \in \overline{\Omega}$ and any $t \in (0,T]$ thus proving (3.42). Letting ∇ denote the derivative in the $\mathbf{x}$ variable we see that $\mathbf{Z}(t,\mathbf{x}) := \nabla \mathbf{z}(t,\mathbf{x})$ satisfies

$$\begin{cases} \dfrac{\partial \mathbf{Z}}{\partial t}(t,\mathbf{x}) = \nabla \mathbf{v}(\mathbf{z}(t,\mathbf{x}))\mathbf{Z}(t,\mathbf{x}) \qquad t > 0 \\ \\ \mathbf{Z}(0,\mathbf{x}) = \mathbf{I}, \end{cases}$$

whence

$$\mathbf{Z}(t,\mathbf{x}) - \mathbf{I} = \int_0^t \nabla \mathbf{v}(\mathbf{z}(s,\mathbf{x}))\, \mathbf{Z}(s,\mathbf{x})\, ds \tag{3.47}$$

for every $\mathbf{x} \in \overline{\Omega}$ and every $t \geq 0$, therefore

$$|\mathbf{Z}(t,\mathbf{x})| \leq 3 + \|\nabla \mathbf{v}\|_\infty \int_0^t |\mathbf{Z}(s,\mathbf{x})|\, ds$$

and by Gronwall Lemma

$$|\mathbf{Z}(t,\mathbf{x})| \leq 3 \exp(t \|\nabla \mathbf{v}\|_\infty).$$

Therefore for every $t \in (0,T]$ and every $\mathbf{x} \in \overline{\Omega}$ we get

$$\begin{aligned} |\mathbf{Z}(t,\mathbf{x}) - \mathbf{I}| &\leq \int_0^t |\nabla \mathbf{v}(\mathbf{z}(s,\mathbf{x}))|\, |\mathbf{Z}(s,\mathbf{x})|\, ds \leq \|\nabla \mathbf{v}\|_\infty \int_0^t |\mathbf{Z}(s,\mathbf{x})|\, ds \\ &\leq 3 \|\nabla \mathbf{v}\|_\infty \int_0^t \exp(s\|\nabla \mathbf{v}\|_\infty)\, ds \leq\ 3(\exp(t\|\nabla \mathbf{v}\|_\infty) - 1) \end{aligned}$$

thus proving (3.44). QED

Lemma 3.5. (Upper bound) *Assume that $\partial\Omega$ has a finite number of connected components, that* (2.5), (2.7), (2.8), (2.9), (2.10), (2.11), (2.14), (2.15) *hold true and that $\mathcal{L}(\mathbf{e}_3) < 0$. Let $p > 3$. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that* $\operatorname{div} \mathbf{u} = 0$ *a.e. in Ω. Then there exists a sequence $(\mathbf{y}_j)_{j\in\mathbb{N}} \subset C^1(\overline{\Omega}, \mathbb{R}^3)$ such that*

$$\limsup_{j\to+\infty} \mathcal{G}_j^I(\mathbf{y}_j) \leq \widetilde{\mathcal{G}}^I(\mathbf{u}).$$

Proof. **Step 1.** We assume without loss of generality that $\mathbf{u} \in \mathcal{A}$ and we let

$$\mathbf{b}^* \in \operatorname{argmin} \left\{ \int_\Omega \mathcal{Q}(\mathbf{x}, \mathbb{E}(\mathbf{u}) + \frac{1}{2} b_\alpha (\mathbf{e}_\alpha \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\alpha))\, d\mathbf{x} : \mathbf{b} \in \mathbb{R}^2 \right\},$$

$$\widetilde{\mathbf{u}}(\mathbf{x}) := \mathbf{u}(\mathbf{x}) + x_3(b_1^* \mathbf{e}_1 + b_2^* \mathbf{e}_2). \tag{3.48}$$

It is readily seen that $\widetilde{\mathbf{u}} \in \mathcal{A}$, that $\operatorname{div} \widetilde{\mathbf{u}} = 0$ a.e. in Ω and that $\mathbb{E}(\widetilde{\mathbf{u}}) = \mathbb{E}(\mathbf{u}) + \frac{1}{2} b_\alpha^* (\mathbf{e}_\alpha \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\alpha)$, hence

$$\mathcal{I}(\mathbf{u}) = \int_\Omega \mathcal{Q}(\mathbf{x}, \mathbb{E}(\widetilde{\mathbf{u}}))\, d\mathbf{x}. \tag{3.49}$$

Moreover, by Lemma 2.2 and Remark 2.3 we obtain

$$\mathcal{L}(\mathbf{R}\widetilde{\mathbf{u}}) = \mathcal{L}(\mathbf{R}\mathbf{u}) + \mathcal{L}(x_3(b_1^* \mathbf{R}\mathbf{e}_1 + b_2^* \mathbf{R}\mathbf{e}_2)) = \mathcal{L}(\mathbf{R}\mathbf{u}) \qquad \forall\, \mathbf{R} \in \mathcal{S}_{\mathcal{L},E}. \tag{3.50}$$

Therefore by choosing

$$\widetilde{\mathbf{R}} \in \operatorname{argmin} \{ -\mathcal{L}(\mathbf{R}\widetilde{\mathbf{u}}) : \mathbf{R} \in \mathcal{S}_{\mathcal{L},E} \}$$

we get

$$\widetilde{\mathcal{G}}^I(\mathbf{u}) = \int_\Omega \mathcal{Q}^I(\mathbf{x}, \mathbb{E}(\widetilde{\mathbf{u}}))\, d\mathbf{x} - \mathcal{L}(\widetilde{\mathbf{R}}\widetilde{\mathbf{u}}). \tag{3.51}$$

Since $\partial\Omega$ has a finite number of connected components and $\operatorname{div} \widetilde{\mathbf{u}} = 0$ a.e. in Ω then by [10, Proposition 3.1, Corollary 3.2] there exists a Sobolev extension of $\widetilde{\mathbf{u}}$, still denoted with $\widetilde{\mathbf{u}} \in W^{1,p}(\mathbb{R}^3;\mathbb{R}^3)$, such that $\operatorname{spt} \widetilde{\mathbf{u}}$ is compact and $\operatorname{div} \widetilde{\mathbf{u}} = 0$ in an open neighborhood Ω' of $\overline{\Omega}$. Since $\widetilde{\mathbf{u}} \in W^{1,p}(\mathbb{R}^3;\mathbb{R}^3)$ with $p > 3$ and $\operatorname{spt} \widetilde{\mathbf{u}}$ is compact then $\widetilde{\mathbf{u}} \in C^{0,\gamma}(\mathbb{R}^3;\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with $\gamma = 1 - 3/p$ and we denote with

$$\|\widetilde{\mathbf{u}}\|_{0,\gamma} := \sup_{\mathbb{R}^3} |\widetilde{\mathbf{u}}| + \sup_{\substack{\mathbf{x} \in \mathbb{R}^3, \mathbf{y} \in \mathbb{R}^3 \\ \mathbf{x} \neq \mathbf{y}}} \frac{|\widetilde{\mathbf{u}}(\mathbf{x}) - \widetilde{\mathbf{u}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\gamma}$$

its Hölder norm. Let now, for every $j \in \mathbb{N}$, $\widetilde{\mathbf{u}}_j := \widetilde{\mathbf{u}} * \rho_j$, where $\rho_j(\mathbf{x}) := \varepsilon_j^{-3} \rho(\varepsilon_j^{-1} |\mathbf{x}|)$ and ρ is the unit symmetric mollifier, so that $\operatorname{spt} \rho \subset B_1(0)$, and

$$\varepsilon_j := h_j^{\gamma/2}.$$

It is well known that $\widetilde{\mathbf{u}}_j \in C_c^1(\mathbb{R}^3;\mathbb{R}^3)$, that $\widetilde{\mathbf{u}}_j \to \widetilde{\mathbf{u}}$ in $W^{1,p}(\mathbb{R}^3;\mathbb{R}^3)$, that $\|\widetilde{\mathbf{u}}_j\|_{L^\infty} \le \|\widetilde{\mathbf{u}}\|_{L^\infty}$ and that

$$\|\nabla \widetilde{\mathbf{u}}_j\|_{L^\infty} \le \|\widetilde{\mathbf{u}}\|_{L^\infty} \int_{B_{\varepsilon_j}(0)} |\nabla \rho_j(\mathbf{y})|\, d\mathbf{y} = K \varepsilon_j^{-1} \|\widetilde{\mathbf{u}}\|_{L^\infty}, \tag{3.52}$$

where we have set

$$K := 4\pi \int_0^1 |\rho'(r)| r^2\, dr.$$

Moreover it is readily seen that $\operatorname{div}\widetilde{\mathbf{u}}_j = 0$ in an open neighbourhood Ω_* of $\overline{\Omega}$ such that $\overline{\Omega_*} \subset \Omega'$ and that for every $\mathbf{x}, \mathbf{q} \in \mathbb{R}^3$ we have

$$\begin{aligned} |\widetilde{\mathbf{u}}_j(\mathbf{x}) - \widetilde{\mathbf{u}}(\mathbf{x})| &\le \int_{B_{\varepsilon_j}(0)} |\widetilde{\mathbf{u}}(\mathbf{x}-\mathbf{y}) - \widetilde{\mathbf{u}}(\mathbf{x})|\rho_j(\mathbf{y})\,d\mathbf{y} \\ &\le \|\widetilde{\mathbf{u}}\|_{0,\gamma} \int_{B_{\varepsilon_j}(0)} |\mathbf{y}|^\gamma |\rho_j(\mathbf{y})\,d\mathbf{y} \le \varepsilon_j^\gamma \|\widetilde{\mathbf{u}}\|_{0,\gamma}, \end{aligned} \tag{3.53}$$

$$\begin{aligned} |\nabla\widetilde{\mathbf{u}}_j(\mathbf{x}) - \nabla\widetilde{\mathbf{u}}_j(\mathbf{q})| &\le \int_{B_{\varepsilon_j}(0)} |\widetilde{\mathbf{u}}(\mathbf{x}-\mathbf{y}) - \widetilde{\mathbf{u}}(\mathbf{q}-\mathbf{y})||\nabla\rho_j(\mathbf{y})|\,d\mathbf{y} \\ &\le \|\widetilde{\mathbf{u}}\|_{0,\gamma}|\mathbf{x}-\mathbf{q}|^\gamma \int_{B_{\varepsilon_j}(0)} |\nabla\rho_j(\mathbf{y})|\,d\mathbf{y} = K\varepsilon_j^{-1}\|\widetilde{\mathbf{u}}\|_{0,\gamma}|\mathbf{x}-\mathbf{q}|^\gamma. \end{aligned} \tag{3.54}$$

Step 2. Let now $\mathbf{z}_j$ be the Lagrangian flow associated to the vector field $\widetilde{\mathbf{u}}_j$ as in Lemma 3.4. Let

$$\mathbf{w}_j(\mathbf{x}) := \mathbf{z}_j(h_j, \mathbf{x}), \qquad \mathbf{y}_j(\mathbf{x}) := \widetilde{\mathbf{R}}\mathbf{w}_j(\mathbf{x}) + \beta_j \mathbf{e}_3, \tag{3.55}$$

where

$$\beta_j = \beta_j(\|\widetilde{\mathbf{u}}\|_{0,\gamma}) := h_j\|\widetilde{\mathbf{u}}\|_{0,\gamma}\left(\varepsilon_j^\gamma + \exp\left(Kh_j\varepsilon_j^{-1}\|\widetilde{\mathbf{u}}\|_{0,\gamma}\right) - 1\right).$$

By (3.41) and by (3.52) we obtain

$$\sup_{\mathbf{x}\in\overline{\Omega}} |\mathbf{z}_j(h_j,\mathbf{x}) - \mathbf{x}| \le h_j\|\widetilde{\mathbf{u}}\|_\infty \exp\{K\|\widetilde{\mathbf{u}}\|_\infty h_j\varepsilon_j^{-1}\} \quad \text{for every } j,$$

where the right hand side goes to zero as $j \to +\infty$ since $h_j\varepsilon_j^{-1} \to 0$, so that for every large enough j we obtain

$$\mathbf{z}_j(h_j, \mathbf{x}) \in \Omega_* \quad \text{for every } \mathbf{x} \in \overline{\Omega}$$

and, as in the proof of Lemma 3.4,

$$\det\nabla\mathbf{w}_j(\mathbf{x}) = \det\nabla\mathbf{y}_j(\mathbf{x}) = 1 \quad \text{for every } \mathbf{x} \in \overline{\Omega}.$$

Moreover, still referring to the proof of Lemma 3.4, and in particular to (3.47), we obtain

$$\begin{aligned} \left|h_j^{-1}(\nabla\mathbf{z}_j(h_j,\mathbf{x}) - \mathbf{I}) - \nabla\widetilde{\mathbf{u}}_j(\mathbf{x})\right| &\le h_j^{-1}\int_0^{h_j} |\nabla\widetilde{\mathbf{u}}_j(\mathbf{x})\nabla\mathbf{z}_j(s,\mathbf{x}) - \nabla\widetilde{\mathbf{u}}_j(\mathbf{x})|\,ds \\ &+ h_j^{-1}\int_0^{h_j} |\nabla\widetilde{\mathbf{u}}_j(\mathbf{z}_j(s,\mathbf{x}))\nabla\mathbf{z}_j(s,\mathbf{x}) - \nabla\widetilde{\mathbf{u}}_j(\mathbf{x})\nabla\mathbf{z}_j(s,\mathbf{x}))|\,ds \\ &=: M_j(\mathbf{x}) + N_j(\mathbf{x}) \end{aligned} \tag{3.56}$$

for every $\mathbf{x} \in \overline{\Omega}$ and we next estimate both terms: from (3.44) and (3.52) we get

$$\begin{aligned} cM_j(\mathbf{x}) &\leq h_j^{-1}\|\nabla\widetilde{\mathbf{u}}_j\|_\infty \int_0^{h_j} |\nabla\mathbf{z}_j(s,\mathbf{x}) - \mathbf{I}|\,ds \\ &\leq h_j^{-1}\|\nabla\widetilde{\mathbf{u}}_j\|_\infty \int_0^{h_j} 3\left(\exp(s\|\nabla\widetilde{\mathbf{u}}_j\|_\infty) - 1\right)\,ds \\ &\leq 3K\varepsilon_j^{-1}\|\widetilde{\mathbf{u}}\|_\infty \Big(\exp(Kh_j\varepsilon_j^{-1}\|\widetilde{\mathbf{u}}\|_\infty) - 1\Big) \end{aligned} \tag{3.57}$$

while (3.54), (3.41) and (3.43) entail

$$\begin{aligned} N_j(\mathbf{x}) &\leq h_j^{-1}\int_0^{h_j} |\nabla\mathbf{z}_j(s,\mathbf{x})||\nabla\widetilde{\mathbf{u}}_j(\mathbf{z}_j(s,\mathbf{x})) - \nabla\widetilde{\mathbf{u}}_j(\mathbf{x})|\,ds \\ &\leq 3K\varepsilon_j^{-1}\|\widetilde{\mathbf{u}}\|_{0,\gamma}h_j^{-1}\int_0^{h_j} \exp(s\|\nabla\widetilde{\mathbf{u}}_j\|_\infty)\,|\mathbf{z}_j(s,\mathbf{x}) - \mathbf{x}|^\gamma\,ds \\ &\leq 3K\varepsilon_j^{-1}\|\widetilde{\mathbf{u}}\|_{0,\gamma}\|\widetilde{\mathbf{u}}\|_\infty^\gamma h_j^{-1}\int_0^{h_j} s^\gamma\,\exp(s(1+\gamma)\|\nabla\widetilde{\mathbf{u}}_j\|_\infty)\,ds \\ &\leq 3K\|\widetilde{\mathbf{u}}\|_{0,\gamma}^{1+\gamma}h_j^\gamma\varepsilon_j^{-1}\,\exp\Big(K(1+\gamma)h_j\varepsilon_j^{-1}\|\widetilde{\mathbf{u}}\|_\infty\Big). \end{aligned}$$

Since $\varepsilon_j = h_j^{\gamma/2}$ and $\gamma \in (0,1)$, we se that both $\sup_{\mathbf{x}\in\overline{\Omega}} M_j(\mathbf{x})$ and $\sup_{\mathbf{x}\in\overline{\Omega}} N_j(\mathbf{x})$ vanish as $j \to +\infty$, so that by setting $\widetilde{\mathbf{v}}_j(\mathbf{x}) := h_j^{-1}(\mathbf{z}_j(h_j,\mathbf{x}) - \mathbf{x})$, from (3.56) we get $\nabla\widetilde{\mathbf{v}}_j - \nabla\widetilde{\mathbf{u}}_j \to 0$ in $L^\infty(\Omega;\mathbb{R}^{3\times3})$, thus $\nabla\widetilde{\mathbf{v}}_j \to \nabla\widetilde{\mathbf{u}}$ in $L^p(\Omega;\mathbb{R}^{3\times3})$, and by taking (3.42) into account we have $\widetilde{\mathbf{v}}_j \to \widetilde{\mathbf{u}}$ in $L^\infty(\Omega;\mathbb{R}^3)$ so that

$$\widetilde{\mathbf{v}}_j \to \widetilde{\mathbf{u}} \qquad \text{in } W^{1,p}(\Omega;\mathbb{R}^3).$$

We also have $h_j\|\nabla\widetilde{\mathbf{v}}_j\|_\infty \to 0$, due to (3.52), since $h_j\varepsilon_j^{-1} \to 0$.

Step 3. We check that $\mathbf{y}_j \in \mathcal{A}$. By taking account that $\widetilde{\mathbf{R}} \in \mathcal{S}_{\mathcal{L},E} \subset \{\mathbf{R} \in SO(3) : \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3\}$, along with (3.42) and (3.52)-(3.53), we have from (3.55)

$$\begin{aligned} y_{j,3} - x_3 &= w_{j,3} - x_3 + \beta_j \geq h_j\widetilde{u}_{j,3} - h_j\|\widetilde{\mathbf{u}}\|_\infty(\exp(h_j\|\nabla\widetilde{\mathbf{u}}_j\|_\infty) - 1) + \beta_j \\ &= h_j(\widetilde{u}_{j,3} - \widetilde{u}_3) - h_j\|\widetilde{\mathbf{u}}\|_\infty(\exp(h_j\|\nabla\widetilde{\mathbf{u}}_j\|_\infty) - 1) + h_j\widetilde{u}_3 + \beta_j \\ &\geq -h_j\varepsilon_j^\gamma\|\widetilde{\mathbf{u}}\|_{0,\gamma} - h_j\|\widetilde{\mathbf{u}}\|_{0,\gamma}(\exp(Kh_j\varepsilon_j^{-1}\|\widetilde{\mathbf{u}}\|_{0,\gamma}) - 1) + h_j\widetilde{u}_3 + \beta_j = h_j\widetilde{u}_3 \end{aligned}$$

for every j, where the last equality follows from the definition of β_j. Hence, by recalling that $\widetilde{u}_3^*(\mathbf{x}) \geq 0$ for q.e. $\mathbf{x} \in E \subset \{x_3 = 0\}$, we deduce $y_{j,3}^* \geq 0$ for q.e. $\mathbf{x} \in E$, that is $\mathbf{y}_j \in \mathcal{A}$ for every j.

Step 4. We conclude by noticing that Lemma 2.6 entails $\mathcal{L}(\widetilde{\mathbf{R}}\mathbf{x} - \mathbf{x}) = 0$, whence

$$\begin{aligned} h_j^{-1}\mathcal{L}(\mathbf{y}_j - \mathbf{x}) &= h_j^{-1}\mathcal{L}(\widetilde{\mathbf{R}}\mathbf{w}_j - \mathbf{x}) + h_j^{-1}\beta_j\mathcal{L}(\mathbf{e}_3) \\ &= h_j^{-1}\mathcal{L}(\widetilde{\mathbf{R}}\mathbf{z}_j(h_j,\mathbf{x}) - \mathbf{x}) + o(1) = \mathcal{L}(\widetilde{\mathbf{R}}\widetilde{\mathbf{v}}_j) + o(1) \quad \text{as } j \to +\infty, \end{aligned}$$

where we have used the fact that $h_j^{-1}\beta_j = o(1)$ as $j \to +\infty$, which follows from the definitions of ε_j and β_j. So by exploiting that $h_j\|\nabla\widetilde{\mathbf{v}}_j\|_\infty \to 0$ along with (2.12), by taking account that for large enough j there holds $\det\nabla\mathbf{y}_j = \det(\mathbf{I} + h_j\nabla\widetilde{\mathbf{v}}_j) = 1$ in Ω and that $\operatorname{div}\widetilde{\mathbf{u}} = 0$ a.e. in Ω, we get

$$\begin{aligned}\limsup_{j\to+\infty}\mathcal{G}_j^I(\mathbf{y}_j) &\le \limsup_{j\to+\infty}\int_\Omega {h_j}^{-2}\left(\mathcal{W}(\mathbf{x},\mathbf{I}+h_j\nabla\widetilde{\mathbf{v}}_j) - \mathcal{Q}(\mathbf{x},\mathbb{E}(h_j\widetilde{\mathbf{v}}_j))\right)d\mathbf{x}\\ &\quad + \limsup_{j\to+\infty}\left(h_j^{-2}\int_\Omega \mathcal{Q}(\mathbf{x},\mathbb{E}(h_j\widetilde{\mathbf{v}}_j))\,d\mathbf{x} - h_j^{-1}\mathcal{L}(\mathbf{y}_j-\mathbf{x})\right)\\ &\le \limsup_{j\to+\infty}\int_\Omega \omega(h_j|\nabla\widetilde{\mathbf{v}}_j|)|\nabla\widetilde{\mathbf{v}}_j|^2\,d\mathbf{x}\\ &\quad + \limsup_{j\to+\infty}\int_\Omega \mathcal{Q}(\mathbf{x},\mathbb{E}(\widetilde{\mathbf{v}}_j))\,d\mathbf{x} - \mathcal{L}(\widetilde{\mathbf{R}}\widetilde{\mathbf{u}})\\ &= \int_\Omega \mathcal{Q}^I(\mathbf{x},\mathbb{E}(\widetilde{\mathbf{u}}))\,d\mathbf{x} - \mathcal{L}(\widetilde{\mathbf{R}}\widetilde{\mathbf{u}}),\end{aligned}$$

which proves the result in view of (3.51). *QED*

We are now in a position to prove our main theorem.

Proof of Theorem 2.8. If $(\overline{\mathbf{y}}_j)_{j\in\mathbb{N}} \subset H^1(\Omega,\mathbb{R}^3)$ is a minimizing sequence for $\mathcal{G}_j^I$ then we may assume that $\mathcal{G}_j^I(\overline{\mathbf{y}}_j) \le \mathcal{G}_j^I(\mathbf{x}) + 1 = 1$ for every j. Moreover, if $\mathbf{R}_j$ belong to $\mathcal{A}(\overline{\mathbf{y}}_j)$ and $\overline{\mathbf{c}}_j$ is defined by the right hand sides of (3.21), (3.22), then Lemma 3.1 entails that the sequence

$$\overline{\mathbf{u}}_j(\mathbf{x}) := {h_j}^{-1}\mathbf{R}_j^T\left\{\left(\overline{\mathbf{y}}_j - \mathbf{R}_j\mathbf{x} - \overline{\mathbf{c}}_j\right)_\alpha\mathbf{e}_\alpha + (\overline{y}_{j,3} - x_3)\mathbf{e}_3\right\}$$

is bounded in $H^1(\Omega;\mathbb{R}^3)$. Therefore up to subsequences $\overline{\mathbf{u}}_j \to \overline{\mathbf{u}}$ weakly in $H^1(\Omega;\mathbb{R}^3)$, so, by Lemma 3.2, we have $\overline{\mathbf{u}} \in \mathcal{A}$, $\operatorname{div}\overline{\mathbf{u}} = 0$ a.e. in Ω and

$$\liminf_{j\to+\infty}\mathcal{G}_j^I(\overline{\mathbf{y}}_j) \ge \widetilde{\mathcal{G}}^I(\overline{\mathbf{u}}). \tag{3.58}$$

On the other hand, by Lemma 3.5, for every $\mathbf{u}_* \in W^{1,p}(\Omega,\mathbb{R}^3)\cap\mathcal{A}$ with $p > 3$, there exists a sequence $\mathbf{y}_j \in C^1(\overline{\Omega},\mathbb{R}^3)$ such that

$$\limsup_{j\to+\infty}\mathcal{G}_j^I(\mathbf{y}_j) \le \widetilde{\mathcal{G}}^I(\mathbf{u}_*). \tag{3.59}$$

Since

$$\mathcal{G}_j^I(\overline{\mathbf{y}}_j) + o(1) = \inf_{H^1(\Omega,\mathbb{R}^3)}\mathcal{G}_j^I \le \mathcal{G}_j^I(\mathbf{y}_j) \qquad \text{as } j \to +\infty, \tag{3.60}$$

by passing to the limit as $j \to +\infty$, we get

$$\widetilde{\mathcal{G}}^I(\overline{\mathbf{u}}) \le \widetilde{\mathcal{G}}^I(\mathbf{u}_*) \qquad \text{for every } \mathbf{u}_* \in W^{1,p}(\Omega,\mathbb{R}^3)\cap\mathcal{A} \text{ with } p > 3. \tag{3.61}$$

Now fix $\mathbf{u} \in \mathcal{A}$ such that $\operatorname{div} \mathbf{u} = 0$ a.e. in Ω and denote again by $\mathbf{u}$ a Sobolev extension of $\mathbf{u}$ to the whole $\mathbb{R}^3$. By [16, Lemma A.11] there exists $\mathbf{v}_j \in C^1(\mathbb{R}^3, \mathbb{R}^3) \cap \mathcal{A}$ such that $\mathbf{v}_j \to \mathbf{u}$ in $H^1(\Omega; \mathbb{R}^3)$ and by Bogowskii's Theorem [3], see also [9, Theorem A.2], there exists $\mathbf{w}_j$ belonging to $W_0^{1,r}(\Omega; \mathbb{R}^3)$ for every $r \geq 1$ such that

$$\operatorname{div} \mathbf{w}_j = -\operatorname{div} \mathbf{v}_j + |\Omega|^{-1} \int_\Omega \operatorname{div} \mathbf{v}_j \, d\mathbf{x} \quad \text{in } \Omega, \tag{3.62}$$

and

$$\|\mathbf{w}_j\|_{H^1(\Omega;\mathbb{R}^3)} \leq C_* \left\| -\operatorname{div} \mathbf{v}_j + |\Omega|^{-1} \int_\Omega \operatorname{div} \mathbf{v}_j \, d\mathbf{x} \right\|_{L^2(\Omega)} \tag{3.63}$$

for some suitable constant $C_* = C_*(\Omega)$. Therefore by setting

$$\mathbf{u}_j := \mathbf{v}_j - \left(|\Omega|^{-1} \int_\Omega \operatorname{div} \mathbf{v}_j \, d\mathbf{x} \right) x_3 \mathbf{e}_3 + \mathbf{w}_j$$

and by taking account of (3.62) we get $\operatorname{div} \mathbf{u}_j = 0$ a.e. in Ω. Moreover since $E \subset \partial\Omega \cap \{x_3 = 0\}$ and $\mathbf{u}_j,\ \mathbf{v}_j \in C^0(\overline{\Omega}; \mathbb{R}^3)$ we have $\mathbf{u}_j(\mathbf{x}) = \mathbf{v}_j(\mathbf{x})$ for every $\mathbf{x} \in E$, hence $\mathbf{u}_j \in \mathcal{A}$. Eventually by (3.63) we get $\mathbf{w}_j \to 0$ in $H^1(\Omega; \mathbb{R}^3)$. By recalling that $\mathbf{v}_j \to \mathbf{u}$ in $H^1(\Omega; \mathbb{R}^3)$ and that $\operatorname{div} \mathbf{v}_j \to 0$ in $L^2(\Omega)$, we have

$$\|\mathbf{u}_j - \mathbf{u}\|_{H^1(\Omega;\mathbb{R}^3)} \leq \left\| \mathbf{v}_j - \mathbf{u} - \left(|\Omega|^{-1} \int_\Omega \operatorname{div} \mathbf{v}_j \, d\mathbf{x} \right) x_3 \mathbf{e}_3 \right\|_{H^1(\Omega;\mathbb{R}^3)} + \|\mathbf{w}_j\|_{H^1(\Omega;\mathbb{R}^3)} \to 0.$$

By (3.61) we have $\widetilde{\mathcal{G}}^I(\overline{\mathbf{u}}) \leq \widetilde{\mathcal{G}}^I(\mathbf{u}_j)$, whence by Remark 2.7 we have

$$\widetilde{\mathcal{G}}^I(\overline{\mathbf{u}}) \leq \lim_{j \to +\infty} \widetilde{\mathcal{G}}^I(\mathbf{u}_j) = \widetilde{\mathcal{G}}^I(\mathbf{u}).$$

The arbitrariness of $\mathbf{u} \in \mathcal{A}$ shows that $\overline{\mathbf{u}} \in \operatorname{argmin}_{H^1(\Omega;\mathbb{R}^3)} \widetilde{\mathcal{G}}^I$.

We claim that $\mathcal{G}_j^I(\overline{\mathbf{y}}_j) \to \widetilde{\mathcal{G}}^I(\overline{\mathbf{u}})$: indeed, by the previous arguments, if $p > 3$ we have $\min_{H^1(\Omega;\mathbb{R}^3)} \widetilde{\mathcal{G}}^I = \inf_{W^{1,p}(\Omega;\mathbb{R}^3)} \widetilde{\mathcal{G}}^I$. Since (3.58), (3.59), (3.60) and (3.61) imply

$$\min_{H^1(\Omega;\mathbb{R}^3)} \widetilde{\mathcal{G}}^I = \widetilde{\mathcal{G}}^I(\overline{\mathbf{u}}) \leq \liminf_{j \to +\infty} \widetilde{\mathcal{G}}_j^I(\overline{\mathbf{y}}_j) \leq \limsup_{j \to +\infty} \widetilde{\mathcal{G}}_j^I(\overline{\mathbf{y}}_j) \leq \widetilde{\mathcal{G}}^I(\mathbf{u}_*)$$

for every $\mathbf{u}_* \in \mathcal{A}$ such that $\mathbf{u}_* \in W^{1,p}(\Omega; \mathbb{R}^3)$, the claim follows.

We are only left to show that $\min_{H^1(\Omega;\mathbb{R}^3)} \widetilde{\mathcal{G}}^I = \min_{H^1(\Omega;\mathbb{R}^3)} \mathcal{G}^I$. To this aim we show first that for every $\mathbf{u} \in \mathcal{A}$ there exists $\widetilde{\mathbf{u}} \in \mathcal{A}$ such that $\mathcal{G}^I(\widetilde{\mathbf{u}}) = \widetilde{\mathcal{G}}^I(\mathbf{u})$.

Indeed if $\widetilde{\mathbf{u}}$ is defined as in (3.48) then by (3.49) and (3.50) we get

$$\widetilde{\mathcal{G}}^I(\mathbf{u}) = \mathcal{I}^I(\mathbf{u}) - \max_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{R}\mathbf{u}) = \int_\Omega \mathcal{Q}^I(\mathbf{x}, \mathbb{E}(\widetilde{\mathbf{u}}))\,d\mathbf{x} - \max_{\mathbf{R}\in\mathcal{S}_{\mathcal{L},E}} \mathcal{L}(\mathbf{R}\widetilde{\mathbf{u}}) = \mathcal{G}^I(\widetilde{\mathbf{u}}) \tag{3.64}$$

as claimed. By recalling that $\widetilde{\mathcal{G}}^I(\overline{\mathbf{u}}) = \min_{H^1(\Omega;\mathbb{R}^3)} \widetilde{\mathcal{G}}^I \leq \inf_{H^1(\Omega;\mathbb{R}^3)} \mathcal{G}^I$ let us assume that inequality is strict. Then by (3.64) there exists $\widetilde{\overline{\mathbf{u}}} \in \mathcal{A}$ such that $\mathcal{G}^I(\widetilde{\overline{\mathbf{u}}}) = \widetilde{\mathcal{G}}^I(\overline{\mathbf{u}}) < \inf_{H^1(\Omega;\mathbb{R}^3)} \mathcal{G}^I$, a contradiction. Thus again by (3.64) $\mathcal{G}^I(\widetilde{\overline{\mathbf{u}}}) = \widetilde{\mathcal{G}}^I(\overline{\mathbf{u}}) = \min_{H^1(\Omega;\mathbb{R}^3)} \mathcal{G}^I$. QED

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