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Generalizations of Baruah and Berndt's Ramanujan-type series for $1/\pi$

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Abstract. We introduce infinite families of generalizations of Ramanujan-type series for $\frac{1}{\pi}$ that had been derived using Eisenstein series identities by Baruah and Berndt.

Keywords: Ramanujan-type series, elliptic lambda function, elliptic alpha function, elliptic singular value

MSC 2022 classification: primary 11Y60, secondary 33C75

Introduction

This article is based on new techniques that we apply to generalize series for $\frac{1}{\pi}$ of Ramanujan type that were introduced in a 2010 research contribution by Baruah and Berndt [5]. As in [5], we are to refer to the famous article in which Ramanujan introduced 17 series expansions for $\frac{1}{\pi}$ [20], along with the 1987 text by Borwein and Borwein [12] in which all 17 of Ramanujan's formulas were first proved. We again refer to the article [5] for references apart from [12, 20] concerning Ramanujan's series for $\frac{1}{\pi}$, including the groundbreaking work by Chudnovsky and Chudnovsky in [17]. In contrast to the Eisenstein series-based techniques employed by Baruah and Berndt [5], we introduce results and applications concerning the elliptic lambda function, the elliptic alpha function, and elliptic singular values to determine infinite families of generalizations of Ramanujan-type series that were discovered by Baruah and Berndt in 2010 [5].

A number of Ramanujan's series for $\frac{1}{\pi}$ are of the form

$$\frac{a}{\pi} = \sum_{k=0}^{\infty} z^k \binom{2k}{k}^3 p_1(k) \tag{0.1}$$

for algebraic values a and z and for a linear polynomial $p_1(k)$ in k with algebraic coefficients. For example, the following formula is due to Ramanujan, with reference to Berndt's listing of Ramanujan's series for $\frac{1}{\pi}$ [10, pp. 352–354]:

$$\frac{32}{\pi} = \sum_{k=0}^{\infty} \left(\frac{47 - 21\sqrt{5}}{8192} \right)^k \binom{2k}{k}^3 \left(\left(30 + 42\sqrt{5} \right) k + 5\sqrt{5} - 1 \right).$$

New equalities that are of the form indicated in (0.1) were introduced in [5]. As a natural way of generalizing such equalities, we consider the problem of constructing and proving equations of the form

$$\frac{a}{\pi} = \sum_{k=0}^{\infty} z^k \binom{2k}{k}^3 p_2(c, k) \tag{0.2}$$

such that $p_2(c, k)$ is a polynomial of degree 3 in k with algebraic coefficients such that $p_2(0, k) = p_1(k)$, for a free parameter c. We succeed in our developing a method for constructing generalizations of the form indicated in (0.2). This method is fundamentally different from the Eisenstein series-based techniques involved in Baruah and Berndt's work in [5], and thus has the advantage of providing new proofs of Baruah and Berndt's Ramanujan-type formulas in [5]. So, it is relevant to briefly review preliminaries regarding Baruah and Berndt's work in [5].

Let the Pochhammer symbol be such that $(a)_0 = 1$ and $(a)_n = a(a + 1) \cdots (a + n - 1)$. We are to employ the following notation for generalized hypergeometric series:

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \,\middle|\, x \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}.$$

In the context of the application of Eisenstein series as in [5], the variables q and x are related to one another in the following manner:

$$q = \exp\left(-\pi \frac{{}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, 1 - x \right]}{{}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, x \right]} \right).$$

The expression ϕ may be defined so that:

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{0.3}$$

Following [5], we stress the importance of the formula

$$\phi^2(q) = {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix} \,\middle|\, x\right] \tag{0.4}$$

in the context of the study of Ramanujan-type series. With regard to (0.4), we may adopt Ramanujan's notation indicated below:

$$z := z(q) := \phi^2(x). \tag{0.5}$$

Again with reference to Baruah and Berndt's work in [5], we may let

$$P(q) := 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} \tag{0.6}$$

denote Ramanujan's Eisenstein series. Baruah and Berndt's evaluation strategy in [5] relied heavily on manipulations of and identities involving Ramanujan's Eisenstein series, as in with the identities reproduced in (2.11) and (2.12). In contrast, our methods do not involve the P-function in (0.6), and instead are based on new applications of the special functions denoted as $\mathbf{K}$, $\mathbf{E}$, α, and λ^* and defined below. There is an argument to be made that our methods may be applied in a more versatile way relative to [5], as evidenced by our infinite families of generalizations of Ramanujan-type series from [5].

The identity in (0.4) leads us to consider the application of results on moduli as in (0.5) using the complete elliptic integrals of the first and second kinds. These functions may be defined, respectively, as below:

$$\mathbf{K}(k) = \frac{\pi}{2} \cdot {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix} \,\middle|\, k^2\right], \quad \mathbf{E}(k) = \frac{\pi}{2} \cdot {}_2F_1\left[\begin{matrix}\frac{1}{2}, -\frac{1}{2} \\ 1\end{matrix} \,\middle|\, k^2\right].$$

In this regard, we are to heavily make use of the generating function identity [12, pp. 178–181]

$$\sum_{n=0}^{\infty} \binom{2n}{n}^3 x^n = \frac{4\mathbf{K}^2\left(\frac{\sqrt{1-\sqrt{1-64x}}}{\sqrt{2}}\right)}{\pi^2}, \tag{0.7}$$

which can be shown to hold from Clausen's hypergeometric product identity [4]

$$\left({}_2F_1\left[\begin{matrix}a, b \\ a+b+\frac{1}{2}\end{matrix} \,\middle|\, z\right]\right)^2 = {}_3F_2\left[\begin{matrix}2a, 2b, a+b \\ a+b+\frac{1}{2}, 2a+2b\end{matrix} \,\middle|\, z\right];$$

see also [2, 21, 28] for related and relevant material.

The elliptic alpha function may be defined so that

$$\alpha(r) = \frac{\pi}{4\left(\mathbf{K}\left(k_r\right)\right)^2} + \sqrt{r} - \frac{\mathbf{E}\left(k_r\right)\sqrt{r}}{\mathbf{K}\left(k_r\right)}, \tag{0.8}$$

as in [12, §5]. This special function is to be of key importance in our work. With regard to (0.8), we are using notation for the the elliptic lambda function [12, §5] [13], where expressions of the form k_r are referred to as *elliptic singular values* [12, §4]. Being consistent with [12, §4], we write $\lambda^*(r) = k_r$, letting λ^* denote the elliptic lambda function. This function may be defined so that

$$\lambda^*(r) := k_r = \frac{\vartheta_2^2(0, e^{-\pi\sqrt{r}})}{\vartheta_3^2(0, e^{-\pi\sqrt{r}})}.$$

As above, we are letting the Jacobi theta functions [26, §21.3] be defined as per usual, writing:

$$\vartheta_2(z, q) = 2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)}\cos((2n+1)z), \text{ and}$$
$$\vartheta_3(z, q) = 1 + 2\sum_{n=0}^{\infty} q^{n^2}\cos(2nz). \tag{0.9}$$

Observe that a series rearrangement applied to (0.3) allows us to express ϕ in terms of ϑ_3.

1 Organization of the article

Our article is structured in a way that is based on the sequence of new Ramanujan-type series presented by Baruah and Berndt in [5]. In particular, Sections 2–7 are respectively based on generalizing the Ramanujan-type series evaluations given as follows in [5]: Eq. 4.1 of Theorem 4.1, Eq. 4.3 of Theorem 4.1, Eq. 5.1 of Theorem 5.1, Eq. 5.3 of Theorem 5.1, Eq. 6.1 of Theorem 6.1, and Eq. 6.2 of Theorem 6.1. As we briefly consider in the Conclusion, our methods also apply to subsequent Ramanujan-type series in [5]. However, the formula following Eq. 6.2 of Theorem 6.1 in [5] is the first Ramanujan-type formula in [5] that requires a new evaluation for the elliptic alpha function.

For each section among Sections 2–7, we briefly review the Eisenstein series-based proof for the corresponding Ramanujan-type series in [5]. We then proceed to construct an infinite family of generalizations of the same Ramanujan-type series from [5], using a completely different approach relative to [5]. Also, in the Conclusion, we briefly consider some further areas of research to pursue.

2 Theorem 4.1 (Eq. 4.1) of [5]

The first equation given in Theorem 4.1 in [5] may be rewritten in the following manner:

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}\left(\frac{5\sqrt{2}-7}{8}\right)^{k}\binom{2k}{k}^{3}\left(\left(8-5\sqrt{2}\right)k+3-2\sqrt{2}\right). \tag{2.10}$$

It is stated in [5] that the above identity is new, as of the publication of the article [5]. This motivates the new techniques we apply to both offer a new proof of (2.10) and to introduce an infinite family of generalizations of (2.10).

The formula in (2.10) is proved in the following manner in [5]. Using a result from [11, p. 281], we have that $x_2=(\sqrt{2}-1)^2$, where $x_n := x\big(e^{-\pi\sqrt{n}}\big)$, and where x is a function of q according to (0.4). The following Eisenstein series identity from [9, p. 127] is also used:

$$f_2(q)=2P(q^4)-P(q^2)=z^2(q)\left(1-\frac{x(q)}{2}\right). \tag{2.11}$$

Also, the following identities recorded in [1, §15] are needed in Baruah and Berndt's proof of (2.10):

$$1-x_n=x_{1/n} \quad \text{and} \quad z_{1/n}=\sqrt{n}z_n.$$

This is used to write $f_2(e^{-\pi/\sqrt{2}})$ in terms of z_2^2, and then the identity

$$nP\left(e^{-2\pi\sqrt{n}}\right)+P\left(e^{-2\pi/\sqrt{n}}\right)=\frac{6\sqrt{n}}{\pi} \tag{2.12}$$

is used to rewrite $P\big(e^{-2\pi\sqrt{2}}\big)$ in terms of z_2^2. The remaining argument relies on the power series identity from (0.7).

Theorem 1. *The identity*

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}\left(\frac{5\sqrt{2}-7}{8}\right)^{k}\binom{2k}{k}^{3}p(c,k)$$

holds for a free parameter c, where $p(c,k){:=}\frac{1}{12}k^3(1-5\sqrt{2})c+k^2c+k(8-5\sqrt{2}+\frac{c}{2})+3-2\sqrt{2}+\frac{c}{12}$. The Baruah–Berndt formula in (2.10) *is a special case of the above identity, for $c=0$.*

Proof. Writing $k' = \sqrt{1-k^2}$, according to the differential equations

$$\frac{d\mathbf{E}}{dk} = \frac{1}{k}(\mathbf{E} - \mathbf{K}),$$
$$\frac{d\mathbf{K}}{dk} = \frac{1}{k(k')^2}(\mathbf{E} - (k')^2\mathbf{K}),$$

this gives us the following. For a parameter $n \in \mathbb{N}_0$, the generating function for

$$\left(\binom{2k}{k}^3 k^n : k \in \mathbb{N}_0\right)$$

is always expressible as a finite combination of

$$\mathbf{K}^2\left(\frac{\sqrt{1-\sqrt{1-64x}}}{\sqrt{2}}\right), \tag{2.13}$$

$$\mathbf{E}^2\left(\frac{\sqrt{1-\sqrt{1-64x}}}{\sqrt{2}}\right), \quad \text{and} \tag{2.14}$$

$$\mathbf{E}\left(\frac{\sqrt{1-\sqrt{1-64x}}}{\sqrt{2}}\right)\mathbf{K}\left(\frac{\sqrt{1-\sqrt{1-64x}}}{\sqrt{2}}\right), \tag{2.15}$$

together with π and algebraic expressions. So, the same holds true a power series of the form

$$\sum_{k=0}^{\infty}\binom{2k}{k}^3 x^k(s_0 + s_1k + s_2k^2 + s_3k^3) \tag{2.16}$$

for algebraic scalars s_i. So, by evaluating

$$\sum_{k=0}^{\infty}\left(\frac{5\sqrt{2}-7}{8}\right)^k\binom{2k}{k}^3$$

according to Clausen's product, this gives us a multiple of $\mathbf{K}^2(\sqrt{2}-1)$, and it is known that $\lambda^*(2) = \sqrt{2}-1$ [25]. So, from the known elliptic integral singular value [24]

$$\mathbf{K}\left(\sqrt{2}-1\right) = \frac{\sqrt{1+\sqrt{2}}\,\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{8\sqrt[4]{2}\sqrt{\pi}},$$

we may determine the following. Using the known elliptic alpha function evaluation [12, p. 172]

$$\alpha(2) = \sqrt{2}-1,$$

we may obtain that

$$\mathbf{E}\left(\sqrt{2}-1\right)=\frac{2^{3/4}\pi^{3/2}}{\sqrt{1+\sqrt{2}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}+\frac{\sqrt{\frac{1+\sqrt{2}}{\pi}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{2^{15/4}}.$$

So, according to the identity for writing the power series in (2.16) in terms of (2.13)–(2.15), this, together with the above values for $\mathbf{K}(\lambda^*(2))$ and $\mathbf{E}(\lambda^*(2))$, gives us the desired result, after much simplification, after setting $x=\frac{5\sqrt{2}-7}{8}$ in (2.16) and setting the scalars s_i accordingly. $\boxed{QED}$

Example 1. Setting $c=12\left(2\sqrt{2}-3\right)$, we obtain

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}\left(\frac{5\sqrt{2}-7}{8}\right)^k\binom{2k}{k}^3 k\left(k\left(\left(17\sqrt{2}-23\right)k+24\sqrt{2}-36\right)+7\sqrt{2}-10\right).$$

An interesting aspect about our infinite families of Ramanujan-type series for $\frac{1}{\pi}$ is due to the presence of a free or continuous variable in such results, as in Theorem 1 above. In contrast, for example, in the work of Bagis and Glasser in [2] on the elliptic singular modulus function k_r, identities such as

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3}4^n\left(k_rk_r'\right)^{2n}\left(n+\frac{\alpha(r)-\sqrt{r}k_r^2}{\sqrt{r}(1-2k_r^2)}\right)=\frac{1}{\pi\sqrt{r}(1-2k_r^2)}$$

are given, writing $k_r'=\sqrt{1-k_r^2}$, but such identities depend on known closed forms for k_r, with the rationality of the index r implying that k_r is algebraic. Identities such as

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3}(40\sqrt{2}-56)^n(an+b)=\frac{4a}{7\pi}+\frac{5a}{7\sqrt{2}\pi}+4(-4a+\sqrt{2}a+14b)\frac{\Gamma^2\left(\frac{9}{8}\right)}{7\pi\Gamma^2\left(\frac{5}{8}\right)}$$

are given by Bagis and Glasser in [2] for free parameters a and b, and these kinds of results may also be proved using our methods.

As in our proof of Theorem 1, the elliptic alpha function is to play an important role in our below proofs. It seems that there has not been much research based on the elliptic alpha function; see [12, §5], along with [3, 27]. This motivates our current research endeavours.

3 Theorem 4.1 (Eq. 4.3) of [5]

In Eq. 4.3 of Theorem 4.1 of [5], Baruah and Berndt introduced an equivalent formulation of the following:

$$\frac{2\sqrt{\sqrt{2}-1}}{\pi} = \sum_{k=0}^{\infty} \left(\frac{7-5\sqrt{2}}{512}\right)^k \binom{2k}{k}^3 \left(\left(4\sqrt{2}-2\right)k + \sqrt{2}-1\right). \tag{3.17}$$

This is proved in [5] in a similar way relative to Baruah and Berndt's proof of (2.10).

Theorem 2. *The identity*

$$\frac{2\sqrt{\sqrt{2}-1}}{\pi} = \sum_{k=0}^{\infty} \left(\frac{7-5\sqrt{2}}{512}\right)^k \binom{2k}{k}^3 p(c,k)$$

holds for a free parameter c*, where* $p(c,k) := k^3(38+\frac{80\sqrt{2}}{3})c + k^2c + \frac{1}{2}k(c+8\sqrt{2}-4) + \frac{c}{12} + \sqrt{2}-1$*. The Baruah–Berndt formula in* (3.17) *is a special case of the above identity for* $c=0$*.*

Proof. By taking the above series and removing the summand factor $p(c,k)$, this gives us, according to Clausen's product, a multiple of

$$\mathbf{K}\left(i\,\frac{1}{2}\sqrt{\frac{1}{2}\left(\sqrt{2\left(1+5\sqrt{2}\right)}-4\right)}\right). \tag{3.18}$$

We may make use of the classic elliptic integral identity

$$\mathbf{K}\left(ik/k'\right) = k'\mathbf{K}(k). \tag{3.19}$$

Explicitly, this gives us that (3.18) equals

$$\sqrt{-112-80\sqrt{2}+40\sqrt{1+5\sqrt{2}}+28\sqrt{2\left(1+5\sqrt{2}\right)}}\times$$
$$\mathbf{K}\left(5+4\sqrt{2}-2\sqrt{2\left(7+5\sqrt{2}\right)}\right), \tag{3.20}$$

but the argument of $\mathbf{K}$ in (3.20) is precisely $\lambda^*(8)$ [25]. From the known elliptic singular value [24]

$$\mathbf{K}(\lambda^*(8)) = \frac{1}{16}\sqrt[4]{\frac{1}{2}\left(1+\sqrt{2}\right)}\sqrt{\frac{2\sqrt{2}+\sqrt{1+5\sqrt{2}}}{\pi}}\,\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right),$$

this gives us an explicit evaluation for (3.18). So, we can use the classical identity

$$\mathbf{E}(ik/k') = \frac{1}{k'}\mathbf{E}(k) \tag{3.21}$$

together with the elliptic alpha function to evaluate the expression obtained by replacing $\mathbf{K}$ with $\mathbf{E}$ in (3.18). Explicitly, using the elliptic alpha function value [12, p. 172]

$$\alpha(8) = 2(10+7\sqrt{2})\left(1-\sqrt{\sqrt{8}-2}\right)^2,$$

we obtain that

$$\mathbf{E}\left(5+4\sqrt{2}-2\sqrt{2\left(7+5\sqrt{2}\right)}\right) = \frac{2^{3/4}\pi^{3/2}}{\sqrt[4]{1+\sqrt{2}}\sqrt{2\sqrt{2}+\sqrt{1+5\sqrt{2}}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} + \frac{\sqrt[4]{\frac{1}{2}\left(1+\sqrt{2}\right)}\sqrt{2\sqrt{2}+\sqrt{1+5\sqrt{2}}}\left(1-\frac{\left(10+7\sqrt{2}\right)\left(1-\sqrt{2\sqrt{2}-2}\right)^2}{\sqrt{2}}\right)\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{16\sqrt{\pi}},$$

which, in turn, according to the classical identity in (3.21), gives us the following:

$$\mathbf{E}\left(i\frac{1}{2}\sqrt{\frac{1}{2}\left(\sqrt{2\left(1+5\sqrt{2}\right)}-4\right)}\right) \frac{\pi^{3/2}}{\sqrt{2}\sqrt[4]{1+\sqrt{2}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} + \frac{\sqrt[4]{\frac{1}{2}\left(1+\sqrt{2}\right)}\sqrt{2\sqrt{2}+\sqrt{1+5\sqrt{2}}}\left(1-\frac{\left(10+7\sqrt{2}\right)\left(1-\sqrt{2\sqrt{2}-2}\right)^2}{\sqrt{2}}\right)\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{16\sqrt{\left(-112-80\sqrt{2}+40\sqrt{1+5\sqrt{2}}+28\sqrt{2\left(1+5\sqrt{2}\right)}\right)\pi}}.$$

So, from the above evaluations for values of $\mathbf{E}$ and $\mathbf{K}$ with complex arguments, together with the identity for expressing the power series in (2.16) in terms of (2.13)–(2.15), this effectively gives us the desired result, after much simplification, after setting $x = \frac{7-5\sqrt{2}}{512}$ in (2.16) and setting the scalars s_i accordingly. QED

Example 2. For $c = -12\left(\sqrt{2}-1\right)$, we obtain

$$-\frac{\sqrt{\sqrt{2}-1}}{\pi} =$$
$$\sum_{k=0}^{\infty} \left(\frac{7-5\sqrt{2}}{512}\right)^k \binom{2k}{k}^3 k \left(\left(68\sqrt{2}+92\right)k^2 + 6\left(\sqrt{2}-1\right)k + \sqrt{2}-2\right).$$

Since the Baruah–Berndt formulas in (2.10) and (3.17) are described as new in the 2010 article [5], it is worthwhile to state that relevant research concerning [5] such as [2, 3, 6, 7, 8, 16, 18, 19, 22, 23] does not involve any material that is meaningfully similar relative to our methods.

4 Theorem 5.1 (Eq. 5.1) of [5]

Eq. 5.1 of Theorem 5.1 of [5] may be formulated in an equivalent way as follows:

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \left(\frac{99\sqrt{2}-140}{4}\right)^k \binom{2k}{k}^3 \left(\left(48\sqrt{2}-66\right)k + 20\sqrt{2}-28\right). \tag{4.22}$$

The formula in (4.22) is proved in [5] in the manner roughly outlined as follows.

An identity from [11, p. 284] is such that $x_4 = (\sqrt{2}-1)^4$. The identity (2.11) is applied to obtain that

$$2P(e^{-4\pi}) - \frac{3}{\pi} = z^2(e^{-\pi})\left(1 - \frac{x(e^{-\pi})}{2}\right).$$

The identities $\phi(q) = \sqrt{z}$ and

$$\phi(q^2) = \sqrt{z}\left(\frac{1}{2}(1+\sqrt{1-x})\right)^{1/2}$$

are used to determine that

$$\phi^4(q) = \frac{4}{(1+\sqrt{1-x(q)})^2}\phi^4(q^2).$$

Various other identities involving z_n and (0.6) are then applied in [5] to prove (4.22), in contrast to our brief proof below of an infinite family of generalizations of (4.22).

Theorem 3. *The identity*

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \left(\frac{99\sqrt{2} - 140}{4} \right)^k \binom{2k}{k}^3 p(c,k)$$

holds for a free parameter c, where $p(c,k) := \frac{-3}{16}(12 + 11\sqrt{2})ck^3 + ck^2 + (-66 + 48\sqrt{2} + \frac{c}{2})k - 28 + 20\sqrt{2} + \frac{c}{12}$. *The Baruah–Berndt formula in* (4.22) *is a special case of the above identity for* $c = 0$.

Proof. Taking the above series and removing the summand factor $p(c,k)$, this gives us, according to Clausen's product, a scalar multiple of $\mathbf{K}^2(3 - 2\sqrt{2})$, and it is known that $\lambda^*(4) = 3 - 2\sqrt{2}$ [25], and the following elliptic singular value is known [24]:

$$\mathbf{K}\left(3 - 2\sqrt{2}\right) = \frac{\left(\sqrt{2} + 1\right) \Gamma\left(\frac{1}{4}\right)^2}{2^{7/2}\sqrt{\pi}}.$$

So, from the known elliptic alpha function value [12, p. 172]

$$\alpha(4) = 2(\sqrt{2} - 1)^2,$$

this, together with the relation in (0.8), gives us an explicit symbolic evaluation for $\mathbf{E}(\lambda^*(4))$. So, by writing the power series (2.16) in terms of (2.13)–(2.15), and by then assigning the value $x = \frac{99\sqrt{2} - 140}{4}$ in (2.16) and setting the scalars s_i appropriately, and using the values for $\mathbf{K}(\lambda^*(4))$ and $\mathbf{E}(\lambda^*(4))$ that we have determined, this gives us the desired result, after much simplification. *QED*

Example 3. Setting $c = -48\left(5\sqrt{2} - 7\right)$, we obtain

$$-\frac{1}{3\pi} = \sum_{k=0}^{\infty} \left(\frac{99\sqrt{2} - 140}{4} \right)^k \cdot$$
$$\binom{2k}{k}^3 k \left(\left(51\sqrt{2} - 78\right) k^2 + 16\left(5\sqrt{2} - 7\right) k + 24\sqrt{2} - 34 \right).$$

5 Theorem 5.1 (Eq. 5.3) of [5]

Eq. 5.3 from Theorem 5.1 from [5] is such that

$$\frac{2\sqrt[4]{2}}{\pi} = \sum_{k=0}^{\infty} \left(\frac{140 - 99\sqrt{2}}{2048} \right)^k \binom{2k}{k}^3 \left(\left(18 - 6\sqrt{2}\right) k + 5 - 3\sqrt{2} \right). \tag{5.23}$$

This is proved in [5] in a similar way relative to the proof of (4.22) presented in [5].

Theorem 4. *The identity*

$$\frac{2\sqrt[4]{2}}{\pi} = \sum_{k=0}^{\infty} \left(\frac{140 - 99\sqrt{2}}{2048} \right)^k \binom{2k}{k}^3 p(c, k)$$

holds for a free parameter c*, where* $p(c,k) {:=} k^3 6 \left(249 + 176\sqrt{2}\right) c + k^2 c + k(18 - 6\sqrt{2} + \frac{c}{2}) + 5 - 3\sqrt{2} + \frac{c}{12}$*. The Baruah–Berndt formula in* (5.23) *is a special case of the above identity for* $c = 0$*.*

Proof. Taking the above series, and removing the factor $p(c,k)$ from its summand, this gives us, according to Clausen's product, a multiple of

$$\mathbf{K}\left(\frac{1}{4} i \sqrt{-8 + 3\sqrt{-24 + 22\sqrt{2}}} \right).$$

According to the elliptic integral identity in (3.19), this gives us that the above expression is equal to

$$\sqrt{1 - \left(33 + 24\sqrt{2} - 4\sqrt{140 + 99\sqrt{2}}\right)^2}\, \mathbf{K}\left(33 + 24\sqrt{2} - 4\sqrt{140 + 99\sqrt{2}}\right).$$

From the known value for $\lambda^*(16)$ [25], the above expression may be written as

$$\sqrt{1 - (\lambda^*(16))^2}\mathbf{K}(\lambda^*(16)).$$

Similarly, from the known value for the elliptic alpha function $\alpha(16)$, this, together with (0.8), gives us the following value:

$$\mathbf{E}\left(\frac{1}{4} i \sqrt{-8 + 3\sqrt{-24 + 22\sqrt{2}}} \right) =$$

$$\frac{2\pi^2}{\left(2 + \sqrt{2} + 2\ 2^{3/4}\right) \sqrt{\pi - 116^2\pi}\Gamma^2\left(\frac{1}{4}\right)} + \frac{\left(1 + \sqrt[4]{2}\right)\left(1 + \sqrt{2}\right)\Gamma^2\left(\frac{1}{4}\right)}{4\left(2 + \sqrt{2} + 2\ 2^{3/4}\right)\sqrt{\pi - 116^2\pi}}.$$

So, by rewriting the power series (2.16) in terms of (2.13)–(2.15), and then plugging in the value $x = \frac{140 - 99\sqrt{2}}{2048}$ along with appropriate values for the scalars s_i, this gives us, thanks to the known symbolic forms for $\mathbf{K}(\lambda^*(16))$ and $\mathbf{E}(\lambda^*(16))$, the desired result, after much simplification. QED

Example 4. Setting $c = 12\left(-5 + 3\sqrt{2}\right)$, we obtain

$$-\frac{1}{3\ 2^{3/4}\pi} =$$

$$\sum_{k=0}^{\infty} \left(\frac{140 - 99\sqrt{2}}{2048} \right)^k \binom{2k}{k}^3 k \left(1 - \sqrt{2} + \left(5 - 3\sqrt{2}\right) k + 42\left(27 + 19\sqrt{2}\right) k^2\right).$$

6 Theorem 6.1 (Eq. 6.1) of [5]

Eq. 6.1 from Theorem 6.1 from [5] may be written as follows:

$$\frac{\sqrt{6}+\sqrt{2}+1}{\pi} =$$
$$\sum_{k=0}^{\infty}\binom{2k}{k}^3\left(\frac{\left(3+2\sqrt{2}\right)\left(\sqrt{2}-\sqrt{3}\right)^3\left(-2+\sqrt{3}\right)^3}{8}\right)^k$$
$$\left(\left(6\sqrt{3}+3\sqrt{6}-6\right)k+2\sqrt{3}+\sqrt{6}-3-\sqrt{2}\right).$$

This is proved in [5] using the relation

$$P(e^{-2\pi\sqrt{6}}) =$$
$$\frac{\sqrt{3}}{\pi\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{2}}(\sqrt{2}+1)^2(2-\sqrt{3})(\sqrt{3}-\sqrt{2})\sum_{k=0}^{\infty}A_k X_6^k,$$

where $X_6 = 4x_6(1-x_6) = 8(\sqrt{2}+1)^2(2-\sqrt{3})^3(\sqrt{3}-\sqrt{2})^3$ and $A_k = \left(\frac{1}{2}\right)_k^3/(k!)^3$, and the following relation is also required:

$$P(e^{-2\pi\sqrt{6}}) =$$
$$(\sqrt{2}+1)(2-\sqrt{3})(\sqrt{3}-\sqrt{2})(2\sqrt{2}+3-\sqrt{6})\sum_{k=0}^{\infty}(3k+1)A_k X_6^k.$$

The proof of the above formula for $\frac{\sqrt{6}+\sqrt{2}+1}{\pi}$ given by Baruah and Berndt [5] requires many other identities involving (0.6), in contrast to our brief proof of the following generalization.

Theorem 5. *The identity*

$$\frac{\sqrt{6}+\sqrt{2}+1}{\pi} = \sum_{k=0}^{\infty}\binom{2k}{k}^3\left(\frac{\left(3+2\sqrt{2}\right)\left(\sqrt{2}-\sqrt{3}\right)^3\left(-2+\sqrt{3}\right)^3}{8}\right)^k p(c,k)$$

holds for a free parameter c, where $p(c,k) := \frac{1}{4}k^3 c\left(-21-16\sqrt{2}-14\sqrt{3}-9\sqrt{6}\right)+k^2 c+k\left(-6+6\sqrt{3}+3\sqrt{6}+\frac{c}{2}\right)-3-\sqrt{2}+2\sqrt{3}+\sqrt{6}+\frac{c}{12}$. The Baruah–Berndt formula in for $\frac{\sqrt{6}+\sqrt{2}+1}{\pi}$ given in [5] is a special case of the above identity for $c = 0$.

Proof. If we take the above series and remove the summand factor $p(c,k)$, this gives us, via Clausen's product, a multiple of

$$\mathbf{K}\left(\frac{1}{3+2\sqrt{2}+2\sqrt{3}+\sqrt{6}}\right),$$

and it is known [25] that

$$\lambda^*(6) = \frac{1}{3 + 2\sqrt{2} + 2\sqrt{3} + \sqrt{6}}.$$

From the known elliptic singular value for $\mathbf{K}(\lambda^*(6))$ [24], together with the known elliptic alpha function value [12, p. 172]

$$\alpha(6) = 5\sqrt{6} + 6\sqrt{3} - 8\sqrt{2} - 11,$$

this gives us an explicit symbolic form for $\mathbf{E}(\lambda^*(6))$. So, by setting

$$x = \frac{\left(3 + 2\sqrt{2}\right)\left(\sqrt{2} - \sqrt{3}\right)^3\left(-2 + \sqrt{3}\right)^3}{8}$$

in (2.16) and setting the scalars in (2.16) accordingly, and by rewriting (2.16) in terms of (2.13)–(2.15), our symbolic forms for $\mathbf{K}(\lambda^*(6))$ and $\mathbf{E}(\lambda^*(6))$ give us the desired result, after much simplification. QED

7 Theorem 6.1 (Eq. 6.2) of [5]

Eq. 6.2 of Theorem 6.1 from [5] may be formulated as follows:

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} \left(\frac{12\sqrt{2} - 17}{64}\right)^k \binom{2k}{k}^3 \left(\left(12\sqrt{2} - 12\right) k + 4\sqrt{2} - 5\right). \tag{7.24}$$

This is proved in [5] in much the same way as Baruah and Berndt's proof of the Baruah–Berndt formula given in Section 6.

Theorem 6. *The identity*

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} \left(\frac{12\sqrt{2} - 17}{64}\right)^k \binom{2k}{k}^3 p(c, k) \tag{7.25}$$

for a free parameter c, where $p(c,k){:=}4(3+2\sqrt{2})ck^3 + ck^2 + \frac{1}{2}(24\left(-1+\sqrt{2}\right) + c)k - 5 + 4\sqrt{2} + \frac{c}{12}$. The Baruah–Berndt formula in (7.24) *is a special case of the above identity for $c = 0$.*

Proof. This can be shown to hold using the same method used in our preceding proofs: We write (2.16) in terms of (2.13)-(2.15), set the x- and s-values accordingly, and then use Clausen's product to determine the corresponding value for $\lambda^*(r)$, and then determine the known singular value for $\mathbf{K}(\lambda^*(r))$, and then use the known value for $\alpha(r)$ to compute $\mathbf{E}(\lambda^*(r))$, and then simplify (7.25) according to the symbolic forms we have determined. For the sake of brevity, we leave it to the reader to verify this. QED

As we discuss in the following section, the Ramanujan-type formula given in [5] after Eq. 6.2 of Theorem 6.1 from [5] is significantly different compared to the previous Ramanujan-type series given in [5], so we find it fitting to conclude our main demonstration of our methods with Theorem 6.

8 Conclusion

The Ramanujan-type formula following (7.24) in [5] is again of the form indicated in (0.1), but, in this case, this Ramanujan-type formula is the first such formula in [5] such that the a-value in (0.1) has a minimal polynomial of degree greater than 4.

Eq. 6.3 in [5] states that

$$\begin{aligned}\frac{2\sqrt{2\sqrt{3}+2\sqrt{2}}}{\pi} &= \sum_{k=0}^{\infty}\{(4\sqrt{6}+6\sqrt{3}-6)k+3\sqrt{2}-3\sqrt{3}+5-\sqrt{6}\} \\ &\quad \times (-1)^k A_k \{8(\sqrt{2}+1)^2(\sqrt{3}+\sqrt{2})^3(2+\sqrt{3})^3\}^{-k}.\end{aligned}$$

Proving this using our methods is of a more computationally challenging nature, relative to our previous results. We encounter similar computational obstacles in the process of determining analogues of Theorems 1–6 according to our methods. So, we find it to be an appropriate place to conclude our demonstration as to how our methods may be applied to generalize the results from [5], as in with Theorem 6 above. However, we discuss the problem of generalizing the above formula for $\frac{2\sqrt{2\sqrt{3}+2\sqrt{2}}}{\pi}$ below.

By mimicking our proofs of Theorems 1–6 above in the hope of generalizing the above formula for $\frac{2\sqrt{2\sqrt{3}+2\sqrt{2}}}{\pi}$, this would require the evaluation of $\alpha(24)$, which is nontrivial, and no value for $\alpha(24)$ is given in references such as [12]. The closed form for the required value $\lambda^*(24)$ is so unwieldy that computational difficulties arise in the linear algebra involved in solving for the s-coefficients in the summand in (2.16), in order to generalize Baruah and Berndt's formula for $\frac{2\sqrt{2\sqrt{3}+2\sqrt{2}}}{\pi}$. The computational difficulty associated with evaluating $\lambda^*(24)$ is evidenced by how the latest version of Mathematica in 2022 is not able to determine any closed form for the following input (see OEIS entry `A084540`).

```
ToRadicals[RootApproximant[N[Sqrt[ModularLambda[I*Sqrt[24]]],
1000]]]
```

More specifically, the required root of the eighth-degree polynomial cannot be computed by Mathematica, but we have experimentally determined that $\lambda^*(24)$ appears to be equal to the following.

```
Sqrt[37745 + 26688*Sqrt[2] + 21792*Sqrt[3] + 15408*Sqrt[6] -
23112*Sqrt[-21 - 16*Sqrt[2] + 14*Sqrt[3] + 9*Sqrt[6]] -
16344*Sqrt[2*(-21 - 16*Sqrt[2] + 14*Sqrt[3] + 9*Sqrt[6])] -
13344*Sqrt[3*(-21 - 16*Sqrt[2] + 14*Sqrt[3] + 9*Sqrt[6])] -
9436*Sqrt[6*(-21 - 16*Sqrt[2] + 14*Sqrt[3] + 9*Sqrt[6])]]
```

Similarly, using a result from [14], we can show that $\mathbf{K}(\lambda^*(24))$ equals the following.

```
((1 + (1 + Sqrt[2])*(-1 + Sqrt[3])*Sqrt[-Sqrt[2] +
Sqrt[3]])* Sqrt[((-1 + Sqrt[2])*(2 + Sqrt[3])*(Sqrt[2] +
Sqrt[3])*Gamma[1/24]*Gamma[5/24]*Gamma[7/24]*Gamma[11/24])/(6*
Pi)])/16
```

We encounter similar obstacles when dealing with results as in Eq. (7.2) from Theorem 7.1 from [5].

8.1 Further research

We may mimic our methods using the hypergeometric identity

$$\sum_{k=0}^{\infty}\binom{2k}{k}^2\binom{4k}{2k}x^k = \frac{4\sqrt{2}\mathbf{K}^2\left(\sqrt{\frac{1}{2}-\frac{\sqrt{\frac{1-\sqrt{1-256x}}{x}}}{16\sqrt{2}}}\right)}{\pi^2\sqrt[4]{-256x+2\sqrt{1-256x}+2}}$$

in place of (0.7). Simialrly, we may generalize our methods by expressing the generating functions for sequences such as

$$\left(\frac{\binom{2k}{k}^3}{k+1} : k \in \mathbb{N}_0\right)$$

and

$$\left(\frac{\binom{2k}{k}^3}{2k-1} : k \in \mathbb{N}_0\right)$$

in terms of (2.13)–(2.15).

We encourage the use of our techniques given in thie article in conjunction with the modified Abel lemma-based technique from [15]. For example, we can show that

$$-\frac{32}{\pi} = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{12k}\binom{2k}{k}^3\frac{p(c,k)}{(84k^2-12k+1)\,(84k^2+156k+73)}$$

holds for a free parameter c, where $p(c,k) = 113799168ck^7 + 192374784ck^6 + 64060416ck^5 - 15828480ck^4 + 232704ck^3 - 28800ck^2 + 9024ck - 2336c - 3556224k^7 - 1566432k^6 + 5879664k^5 + 3671856k^4 - 312240k^3 + 15906k^2 + 1347k - 768$.

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