Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **38** (2018) no. 1, 17–22. doi:10.1285/i15900932v38n1p17

A spaceability result in the context of hypergroups

Seyyed Mohammad Tabatabaie
Department of Mathematics, University of Qom, Qom, Iran
sm.tabatabaie@qom.ac.ir

Bentol Hoda Sadathoseyni
Department of Mathematics, University of Qom, Qom, Iran
sb.sadathosseini@stu.qom.ac.ir

Received: 14.8.2017; accepted: 4.12.2017.

Abstract. In this paper, by an elementary constractive technique, it is shown that $L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+)$ is non-empty, where $\mathbb{Z}_+$ is the dual of a compact countable hypergroup introduced by Dunkl and Ramirez. Also, we prove that for each $r > 1$, $L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+)$ is spaceable.

Keywords: locally compact hypergroup, spaceability, L^p–space

MSC 2000 classification: primary 46A45, secondary 43A62, 46B45

1 Introduction and Notations

Suppose that X is a topological vector space. A subset $S \subseteq X$ is called *spaceable* if $S \cup \{0\}$ is large enough to contain a closed infinite dimensional subspace of X. This concept was first introduced in [6] and then the term *spaceable* was used in [1]. There have been several further works on this notion (for example see [4], [7] and [9]). It is well-known that for each $0 < p$, there are sequences in l_p not belonging to $\bigcup_{q<p} l_q$. Usually, this fact is proved by non-constructive techniques, althought a constructive proof, depending on the Principle of Uniform Boundedness can be found in [8]. Actually, in [3] it is proved that for every $p > 0$, the set $l_p - \bigcup_{q<p} l_q$ is even spaceable.

Let $\mathbb{Z}_+$ be the set of non-negative integers equipped with discrete topology, $M(\mathbb{Z}_+)$ be the space of all Radon measures on $\mathbb{Z}_+$, and p_0 be a fixed prime number. For any $k \in \mathbb{Z}_+$ and distinct non-zero $m, n \in \mathbb{Z}_+$ we put

$$\delta_k * \delta_0 = \delta_0 * \delta_k := \delta_k,$$

$$\delta_n * \delta_n := \frac{1}{p_0^{n-1}(p_0 - 1)}\delta_0 + \sum_{k=1}^{n-1} p_0^{k-1}\delta_k + \frac{p_0 - 2}{p_0 - 1}\delta_n,$$

and $\delta_m * \delta_n := \delta_{\max\{m,n\}}$. Then, by [5], $\mathbb{Z}_+$ equipped with the convolution $* : M(\mathbb{Z}_+) \times M(\mathbb{Z}_+) \to M(\mathbb{Z}_+)$ defined by

$$\mu * \nu := \int_{\mathbb{Z}_+} \int_{\mathbb{Z}_+} \delta_m * \delta_n \, d\mu(m) d\nu(n), \qquad (\mu, \nu \in M(\mathbb{Z}_+),$$

and the identity mapping on $\mathbb{Z}_+$ as involution, is a commutative discrete hypergroup. Also, the measure m defined by

$$m(\{k\}) := \begin{cases} 1, & \text{if } k = 0, \\ (p_0 - 1)p_0^{k-1}, & \text{if } k \geq 1, \end{cases}$$

is a Haar measure for $\mathbb{Z}_+$. For more details about locally compact hypergroups see [2].

In the sequel, we consider $\mathbb{Z}_+$ with above structure, and for each $s > 0$ we denote $L^s(\mathbb{Z}_+) := L^s(\mathbb{Z}_+, m)$.

In this paper, by an elementary technique, we give an algorithm that generates lots of sequences in $L^p(\mathbb{Z}_+)$ but not in $L^q(\mathbb{Z}_+)$ for every $q < p$. This argument can be regarded as a variant of the technique that was used to prove the Banach-Steinhaus Theorem in [8]. Also, we prove that for each $r > 1$, the set $L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+)$ is spaceable in $L^r(\mathbb{Z}_+)$.

2 Main Results

This is well-known that:

Lemma 1. *There is a sequence $(a_n)_{n=1}^{\infty}$ of real numbers such that $a_n > 1$ for all n, $\lim_{n\to\infty} a_n = 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^{a_n}}$ converges.*

Lemma 2. *If $0 < q < r < \infty$, then $L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+)$.*

Proof. Let $f := (x_n)_{n=0}^{\infty} \in L^q(\mathbb{Z}_+)$. Since

$$\|f\|_q^q = |x_0|^q + \sum_{n=1}^{\infty} |x_n|^q \, (p_0 - 1)p_0^{n-1} < \infty, \tag{2.1}$$

there is a positive number N such that for each $n \geq N$,

$$|x_n| \left((p_0 - 1)p_0^{n-1}\right)^{\frac{1}{q}} < 1.$$

But for each $n \in \mathbb{N}$,

$$\left((p_0 - 1)p_0^{n-1}\right)^{\frac{1}{q}} \geq 1.$$

Hence, for all $n \geq N$, $|x_n| < 1$, and then $|x_n|^r < |x_n|^q$. Therefore, for each $n \geq N$ we have

$$|x_n|^r (p_0 - 1)p_0^{n-1} < |x_n|^q (p_0 - 1)p_0^{n-1}.$$

By (2.1) we get $\sum_{n=1}^{\infty} |x_n|^r (p_0 - 1)p_0^{n-1} < \infty$, i.e. $f \in L^r(\mathbb{Z}_+)$. QED

Lemma 3. *Let $q < r$, p_0 be a prime number, and $(a_n)_{n=1}^{\infty}$ be a sequence as in Lemma 1. If the sequence $\boldsymbol{x} = (x_n)_{n=0}^{\infty}$ is defined by*

$$x_n := \begin{cases} 1 & \text{if } n = 0, \\ \frac{1}{(p_0^{n-1}(p_0-1)n^{a_n})^{\frac{1}{r}}} & \text{if } n \geq 1, \end{cases}$$

then $\boldsymbol{x} \in L^r(\mathbb{Z}_+) - L^q(\mathbb{Z}_+)$.

Proof. From

$$\begin{aligned} \| \boldsymbol{x} \|_r^r &= \sum_{n=0}^{\infty} |x_n|^r m(\{n\}) \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{p_0^{n-1}(p_0 - 1)n^{a_n}} (p_0 - 1)p_0^{n-1} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n^{a_n}} < \infty \end{aligned}$$

we conclude that $\boldsymbol{x} \in L^r(\mathbb{Z}_+)$. Let $0 < q < r$. Since $\frac{r}{a_n} < r$ for every n and $\lim_{n\to\infty} \frac{r}{a_n} = r$, there is some $t_0 \in \mathbb{N}$ such that $q < \frac{r}{a_{t_0}}$. Since

$$\lim_{n\to\infty} \frac{a_n}{a_{t_0}} = \frac{1}{a_{t_0}} < 1,$$

we can choose $s \in (\frac{1}{a_{t_0}}, 1)$, and there is $N_0 \in \mathbb{N}$ such that $\frac{a_n}{a_{t_0}} < s < 1$ for all $n \geq N_0$. Hence,

$$\frac{1}{n^s} < \frac{1}{n^{\frac{a_n}{a_{t_0}}}}$$

for all $n \geq N_0$. We have

$$\begin{aligned}\|\boldsymbol{x}\|_{\frac{r}{a_{t_0}}}^{\frac{r}{a_{t_0}}} &= \sum_{n=0}^{\infty} |x_n|^{\frac{r}{a_{t_0}}}\, m(\{n\}) \\ &= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{\left(p_0^{n-1}(p_0-1)n^{a_n}\right)^{\frac{1}{r}}} \right)^{\frac{r}{a_{t_0}}} (p_0-1)p_0^{n-1} \\ &= 1 + \sum_{n=1}^{\infty} \left((p_0-1)p_0^{n-1}\right)^{1-\frac{1}{a_{t_0}}} \frac{1}{n^{\frac{a_n}{a_{t_0}}}}.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^s} = \infty$ and

$$\left((p_0-1)p_0^{n-1}\right)^{1-\frac{1}{a_{t_0}}} \geq 1,$$

we have

$$\sum_{n=N_0}^{\infty} \frac{1}{n^s} \leq \sum_{n=N_0}^{\infty} \frac{1}{n^{\frac{a_n}{t_0}}} \leq 1 + \sum_{n=1}^{\infty} \left((p_0-1)p_0^{n-1}\right)^{1-\frac{1}{a_{t_0}}} \frac{1}{n^{\frac{a_n}{a_{t_0}}}} \leq \|\,\boldsymbol{x}\|_{\frac{r}{a_{t_0}}}^{\frac{r}{a_{t_0}}},$$

and so, $\boldsymbol{x} \notin L^{\frac{r}{a_{t_0}}}(\mathbb{Z}_+)$. Therefore, $\boldsymbol{x} \notin L^q(\mathbb{Z}_+)$ because $q < \frac{r}{a_{t_0}}$ and $L^q(\mathbb{Z}_+) \subseteq L^{\frac{r}{a_{t_0}}}(\mathbb{Z}_+)$. QED

Theorem 1. *If $1 \leq q < r < \infty$, then $L^q(\mathbb{Z}_+)$ is not closed in $L^r(\mathbb{Z}_+)$.*

Proof. First Proof: Let $C_c(\mathbb{Z}_+)$ be the space of all functions from $\mathbb{Z}_+$ into $\mathbb{C}$ with finite support. We have $C_c(\mathbb{Z}_+) \subseteq L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+)$, and $C_c(\mathbb{Z}_+)$ is dense in $L^q(\mathbb{Z}_+)$ and in $L^r(\mathbb{Z}_+)$. So, if $L^q(\mathbb{Z}_+)$ is closed in $L^r(\mathbb{Z}_+)$, then we have $L^q(\mathbb{Z}_+) = L^r(\mathbb{Z}_+)$, a contradiction.

Second Proof: Let $T : L^q(\mathbb{Z}_+) \longrightarrow L^r(\mathbb{Z}_+)$ be the inclusion mapping. We claim that T is continuous. For this, suppose that $(f_n)_{n=0}^{\infty}$ is a sequence in $L^q(\mathbb{Z}_+)$ with $f_n := (x_{n,i})_{i=0}^{\infty}$, and $f_n \to f$ in $L^q(\mathbb{Z}_+)$, where $f := (x_i)_{i=0}^{\infty} \in L^q(\mathbb{Z}_+)$. For each $0 < \varepsilon < 1$ there is $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|f_n - f\|_q^q = |x_{n,0} - x_0|^q + \sum_{i=1}^{\infty} |x_{n,i} - x_i|^q (p_0-1)p_0^{i-1} < \varepsilon < 1.$$

Hence, $|x_{n,i} - x_i|^q < 1$ for all $n \geq N$ and $i \in \mathbb{Z}_+$. Since $q < r$, we get

$$|x_{n,i} - x_i|^r \leq |x_{n,i} - x_i|^q,$$

where $n \geq N$ and $i \in \mathbb{Z}_+$, and so $\|f_n - f\|_r \leq \|f_n - f\|_q$ for all $n \geq N$. Therefore, $f_n \to f$ with respect to the norm $\|\cdot\|_r$, and T is continuous.

Now, we show that $\|\cdot\|_q$ and $\|\cdot\|_r$ are not equivalent. In contrast, suppose that there exists $M > 0$ such that for each $f \in L^q(\mathbb{Z}_+)$, $\|f\|_q \leq M\|f\|_r$. For any $n \in \mathbb{N}$, let χ_n be the characteristic function of the set $\{n\}$. Then for all $n \in \mathbb{N}$, we have

$$\left((p_0 - 1)p_0^{n-1}\right)^{\frac{1}{q}} = \|\chi_n\|_q \leq M\,\|\chi_n\|_r = M\left((p_0 - 1)p_0^{n-1}\right)^{\frac{1}{r}},$$

and so,

$$\left((p_0 - 1)p_0^{n-1}\right)^{\frac{1}{q}-\frac{1}{r}} \leq M,$$

which is a contradiction, since $\frac{1}{q} - \frac{1}{r} > 0$. Therefore, the norms $\|\cdot\|_q$ and $\|\cdot\|_r$ are not equivalent. Finally, suppose that $L^q(\mathbb{Z}_+)$ is closed in $L^r(\mathbb{Z}_+)$. Since T is continuous, the induced mapping

$$\widetilde{T} : L^q(\mathbb{Z}_+) \longrightarrow T(L^q(\mathbb{Z}_+)) = (L^q(\mathbb{Z}_+), \|\cdot\|_r), \quad \widetilde{T}(f) := T(f) = f,$$

is a bijective continuous linear operator between Banach spaces. So, by the Open Mapping Theorem, $\widetilde{T}$ is an isomorphism, and in particular the norms $\|\cdot\|_q$ and $\|\cdot\|_r$ are equivalent on $L^q(\mathbb{Z}_+)$, a contradiction. Therefore, $L^q(\mathbb{Z}_+)$ is not closed in $L^r(\mathbb{Z}_+)$. *QED*

By [9, Theorem 2.4], we can conclude:

Corollary 1. *If $0 < q < r < \infty$, then $L^r(\mathbb{Z}_+) - L^q(\mathbb{Z}_+)$ is spaceable.*

Here, we recall the following result (see [9, Theorem 3.3]).

Theorem 2. *Suppose that X is a Frechet space, and for each $n = 1, 2, \ldots$, Z_n is a Banach space. Let for each $n \in \mathbb{N}$, $T_n : Z_n \longrightarrow X$ be bounded linear operators. If $Y := span(\bigcup_{n=1}^{\infty} T_n(Z_n))$ is not closed in X, then $X - Y$ is spaceable.*

Theorem 3. *If $r > 1$, then*

$$L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+)$$

is spaceable.

Proof. The sequence $\boldsymbol{x} \in L^r(\mathbb{Z}_+)$ constructed in Lemma 3 does not belong to $L^q(\mathbb{Z}_+)$ for all $q < r$, i.e. $L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+)$ is non-empty.

Let $q < r$. Since $C_c(\mathbb{Z}_+)$ is dense in $L^r(\mathbb{Z}_+)$, and

$$C_c(\mathbb{Z}_+) \subseteq L^q(\mathbb{Z}_+) \subseteq L^r(\mathbb{Z}_+),$$

$L^q(\mathbb{Z}_+)$ is dense in $L^r(\mathbb{Z}_+)$, and so $\bigcup_{q<r} L^q(\mathbb{Z}_+)$ is not closed in $L^r(\mathbb{Z}_+)$. By Lemma 2, the set $\bigcup_{q<r} L^q(\mathbb{Z}_+)$ is actually a countable union:

$$\bigcup_{q<r} L^q(\mathbb{Z}_+) = \bigcup_{n=1}^{\infty} L^{q_n}(\mathbb{Z}_+), \tag{2.2}$$

where $q_1 < q_2 < \cdots < r$, and moreover, the sequence $(q_n)_n$ converges to r. Easily, one can see that span $(\bigcup_{n=1}^{\infty} L^{q_n}(\mathbb{Z}_+)) = \bigcup_{n=1}^{\infty} L^{q_n}(\mathbb{Z}_+)$. So, applying Theorem 2 (put $Z_n := (L^{q_n}(\mathbb{Z}_+), \|\cdot\|_{q_n})$ and $T_n : Z_n \hookrightarrow L^r(\mathbb{Z}_+)$), $L^r(\mathbb{Z}_+) - \bigcup_{q<r} L^q(\mathbb{Z}_+)$ is spaceable. QED

Acknowledgements. We would like to thank the referee of this paper for a careful reading and helpful comments. Also, we are specially grateful to Professor G. Botelho for his effective comments and suggestions on this paper.

References

[1] R. M. Aron, V. I. Gurariy, J. B. Seoane-Sepúlveda: *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc., **133**, n. 3 (2005), 795–803.

[2] W. R. Bloom, H. Heyer: Harmonic analysis of probability measures on hypergroups, De GruYte, Berlin, 1995.

[3] G. Botelho, D. Diniz, V. V. Fávaro, D. Pellegrino: *Spaceability in Banach and quasi-Banach sequence spaces*, Linear Algebra Appl., **434**, n. 5 (2011), 1255–1260.

[4] G. Botelho, V. V. Fávaro, D. Pellegrino, J. B. Seoane-Sepúlveda: $L_p[0,1]\setminus\bigcup_{q>p} L_q[0,1]$ *is spaceable for every* $p > 0$, Linear Algebra Appl., **436** (2012), 2963–2965.

[5] C. F. Dunkl, D. E. Ramirez: *A family of countably compact* P_**-hypergroups*, Trans. Amer. Math. Soc., **202** (1975), 339–356.

[6] V. P. Fonf, V. I. Gurariy, M. I. Kadets: *An infinite dimensional subspace of* $C[0,1]$ *consisting of nowhere differentiable functions*, C. R. Acad. Bulgare Sci., **52** (1999), 13–16.

[7] V. I. Gurariy, L. Quarta: *On lineability of sets of continuous functions*, J. Math. Anal. Appl., **294** (2004), 62–72.

[8] H. Hahn: *Uber folgen linearer operationen*, Monatsh. Math. Phys., **32** (1922), 3–88.

[9] D. Kitson, R. M. Timoney: *Operator ranges and spaceability*, J. Math. Anal. Appl., **378** (2011), 680–686.