Note di Matematica
Note Mat. 1(2008), suppl. n. 1, 169-190
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n1supplp169
http://siba2.unile.it/notemat

Solvmanifolds and Generalized Kähler Structures

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Received: 12/11/07; accepted: 6/3/08.

Abstract. We review generalized complex geometry in relation with solvmanifolds and we describe the construction in [17] of a generalized Kähler structure on a 6-dimensional solvmanifold. The compact manifold is the total space of a $\mathbb{T}^2$-bundle over an Inoue surface and does not admit any Kähler structure.

Keywords: generalized complex structure, generalized Kähler structure, solvmanifold, Hermitian metric, torsion

MSC 2000 classification: primary 53C15, secondary 22E25, 53C56

Introduction

The generalized Kähler structures were introduced and studied by M. Gualtieri in his PhD thesis [23] in the more general context of generalized geometry started by N. Hitchin in [28] for generalized Calabi-Yau manifolds. Both generalized Kähler and generalized Calabi-Yau structures are related to generalized complex geometry, which contains complex and symplectic geometry as extremal special cases and shares important properties with them.

Even if there are many explicit constructions [2, 3, 29, 32, 35, 6, 11, 17] the problem about the existence of non-trivial generalized Kähler structures on compact manifolds which do not admit any Kähler structure is interesting.

Some obstructions and conditions on the underlying complex manifolds were determined (as [3, 8, 23] and related references show).

[i]This work was supported by the Project MIUR ”Riemannian Metrics and Differentiable Manifolds” and by GNSAGA of INdAM.

[ii]This work was supported by the Project MIUR “Geometric Properties of Real and Complex Manifolds’ and by GNSAGA of INdAM.

By [23, 3] it follows that a generalized Kähler structure on a $2m$-dimensional manifold M is equivalent to a pair of Hermitian structures (J_+, g) and (J_-, g), where $J_\pm$ are two integrable almost complex structures on M and g is a Hermitian metric with respect to $J_\pm$, such that the 3-form $H = d^c_+ F_+ = -d^c_- F_-$ is closed, where $F_\pm(\cdot,\cdot) = g(J_\pm \cdot, \cdot)$ are the fundamental 2-forms associated with the Hermitian structures $(J_\pm, g)$ and $d^c_\pm = i(\overline{\partial}_\pm - \partial_\pm)$. In particular, any Kähler metric (J, g) determines a generalized Kähler structure by setting $J_+ = J$ and $J_- = \pm J$.

If $(J_\pm, g)$ is a generalized Kähler structure, then the two fundamental 2-forms $F_\pm$ are $\partial_\pm \overline{\partial}_\pm$-closed. Therefore the Hermitian structures $(J_\pm, g)$ are *strong Kähler with torsion* and the closed 3-form H can be also identified with the torsion of the Bismut connections associated with the two Hermitian structures $(J_\pm, g)$ (see [5, 22]). The 3-form H is called the *torsion* of the generalized Kähler structure and the structure is said *untwisted* or *twisted* according to the fact that the cohomology class $[H] \in H^3(M, \mathbb{R})$ vanishes or not.

The strong Kähler with torsion structures have been studied by many authors in [15, 16, 19, 31, 39]. In the case of a complex surface, a Hermitian metric g which satisfies the strong Kähler condition is *standard* in the terminology of Gauduchon ([21]) and if, in addition, M is compact, then any Hermitian conformal class contains a standard metric.
Compact homogeneous examples in six dimensions are given in [16], where the Hermitian manifolds are provided by *nilmanifolds*, i.e. compact quotients of nilpotent Lie groups by uniform discrete subgroups, endowed with an Hermitian structure (J, g) in which J and g arise from left-invariant tensors. In [8] it was proved that these compact manifolds cannot admit any invariant generalized Kähler structure unless they are tori. Nevertheless, Cavalcanti and Gualtieri in [9] proved that all 6-dimensional nilmanifolds admit both left-invariant generalized complex structures and left-invariant generalized Calabi-Yau structures.

No general restrictions for the existence of generalized Kähler and generalized Calabi-Yau structures are known in the case of *solvmanifolds*, i.e. for compact quotients of solvable Lie groups by uniform discrete subgroups. A structure theorem by [26] states that a solvmanifold carries a Kähler structure if and only it is covered by a complex torus which has a structure of a complex torus bundle over a complex torus.

Generalized complex structures on 4-dimensional solvable Lie groups have been studied in [12], but no result is known in higher dimensions.

As far as we know, the only known solvmanifolds carrying a generalized Kähler structure are the Inoue surface of type S_M defined in [30] and the $\mathbb{T}^2$-bundle over the Inoue surface of type S_M constructed in [17]. In [3] the complex solvmanifold from [13] was considered and it was shown that this manifold does not admit any left-invariant strong Kähler with torsion metric compatible with the natural left-invariant complex structure.

This note consists essentially of two parts. The first collects some definitions and results on generalized complex, generalized Calabi-Yau and generalized Kähler manifolds. These structures are reviewed also in relation with solvmanifolds, discussing what is known about their existence on this type of homogeneous manifolds.

The second part reviews the construction of the Inoue surface of type S_M and of the 6-dimensional solvmanifold found in [17].

1 Pure spinors

Before reviewing the definition of a generalized complex structure on a $2m$-dimensional manifold, let us start by recalling some definitions and results by Cavalcanti [7], Gualtieri [23] and Hitchin [28] on pure spinors related to linear generalized complex structures on a real vector space. Both generalized complex structures and generalized Calabi-Yau structures can be defined in terms of pure spinors.

Let V be an n-dimensional real vector space. A natural symmetric bilinear form $(,)$ on $V \oplus V^*$ is defined by setting

$$(X + \xi, w + \eta) = \frac{1}{2}(\xi(w) + \eta(X)).$$

Then the *natural pairing* $(\, , \,)$ on $V \oplus V^*$ is non-degenerate and with signature (n, n). The group $O(V \oplus V^*)$ of transformations of $V \oplus V^*$ preserving the symmetric bilinear form $(\, , \,)$ is isomorphic to the non-compact group $O(n, n)$. Note that $V \oplus V^*$ has a canonical orientation. Indeed, we have the identification

$$\mathbb{R} = \Lambda^{2n}(V \oplus V^*) = \Lambda^n V \otimes \Lambda^n V^*$$

by using the natural pairing $(\, , \,) : \Lambda^k V \times \Lambda^k V^* \to \mathbb{R}$ defined on simple elements $\eta = \eta^1 \ldots \wedge \eta^k$, $u = u_1 \wedge \ldots \wedge u_k$ as

$$(\eta, u) = \det(\eta^i(u_j))\,.$$

Hence, $\Lambda^{2n}(V \oplus V^*) = \mathbb{R}$. The Lie group preserving the bilinear form $(\, , \,)$ and the orientation is the *special orthogonal group* $SO(V \oplus V^*) \cong SO(n, n)$.

By using a block representation, the elements of the Lie algebra $\mathfrak{so}(V \oplus V^*)$ can be written in the form

$$T = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}$$

where $A \in \mathrm{End}(V)$, $\beta : V^* \to V$, $B : V \to V^*$ and such that $B = -B^*$, $\beta = -\beta^*$. Hence, B and β give rise to elements of $\Lambda^2 V^*$ and $\Lambda^2 V$ respectively, by setting

$$B(Y, X) = \iota_X B(Y)\,, \quad \beta(\xi, \eta) = \eta(\beta(\xi))\,,$$

where ι_X denotes the contraction along X.

Therefore, one gets the following splitting

$$\mathfrak{so}(V \oplus V^*) = \mathrm{End}(V) \oplus \Lambda^2 V^* \oplus \Lambda^2 V\,.$$

The space $\Lambda^* V^*$ can be viewed as a module over the Clifford algebra $CL(V \oplus V^*)$, defined by the relation

$$v^2 = (v, v)\,, \text{ for any } v \in V \oplus V^*\,.$$

The elements of the Clifford algebra act on the exterior algebra $\Lambda^* V^*$ as

$$(X + \xi) \cdot \varphi = \iota_X \varphi + \xi \wedge \varphi$$

and

$$\begin{aligned} (X + \xi)^2 \cdot \varphi &= (X + \xi) \cdot (\iota_X \varphi + \xi \wedge \varphi) = \iota_X(\xi \wedge \varphi) + \xi \wedge \iota_X \varphi \\ &= (\iota_X \xi)\varphi = (X + \xi, X + \xi)\varphi\,. \end{aligned}$$

This gives rise to the spin representation $\mathbb{S}$ of the group $Spin(V \oplus V^*)$. Since the volume form η of the Clifford algebra on $V \oplus V^*$ satisfies the condition $\eta^2 = 1$, the spin representation $\mathbb{S}$ decomposes into the ± 1-eigenspaces of the volume form

$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$$

and the above splitting corresponds in terms of differential forms to

$$\Lambda^* V^* = \Lambda^{ev} V^* \oplus \Lambda^{od} V^*\,,$$

since

$$\begin{aligned} \mathbb{S}^+ &= \Lambda^{ev} V^* \otimes (\Lambda^n V^*)^{\frac{1}{2}}, \\ \mathbb{S}^- &= \Lambda^{od} V^* \otimes (\Lambda^n V^*)^{\frac{1}{2}}. \end{aligned}$$

If V has even dimension $n = 2m$, it is possible to define the invariant bilinear form on $\mathbb{S}^\pm$ given for even spinors by

$$\langle \varphi, \psi \rangle = \sum_k (-1)^k \varphi_{2k} \wedge \psi_{n-2k} \in \Lambda^n V^* \otimes ((\Lambda^n V^*)^{\frac{1}{2}})^2 = \mathbb{R}\,,$$

and for odd spinors by

$$\langle \varphi, \psi \rangle = \sum_k (-1)^k \varphi_{2k+1} \wedge \psi_{n-2k-1} \in \Lambda^n V^* \otimes ((\Lambda^n V^*)^{\frac{1}{2}})^2 = \mathbb{R} \,.$$

A non-zero spinor $\varphi \in \mathcal{S}^\pm$ is called *pure*, if its *null space*, i.e. the subspace of $V \oplus V^*$ defined as

$$E_\varphi = \{X + \xi \in V \oplus V^* \mid (X + \xi) \cdot \varphi = 0\}$$

has dimension n , or equivalently if E_φ is maximally isotropic.

Indeed, the null space of a spinor φ is isotropic, since for any $v, w \in E_\varphi$,

$$2\langle v, w \rangle \varphi = (vw + wv) \cdot \varphi = 0 \,,$$

where we used the following

$$(v + w)^2 = (v + w, v + w) = v^2 + w^2 + 2(v, w)$$

and $v \cdot \varphi = w \cdot \varphi = 0$.
For example, let 1 be the spinor in $\Lambda^0 V^* \subset \Lambda^{ev}$; then 1 is pure, since

$$(X + \xi) \cdot 1 = \xi$$

and consequently $E_1 = V \subset V \oplus V^*$.
If we apply any element of the group $\mathrm{Spin}(V \oplus V^*)$ to the spinor 1, then we obtain another pure spinor. In particular, if $B \in \Lambda^2 V^*$, then

$$\exp B = 1 + B + \frac{1}{2} B^2 + \dots$$

is a pure spinor. In such a case, we have

$$E_{\exp B} = \{X + \iota(X) B \in V \oplus V^* \mid X \in V\} \,.$$

By Chevalley [10, III 2.4], if φ, ψ are pure spinors, then

$$\langle \varphi, \psi \rangle = 0 \tag{1}$$

if and only if $E_\varphi \cap E_\psi \neq \{0\}$. Moreover, any maximal isotropic subspace of $V \oplus V^*$ is represented by a unique pure line in the spin bundle $\mathcal{S}$. The line is contained in $\mathcal{S}^+$ for even maximal isotropic subspaces and in $\mathcal{S}^-$ for odd ones.

2 Linear generalized complex structures

By [28, 23, 8, 20] it is possible to define an endomorphism of $V \oplus V^*$ which generalizes both complex and symplectic structure.

1 Definition. A *generalized complex structure* on a real vector space V is an endomorphism $\mathcal{J}$ of $V \oplus V^*$ such that

- $\mathcal{J}^2 = -\mathrm{id}_{V\oplus V^*}$
- $\mathcal{J}^* = -\mathcal{J}$,

where $\mathcal{J}^* : (V \oplus V^*)^* \to V \oplus V^*$ is the dual map of $\mathcal{J}$ and $V \oplus V^*$ is identified to its dual space via the natural pairing $(\ ,\ )$ on $V \oplus V^*$.

The following propositions by [23, Proposition 4.2 and 4.3] characterize generalized complex structures.

2 Proposition. *A generalized complex structure on V is equivalent to assign a complex structure on $V \oplus V^*$ that is orthogonal with respect to the natural pairing on $V \oplus V^*$.*

Indeed, if $\mathcal{J}$ is a generalized complex structure on V, one has

$$(\mathcal{J}v, \mathcal{J}w) = (\mathcal{J}^*\mathcal{J}v, w) = -(\mathcal{J}^2 v, w) = (v, w),$$

for any $v, w \in V \oplus V^*$.

In terms of isotropic subspaces with respect to the natural pairing $(\ ,\ )$, as for the case of complex structures, we have the following

3 Proposition. *A generalized complex structure $\mathcal{J}$ on V is equivalent to assign a maximal isotropic subspace E of $(V \oplus V^*) \otimes \mathbb{C}$ such that $E \cap \overline{E} = \{0\}$.*

As in the case of the complex structures, if $\mathcal{J}$ is a generalized complex structure on V, the corresponding maximal isotropic subspace is

$$E = \{v \in (V \oplus V^*) \otimes \mathbb{C} \mid \mathcal{J}v = iv\}\,.$$

Viceversa, if E is maximal isotropic and such that $E \cap \overline{E} = \{0\}$, the generalized complex structure $\mathcal{J}$ on $V \oplus V^*$ is given by

$$\begin{cases} \mathcal{J}v = iv, & \forall v \in E \\ \mathcal{J}v = -iv, & \forall v \in \overline{E}\,. \end{cases}$$

By [23, Theorem 4.8] any maximal isotropic subspace E in $(V \oplus V^*) \otimes \mathbb{C}$ corresponds to a pure spinor line generated by

$$\varphi_E = \exp(B + i\omega)\Omega, \tag{2}$$

where B, ω are real 2-forms and $\Omega = \theta^1 \wedge \cdots \wedge \theta^n$, with $(\theta^1, \ldots, \theta^k)$ linearly independent complex 1-forms. The integer k is called the *type* of the maximal isotropic subspace or equivalently of the generalized complex structure.

The complex and symplectic structures are generalized complex structures of "extreme" type.

4 Example (*Complex structures*)**.** Let J be a complex structure on V. Define $\mathcal{J}$ as

$$\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

Then $\mathcal{J}$ is a complex structure on $V \oplus V^*$ and $\mathcal{J}^* = -\mathcal{J}$, i.e. $\mathcal{J}$ is a generalized complex structure on V. The corresponding pure spinor line is generated by $\varphi_E = \Omega^{n,0}$, where $\Omega^{n,0}$ is a generator of the space of $(n,0)$-forms on the complex space (V, J) and therefore the generalized complex structure is of type n.

5 Example (*Symplectic structures*)**.** Let ω be a symplectic structure on V. Set

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Then $\mathcal{J}^2 = -\mathrm{id}_{V \oplus V^*}$ and $\mathcal{J}^* = -\mathcal{J}$, i.e. $\mathcal{J}$ is a generalized complex structure on V. The associated pure spinor line is generated by $\varphi_E = \exp(i\omega)$ and thus the generalized complex structure is of type 0.

3 Twisted Courant bracket and generalized complex structures

In this section, following [23], we recall the definition of generalized complex structure.

Let M be a manifold of dimension n and H be a closed 3-form on M. Denote by TM and T^*M the tangent and cotangent bundle of M respectively. The *twisted Courant bracket* is the skew-symmetric map

$$[\,,\,] : TM \oplus T^*M \to TM \oplus T^*M$$

defined as

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi) + \iota_X \iota_Y H\,,$$

for any $X, Y \in \Gamma(TM)$ and any $\xi, \eta \in \Gamma(T^*M)$, where $\mathcal{L}_X$ denotes the Lie derivative along the vector field X. By definition, the 3-form H is said to be the *torsion* or the *twisting* of the Courant bracket.

In particular, if $H = 0$, then $[\,,\,]$ is the usual *Courant bracket* on $TM \oplus T^*M$.

Observe that, in general, the twisted Courant bracket does not satisfy the Jacobi identity. Indeed, for any A, B, C sections of $TM \oplus T^*M$, let

$$\mathrm{Jac}_H(A,B,C) = [[A,B],C] + [[B,C],A] + [[C,A],B]$$

the *Jacobiator* and

$$\mathrm{Nij}_H(A,B,C) = \frac{1}{2}(([A,B],C) + ([B,C],A) + ([C,A],B)) + H(A,B,C)$$

the *Nijenhuis operator*, where $(\,,\,)$ denotes the natural pairing on $TM \oplus T^*M$. Then the following holds [23, p. 43]

$$\mathrm{Jac}_H(A,B,C) = d(\mathrm{Nij}_H(A,B,C))\,.$$

In the case M is a Lie group, the twisted Courant bracket when restricted to left-invariant vector fields and left-invariant 1-forms satisfies the Jacobi identity. We are ready to recall the following

6 Definition. A *generalized complex structure* on M is the datum of a subbundle $E \subset (TM \oplus T^*M) \otimes \mathbb{C}$ such that

(1) $E \oplus \overline{E} = (TM \oplus T^*M) \otimes \mathbb{C}$;

(2) the space of sections of E is closed under the Courant bracket $[\,,\,]$;

(3) E is isotropic.

By Proposition 2, it follows that a generalized complex structure on M is equivalent to give an almost complex structure $\mathcal{J}$ on the bundle $TM \oplus T^*M$, compatible with the natural pairing, which is integrable with respect to the Courant bracket. Moreover, by the results reviewed in Section 1, the maximal isotropic subbundle E of $(TM \oplus T^*M) \otimes \mathbb{C}$ can be expressed as the Clifford annihilator of the unique line subbundle U_E of the complex spinors for the metric bundle $TM \oplus T^*M$. By definition U_E is a pure spinor line and it is called the *canonical line bundle* of the generalized complex structure $\mathcal{J}$ associated with E. In terms of complex differential forms, U_E can be defined by

$$E = \{X + \xi \in (TM \oplus T^*M) \otimes \mathbb{C} \,:\, (X + \xi) \cdot U_E = 0\}.$$

Therefore, at any point the line U_E is generated by a pure spinor of the form (2) and the pure spinor is a local trivialization of U_E.

Now, we will describe the two basic examples, namely complex and symplectic structures.

7 Example (*Complex Structures*)**.** Let (M, J) be a complex manifold. Denote, as usual, by

$$\begin{aligned} T^{0,1}M &= \{Z \in T^{\mathbb{C}}M \mid JZ = -iZ\} = \{X + iJX \mid X \in TM\} \\ T^{*1,0}M &= \{\varphi \in T^{*\mathbb{C}}M \mid J\varphi = i\varphi\} = \{\alpha - iJ\alpha \mid \alpha \in T^*M\} \end{aligned}$$

Set

$$E = T^{0,1}M \oplus T^{*1,0}M\,.$$

We check that $E \subset (TM \oplus T^*M) \otimes \mathbb{C}$ gives rise to a generalized complex structure on M.
First of all, $E \oplus \overline{E} = (TM \oplus T^*M) \otimes \mathbb{C}$.
If $Z + \varphi$, $W + \psi$ are any elements of E, then, by taking the $\mathbb{C}$-bilinear extension of the natural pairing $(\,,\,)$ on $TM \oplus T^*M$ to $(TM \oplus T^*M) \otimes \mathbb{C}$, we get:

$$(Z + \varphi, W + \psi) = \frac{1}{2}(\varphi(W) + \psi(Z)) = 0\,,$$

since φ, $\psi \in T^{*1,0}M$ and Z, $W \in T^{0,1}M$. Hence E is isotropic.
Now we check the involutivity of the sections of E with respect to the Courant bracket. We have

$$\begin{aligned} [Z + \varphi, W + \psi] &= [Z, W] + \mathcal{L}_Z\psi - \mathcal{L}_W\varphi - \tfrac{1}{2}d(\iota_Z\psi - \iota_W\varphi) \\ &= [Z, W] + \mathcal{L}_Z\psi - \mathcal{L}_W\varphi\,. \end{aligned}$$

Observe that given any vector field ζ of type $(0,1)$ and any 1-form η of type $(1,0)$, we have, for any vector field ξ of type $(0,1)$

$$\mathcal{L}_\zeta\eta(\xi) = \zeta(\eta(\xi)) - \eta([\zeta, \xi]) = 0\,,$$

i.e. $\mathcal{L}_\zeta\eta$ is of type $(1,0)$. Therefore, the involutivity of E follows.
Equivalently, we can think to the generalized complex structure $\mathcal{J}_J$ induced by a complex one J as the endomorphism of $TM \oplus T^*M$ defined as

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

8 Example (*Symplectic Structures*)**.** Let (M, ω) be a symplectic manifold. Set

$$E = \{Z - i\,\iota_Z\omega \mid Z \in TM \otimes \mathbb{C}\}\,.$$

defines a generalized complex structure on M. Indeed, E is isotropic. To see this, let $Z - i\,\iota_Z\omega$, $W - i\,\iota_W\omega$ be sections of E. Then

$$(Z - i\,\iota_Z\omega, W - i\,\iota_W\omega) = \frac{1}{2}(-i\,\iota_W\omega(Z) - i\,\iota_Z\omega(W)) = 0\,,$$

since ω is skew. Moreover, by the fact that ω is non-degenerate, it follows that $E \cap \overline{E} = \{0\}$.
Finally, we check the involutivity of E. Let $Z - i\,\iota_Z\omega$, $W - i\,\iota_W\omega$ be sections of E. Then

$$\begin{aligned}[][Z - i\,\iota_Z\omega, W - i\,\iota_W\omega] &= [Z,W] + i(-\mathcal{L}_Z\iota_W\omega + \mathcal{L}_W\iota_Z\omega + d(\iota_W\iota_Z\omega)) \\ &= [Z,W] + i(-\mathcal{L}_Z\iota_W\omega + \mathcal{L}_W\iota_Z\omega + \mathcal{L}_Z\iota_W\omega - \iota_Z\mathcal{L}_W\omega) \\ &= [Z,W] - i\iota_{[Z,W]}\omega\,,\end{aligned}$$

where we used that $d\omega = 0$ and $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$. Again, the previous generalized complex structure can be viewed as

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

In the case of a Lie group G, by [12] there is a correspondence between left-invariant complex structures on G and complex structures on the cotangent bundle T^*G which are left-invariant with respect to the Lie group structure on T^*G induced by the coadjoint action. The corresponding Lie bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$ is just the Courant bracket. By [12, Theorem 4.7], if G is solvable and 4-dimensional, then G has neither left-invariant symplectic nor complex structures if and only if G does not admit left-invariant generalized complex structures. Using this correspondence, in [12] they are able to distinguish the 4-dimensional solvable Lie groups admitting a generalized complex structure of type 0, 1 or 2.

In higher dimension, in the case G is nilpotent, by [9] any 6-dimensional nilpotent Lie group has a left-invariant generalized complex structure, but already in dimension 8 there exists an example of nilmanifold which do not admit any left-invariant generalized complex structure.

4 Generalized Calabi-Yau manifolds

In this section we will shortly recall the definition of *generalized Calabi-Yau manifolds*, given by Hitchin in [28]. The two main examples of generalized Calabi-Yau manifolds are furnished by Calabi-Yau manifolds and symplectic manifolds. Furthermore, a generalized Calabi-Yau manifold is a generalized complex manifold in a natural way.

9 Definition. A *generalized Calabi-Yau structure* on a $2m$-dimensional manifold M is given by a form $\varphi \in \Lambda^{ev} \otimes \mathbb{C}$ or $\Lambda^{od} \otimes \mathbb{C}$ which is a complex pure spinor for $TM \oplus T^*M$ endowed with the natural pairing $(\;,\;)$ such that φ is closed and $\langle \varphi, \overline{\varphi} \rangle \neq 0$ at any point.

A *generalized Calabi-Yau manifold* is a pair (M, φ) where M is a $2m$-dimensional manifold and φ is a generalized Calabi-Yau structure.

10 Proposition ([28]). *Let (M, φ) be a generalized Calabi-Yau manifold. Then the null space of φ*

$$E_\varphi = \{X + \xi \in TM \oplus T^*M \mid (X + \xi) \cdot \varphi = 0\}$$

gives rise to a generalized complex structure on M.

Therefore, in the case of a generalized Calabi-Yau structure, the canonical line bundle is a trivial bundle admitting a nowhere-vanishing closed section.

11 Example (Calabi-Yau manifolds). Let (M, g, J, ω, ψ) be a Calabi-Yau manifold, namely a compact Kähler manifold (M, g, J, ω) of dimension $2m$ endowed with a complex $(m, 0)$-form ψ, parallel with respect to the Levi-Civita connection of g and such that

$$\psi \wedge \overline{\psi} = (-1)^{\frac{m(m+1)}{2}} i^m \frac{\omega^m}{m!} \,.$$

Then the complex $(m, 0)$-form ψ defines a structure of generalized Calabi-Yau on M. Indeed, for any $Z + \xi$, with $Z \in T^{0,1}M$ and $\xi \in T^{*1,0}M$ we have

$$(Z + \xi) \cdot \psi = \iota_Z \psi + \xi \wedge \psi = 0 \,,$$

i.e. ψ is a pure spinor. Furthermore, by the conditions above,

$$(\psi, \overline{\psi}) \neq 0 \,,$$

and $d\psi = 0$, i.e. ψ defines a generalized Calabi-Yau on M.

12 Example (Symplectic manifolds). Let (M, ω) be a symplectic manifold of dimension $2m$. Then the spinor $1 \in \Lambda^0 T^*M$ is pure. Hence $\varphi = \exp(i\omega)$ defines a pure spinor on M such that

$$(\varphi, \overline{\varphi}) = (-2i)^m \frac{\omega^m}{m!} \neq 0$$

and $d\varphi = 0$, since $d\omega = 0$. Therefore, φ gives rise to a generalized Calabi-Yau structure on M.

Examples of generalized Calabi-Yau manifolds of even and odd type are given in [28].

If M is a nilmanifold, i.e. a compact quotient of a nilpotent Lie group by a uniform discrete subgroup, then the following result holds

13 Theorem ([9]). *Any left-invariant generalized complex structure on a nilmanifold is generalized Calabi-Yau, i.e. any left-invariant global trivialization of the canonical bundle must be a closed differential form.*

In the more general case of solvmanifolds as far as we know no general result about generalized Calabi-Yau structures is known.

5 Generalized Kähler structures

Generalized Kähler structures have been introduced in [23] in the context of generalized geometries as generalization of Kähler structures. These structures, as showed in [20, 3], are strictly related to bi-Hermitian geometry.

We start by recalling the following

14 Definition. A *generalized Kähler structure* on a $2m$-dimensional manifold M is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of generalized complex structures on M such that

(1) $\mathcal{J}_1$ and $\mathcal{J}_2$ commute;

(2) $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible with the indefinite metric $(\ ,\ )$ on $TM \oplus T^*M$;

(3) the bilinear form $-(\mathcal{J}_1\mathcal{J}_2\,\cdot\,,\,\cdot)$ is positive definite.

In terms of bi-Hermitian geometry, Apostolov and Gualtieri proved in [3] the following

15 Theorem. *A generalized Kähler structure on M is equivalent to a triple (g, J_+, J_-) where:*

(1) g is a Riemannian metric on M;

(2) J_+ and J_- are two complex structures on M compatible with g such that

$$d^c_+ F_+ + d^c_- F_- = 0\,, \quad dd^c_+ F_+ = dd^c_- F_- = 0\,, \tag{3}$$

where $d^c_\pm = i(\overline{\partial}_\pm - \partial_\pm)$ and $F_\pm$ is the fundamental form of the Hermitian structure $(J_\pm, g)$.

In the context of theoretical physics the equations (3) appeared on the general target space geometry for a $(2,2)$ supersymmetric sigma model [20].

The 3-form $d^c_+ F_+$ is called the *torsion form* of the generalized Kähler structure and the generalized Kähler structure is said to be *untwisted* if $[d^c_+ F_+] \in H^3(M)$ vanishes and *twisted* if $[d^c_+ F_+] \neq 0$.

A trivial example of generalized Kähler manifold is given by a Kähler manifold (M, g, J). Indeed, setting $J_+ = J\,, J_- = \pm J$, the triple (g, J_+, J_-) gives a generalized Kähler structure on M.

A natural problem is to see when a compact complex (M, J) admits a generalized Kähler structure (g, J_+, J_-) with $J = J_+$.

The interesting case is the one for which $J_+ \neq \pm J_-$, i.e. the generalized Kähler structure is not induced by a Kähler one on (M, J).

Constructions of non-trivial generalized Kähler structures are given for instance in [23, 3, 6, 29, 35, 36, 11]. For example in [35] the generalized Kähler quotient construction is considered in relation with the hyperkähler quotient construction and generalized Kähler structures are given on $\mathbb{CP}^n$, on some toric varieties and on the complex Grassmannian.

An interesting problem is thus to look for compact examples of generalized Kähler manifolds which do not admit any Kähler structure.

The requirement that a compact manifold admits a Kähler structure has many topological implications, like for instance the evenness of the odd Betti numbers, the hard Lefschetz property and the formality in the sense of Sullivan [14, 40]. In the case of generalized Kähler structures some results about formality have been proved by Cavalcanti in [8], by using the Hodge theory developed in [23] in the context of generalized Kähler structures and in the case the generalized complex structure of the pair giving the generalized Kähler structure has holomorphically trivial canonical bundle.

If (M, J) has a generalized Kähler structure such that $J = J_+$ and the commutator $[J_+, J_-] \neq 0$, then $[J_+, J_-]$ defines a *holomorphic Poisson structure* on (M, J) (see [29]).

By [3], if $[J_+, J_-] = 0$, i.e. J_+ and J_- commute, then $Q = J_+J_-$ is an involution of the tangent bundle TM and one has the splitting

$$TM = T_+M \oplus T_-M,$$

where the (± 1)-eigenspaces $T_\pm M$ are holomorphic and integrable sub-bundles of TM.

Related to Kähler-Einstein manifolds, in [3] it was proved that if (M, J, g) is a compact Kähler-Einstein manifold such that the tangent bundle TM splits as a direct sum of two holomorphic sub-bundles, then (M, J) admits generalized Kähler structures compatible with $J_+ = J$ and $J_- = -J|_{T_+M} + J|_{T_-M}$.

6 Link with strong Kähler with torsion geometry

If a $2m$-dimensional manifold M has a generalized Kähler structure defined by the pair of Hermitian structures $(J_\pm, g)$, then the two fundamental forms $F_\pm$ are $\partial_\pm\overline{\partial}_\pm$-closed and there exists two Hermitian connections whose torsion is the 3-form $\pm H$, where $H = d^cF_+$. Therefore generalized Kähler geometry is related with what in literature is called *Kähler with torsion geometry*.

In general, if (M, J, g) is an Hermitian manifold, then Gauduchon proved in [22] that there is a 1-parameter family of canonical Hermitian connections on M which can be distinguished by the properties of their torsion tensor. In particular, the *Bismut connection* is the unique connection ∇ such that

$$\nabla g = 0\,, \quad \nabla J = 0$$

and $g(X, T^\nabla(Y,Z))$ is totally skew-symmetric, where by T^∇ we denote its torsion tensor.

Then the resulting torsion form

$$T(X,Y,Z) = g(X, T^\nabla(Y,Z))$$

coincides with JdF, where F is the fundamental form of the Hermitian structure (J,g).

We recall the following (see [31])

16 Definition. An Hermitian metric g on (M,J) is said to be *strong Kähler with torsion* if the 3-form JdF is d-closed, or equivalently if

$$dd^cF = 0 \quad \text{or} \quad \partial\overline{\partial}F = 0\,.$$

In the literature a strong Kähler with torsion metric is also called *pluriclosed* (see [15]).

We recall that an Hermitian metric on a $2m$-dimensional compact complex manifold (M,J) is *standard* in the sense of [21] if the $(2m-2)$-form F^{m-1} is dd^c-closed. By [21] the conformal class of every Hermitian structure contains a standard structure, which is unique if $m > 2$. Examples of such structures are given by the semi-Kähler and the homogeneous Hermitian structures.

In particular, if (M,J) is a compact complex surface, then a metric g satisfying the strong Kähler with torsion condition is standard and a strong Kähler with torsion metric can be found in the conformal class of any given Hermitian metric. The theory is completely different in higher dimensions and not many examples are known.

By [39] any real compact semisimple Lie group G of even dimension admits a natural strong Kähler with torsion metric and a twisted generalized Kähler structure (see [23]). Indeed, by [38] G has left and right invariant complex structures J_L and J_R, which can be chosen to be Hermitian with respect to the bi-invariant metric induced by the Killing form. Both (J_L, g) and (J_R, g) are strong Kähler with torsion structures and the Bismut connection ∇ is the flat connection with skew-symmetric torsion $g(X,[Y,Z])$ corresponding to an invariant 3-form on the Lie algebra of the compact Lie group. Moreover, (J_L, J_R, g) is a generalized Kähler structure.

Other examples of homogeneous strong Kähler with torsion manifolds have been found in [16] and they are 6-dimensional nilmanifolds N/Γ, i.e. compact quotients of 6-dimensional nilpotent Lie groups N. In this case both the complex structure J and the Riemannian metric g arise from corresponding left-invariant tensors on N. By the classification obtained in [16] it turns out that only 4 classes of 6-dimensional nilpotent Lie algebras admit a strong Kähler with torsion structure and that the existence of an invariant strong Kähler with torsion structure on the nilmanifold N/Γ depends only on the complex structure on the Lie algebra.

In general, it is well known that a nilmanifold cannot admit any Kähler structure unless it is a torus (see for instance [4]). Moreover, by [8] there are no nilmanifolds, except tori, carrying an invariant generalized Kähler structure, since every generalized complex structure on a nilpotent Lie algebra has holomorphically trivial canonical bundle.

Very few examples of solvmanifolds which admit a strong Kähler with torsion structure are known. Indeed, as far as we know, the known strong Kähler with torsion solvmanifolds are either complex surfaces or the 6-dimensional manifold constructed in [17], that will be reviewed in the last section. Complex surfaces diffeomorphic to solvmanifolds have been classified by Hasegawa in [25], showing that 4-dimensional solvmanifolds admit only left-invariant complex structures. By its classification a complex surface, diffeomorphic to a solvmanifold, has to be one of the following: a Complex torus, an Hyperelliptic surface, an Inoue surface of type S_M and $S^{\pm}$ [30], a Primary and Secondary Kodaira surface [33, 34].

By [17] a $\mathbb{T}^2$-bundle over a Inoue surface admits a generalized Kähler structure, so in particular a Kähler with torsion structure. In [18] blow ups and resolutions of orbifolds are investigated in relation to Kähler with torsion geometry.

In the case of solvmanifolds, Hasegawa proved in [25, 26] that a solvmanifold carries a Kähler metric if and only if it is covered by a finite quotient of a complex torus, which has the structure of a complex torus bundle over a complex torus. No general result is known in general for the existence of a generalized Kähler structure on a solvmanifold except the case of a complex surface [3].

7 Generalized Kähler structures on complex surfaces

Not any compact complex surface admits a generalized Kähler structure. Indeed, there is a classification theorem by Apostolov and Gualtieri [3]:

17 Theorem. *A compact complex surface (M, J) carries a generalized Kähler structure $(J_{\pm}, g)$ with $J_+ = J$, $J_- \neq \pm J$ and $[J_+, J_-] = 0$ if and only if the holomorphic tangent bundle of (M, J) splits as a direct sum of two holomorphic sub-bundles. Moreover, (M, J) is biholomorphic to one of the following:*

(1) a geometrically ruled complex surface or;

(2) a bi-elliptic complex surface or;

(3) a compact complex surface of Kodaira dimension 1 and even first Betti number b_1 or;

(4) a compact complex surface of general type, uniformized by the product of two hyperbolic planes $\mathcal{H} \times \mathcal{H}$ or;

(5) a Hopf surface or;

(6) an Inoue surface of type S_M.

By using the previous result and the classification of complex solvmanifolds in [25], it turns out that an Inoue surface of type S_M is the only solvmanifold admitting a generalized Kähler structure.

The Inoue surfaces S_M are quotients of the form $\mathcal{H} \times \mathbb{C}/G_M$, where

$$\mathcal{H} = \{w \in \mathbb{C} \quad | \quad \Im\mathfrak{m}\, w > 0\}$$

is the upper-half of the complex plane $\mathbb{C}$ and $M \in SL(3, \mathbb{Z})$ is an unimodular matrix with eigenvalues $\alpha, \overline{\alpha}$ and a irrational eigenvalue $c > 1$ such that $|\alpha|^2 c = 1$ (see [30]).

If we denote by $(\alpha_1,\alpha_2,\alpha_3)$ an eigenvector corresponding to α and by (c_1,c_2,c_3) the real eigenvector corresponding to c, then the group G_M is generated by the transformations

$$\begin{aligned} &\varphi_0 : (w, z) \mapsto (cw, \alpha z), \\ &\varphi_j : (w, z) \mapsto (w + c_j, z + \alpha_j),\ j = 1, 2, 3. \end{aligned}$$

By [30] G_M acts freely and properly discontinuously on $\mathcal{H} \times \mathbb{C}$. Therefore, S_M is a compact complex surface, the total space of a $\mathbb{T}^3$-bundle over the circle S^1.

In [30] Inoue proved that the surfaces S_M do not admit any Kähler metric and any symplectic structure since their Betti numbers are

$$b_1(S_M) = 1, \quad b_2(S_M) = 0.$$

In [41] Tricerri showed that the Hermitian metric

$$\frac{1}{w_2^2} dw \otimes d\overline{w} + w_2 dz \otimes d\overline{z}$$

on $\mathcal{H} \times \mathbb{C}$ induces a locally conformally Kähler metric on S_M with non-parallel Lee form.

Due to [25] an Inoue surface S_M can be also viewed also as solvmanifold $\mathbb{R} \ltimes (\mathbb{R} \times \mathbb{R}^2)/\mathbb{Z} \ltimes (\mathbb{Z} \times \mathbb{Z}^2)$, as we already remarked previously. The corresponding 4-dimensional solvable Lie group has structure equations

$$\begin{cases} de^1 = a\, e^1 \wedge e^2\,, \\ de^2 = 0\,, \\ de^3 = \frac{1}{2} a\, e^2 \wedge e^3\,, \\ de^4 = \frac{1}{2} a\, e^2 \wedge e^4\,, \end{cases}$$

where a is a non-zero real number.

By the classification obtained in [12] this solvable Lie group admits only generalized complex structures of type 1 and 2.

8 A 6-dimensional generalized Kähler solvmanifold

In this section we review the construction of the 6-dimensional generalized Kähler example given in [17] as $\mathbb{T}^2$-bundle over an Inoue surface od type S_M.

Let $\mathfrak{s}_{a,b}$ be the 2-step solvable Lie algebra with structure equations:

$$\begin{cases} de^1 = a\, e^1 \wedge e^2\,, \\ de^2 = 0\,, \\ de^3 = \frac{1}{2} a\, e^2 \wedge e^3\,, \\ de^4 = \frac{1}{2} a\, e^2 \wedge e^4\,, \\ de^5 = b\, e^2 \wedge e^6\,, \\ de^6 = -b\, e^2 \wedge e^5\,, \end{cases}$$

where a and b are non-zero real numbers.

If we denote by $S_{a,b}$ the simply-connected solvable Lie group with Lie algebra $\mathfrak{s}_{a,b}$, then the product on the Lie group, in terms of the global coordinates $(t, x_1, x_2, x_3, x_4, x_5)$ on $\mathbb{R}^6$, is expressed by:

$$\begin{aligned} &(t, x_1, x_2, x_3, x_4, x_5) \cdot (t', x_1', x_2', x_3', x_4', x_5') = (t + t',\, e^{-a\,t} x_1' + x_1,\, e^{\frac{a}{2} t} x_2' + x_2, \\ &e^{\frac{a}{2} t} x_3' + x_3, x_4' \cos(b\,t) - x_5' \sin(b\,t) + x_4, x_4' \sin(b\,t) + x_5' \cos(b\,t) + x_5). \end{aligned} \tag{4}$$

Note that by the previous structure equations it turns out that the Lie group $S_{a,b}$ is unimodular and $S_{a,b}$ can be viewed as a semi-direct product of the form

$$\mathbb{R} \ltimes_\varphi (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2)\,,$$

where $\varphi = (\varphi_1, \varphi_2)$ is the diagonal action of $\mathbb{R}$ on $(\mathbb{R} \times \mathbb{R}^2) \times \mathbb{R}^2$ described above by (4).

In contrast with the nilpotent case, there are no existence theorems for uniform discrete subgroups of a solvable Lie group and, if the Lie group is non-completely solvable and admits a compact quotient, one cannot apply Hattori's theorem [27] to compute the de Rham cohomology of the solvmanifold.

In our case the Lie group $S_{a,b}$ is non-completely solvable, since ad_{e_2} has complex eigenvalues, and as shown in [17] it admits a compact quotient. Indeed one has the following

18 Theorem ([17])**.** *Let $S_{1,\frac{\pi}{2}}$ be the simply-connected solvable Lie group with Lie algebra $\mathfrak{s}_{1,\frac{\pi}{2}}$. Then*

(1) $S_{1,\frac{\pi}{2}}$ has a compact quotient $M^6 = S_{1,\frac{\pi}{2}}/\Gamma$ by a uniform discrete subgroup Γ.

(2) The compact manifold M^6 is the total space of a $\mathbb{T}^2$-bundle over an Inoue surface S_M.

(3) M^6 has first Betti number equal to 1, thus it has no Kähler structures.

(4) The solvmanifold M^6 carries a left-invariant (non-trivial) twisted generalized Kähler structure.

We will give a sketch of the proof.

First of all in order to obtain an explicit description of the uniform discrete subgroup Γ, in [17] we showed that the solvable Lie group $S_{1,\frac{\pi}{2}}$ is isomorphic to $(\mathbb{R}^6 = \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}), *)$ with product $*$ given by

$$(t, u, z, w) * (t', u', z', w') = (t + t', c^t u' + u, \alpha^t z' + z, e^{i\frac{\pi}{2}t} w' + w),$$

for any $t, t', u, u' \in \mathbb{R}$ and $z, z', w, w' \in \mathbb{C}$.

It turns out that the discrete subgroup Γ is isomorphic to $\mathbb{Z} \ltimes (\mathbb{Z}^3 \times \mathbb{Z}^2)$ and it is generated by the transformations

$$\begin{aligned} g_0 &: (t, u, z, w) \mapsto (t + 1, cu, \alpha z, iw), \\ g_j &: (t, u, z, w) \mapsto (t, u + c_j, z + \alpha_j, w),\ j = 1, 2, 3, \\ g_4 &: (t, u, z, w) \mapsto (t, u, z, w + 1), \\ g_5 &: (t, u, z, w) \mapsto (t, u, z, w + i). \end{aligned}$$

It can be checked that the previous subgroup Γ acts freely and properly discontinuously on $S_{1,\frac{\pi}{2}}$ and that the quotient manifold is compact. Moreover, the map

$$\begin{aligned} \pi : \quad & \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}) \to \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C}), \\ & (t, u, z, w) \mapsto (t, u, z) \end{aligned}$$

inherits to M^6 the structure of a $\mathbb{T}^2$-bundle over an Inoue surface of type S_M.

The generators of Γ satisfy the following relations:

$$g_j g_k = g_k g_j,$$

for any $j, k = 1, \ldots, 5$.

Moreover,

$$\begin{aligned}
&[g_0, g_j] = g_0 g_j g_0^{-1} g_j^{-1} : (t, u, z, w) \mapsto (t, u - c_j + c\, c_j, z - \alpha_j + \alpha\, \alpha_j, w), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, 3, \\
&[g_0, g_4] = g_0 g_4 g_0^{-1} g_4^{-1} : (t, u, z, w) \mapsto (t, u, z, w - 1 + i), \\
&[g_0, g_5] = g_0 g_5 g_0^{-1} g_5^{-1} : (t, u, z, w) \mapsto (t, u, z, w - 1 - i)
\end{aligned}$$

Therefore

$$\begin{aligned}
&[g_0, g_j] = g_1^{m_{j1}} g_2^{m_{j2}} g_3^{m_{j3}} g_j^{-1},\ j = 1, 2, 3, \\
&[g_0, g_4] = {g_4}^{-1} g_5, \\
&[g_0, g_5] = {g_4}^{-1} {g_5}^{-1}.
\end{aligned}$$

By the fact that the discrete subgroup Γ is 2-step solvable, it follows that $[\Gamma, \Gamma]$ is a torsion-free abelian subgroup of Γ and the rank of $[\Gamma, \Gamma]$ is equal to 5. By definition (see [37])

$$\operatorname{rank} \Gamma = \operatorname{rank} \Gamma / [\Gamma, \Gamma] + \operatorname{rank} [\Gamma, \Gamma].$$

Then

$$\Gamma / [\Gamma, \Gamma] \cong \mathbb{Z}$$

and consequently the first Betti number of M^6 is equal to 1.

This ends the sketch of the proof of $(1), (2), (3)$.

In order to prove (4), we consider the pair of Hermitian structures $(J_\pm, g)$ on the Lie algebra $\mathfrak{s}_{1,\frac{\pi}{2}}$, defined by the two integrable complex structures $J_\pm$ associated with the $(1, 0)$-forms

$$\begin{aligned}
&\omega_+^1 = e^1 + ie^2, \quad \omega_+^2 = e^3 + ie^4, \quad \omega_+^3 = e^5 + ie^6, \\
&\omega_-^1 = e^1 - ie^2, \quad \omega_-^2 = e^3 + ie^4, \quad \omega_-^3 = e^5 + ie^6.
\end{aligned}$$

and compatible with the inner product

$$g = \sum_{\alpha=1}^{6} e^\alpha \otimes e^\alpha.$$

By a direct computation we get that

$$d^c_+ F_+ = -d^c_- F_- = e^1 \wedge e^3 \wedge e^4,$$

which is a closed (but non-exact) 3-form. Therefore, the corresponding left-invariant generalized Kähler structure on M^6 is twisted.

The metric g is not flat since the Ricci component $\mathrm{Ric}(e_2, e_2) = -\frac{3}{2}$. Moreover, the Hermitian structures $(J_\pm, g)$ are not locally conformally Kähler since

$$dF_\pm = e^2 \wedge e^3 \wedge e^4,$$

according to the general result in [1, Remark 1] and [16] that a strong Kähler with torsion (non-Kähler) metric on a compact manifold of dimension greater than 4 cannot be locally conformally Kähler.

The previous construction of the 6-dimensional solvmanifold can be extended in order to get a non-trivial generalized Kähler structure on a $\mathbb{T}^{2n}$-bundle over an Inoue surface S_M, by considering the solvable Lie algebra of dimension $2n + 4$

$$\begin{cases} de^1 = a\, e^1 \wedge e^2, \\ de^2 = 0, \\ de^3 = \frac{1}{2} a\, e^2 \wedge e^3, \\ de^4 = \frac{1}{2} a\, e^2 \wedge e^4, \\ de^{2k+3} = b\, e^2 \wedge e^{2k+4}, \\ de^{2k+4} = -b\, e^2 \wedge e^{2k+3}, \quad k = 1, \dots, n, \end{cases}$$

with $a = 1$ and $b = \frac{\pi}{2}$.

In this case the two integrable complex structures $J_\pm$ are given by setting

$$\begin{aligned} &\omega^1_+ = e^1 + ie^2, \quad \omega^2_+ = e^3 + ie^4, \quad \omega^{k+2}_+ = e^{2k+3} + ie^{2k+4}, \\ &\omega^1_- = e^1 - ie^2, \quad \omega^2_- = e^3 + ie^4, \quad \omega^{k+2}_- = e^{2k+3} + ie^{2k+4}, \quad k = 1, \dots, n, \end{aligned}$$

as the associated $(1, 0)$-forms.

Acknowledgements. Special thanks go to the organizers of the Workshop "Recent advances in Differential Geometry, International Conference in honour of Prof. O. Kowalski".

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