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# ABOUT THE EXISTENCE OF MULTIPLE SOLUTIONS FOR A CLASS OF BOUNDARY VALUE PROBLEMS WITH DISCONTINUOUS NONLINEARITIES (°).

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Sunto. *In questo lavoro si estende al caso in cui la nonlinearità è discontinua alcuni recenti risultati sull'esistenza di soluzioni multiple per il problema al contorno (di Dirichlet o di Neumann)*

$$Au(x) = f(u(x)) + k(x) + t\,\Psi(x) \quad \text{in } \Omega\,, \quad Bu = 0 \quad \text{su } \partial\Omega\,,$$

*in funzione del parametro reale* t.

*Si prova anche con un esempio che tale estensione non si ha se non si intende la soluzione in un opportuno senso multivoco, precisamente nel senso che*

$$Au(x) - k(x) - t\Psi(x) \in \hat{f}(u(x)),$$

*dove* $\hat{f}(s)$ *è l'intervallo chiuso avente per estremi i limiti di* f *per* $\sigma \to s$ *da destra e da sinistra.*

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## INTRODUCTION AND STATEMENT OF THE MAIN RESULTS.

- $\Omega$ be a bounded open domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$ ,(for example of class $C^2$), and let A be a linear uniformly elliptic differential operator of the second order formally defined by

$$A(u) = -\sum_{ij} D_i(a_{ij}D_ju) + \sum_i D_i(a_iu) + a_0u.$$

where $a_0 \in L^\infty(\Omega)$ and the coefficients $a_i$ and $a_{ij}=a_{ji}$ are continuosly differentiable in $\Omega$ . Let us denote by B either the Dirichlet or the Neumann boundary operator.

Moreover let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a mapping of bounded variation on all compact intervals such that f(s) belongs to the interval whose extremes are $f(s-) = \lim_{\varepsilon\to 0^+} f((s-\varepsilon)$ and $f(s+) = \lim_{\varepsilon\to 0^+} f(s+\varepsilon)$ for all $s \in \mathbb{R}$.

For a fixed $k \in L^p(\Omega)$ with $p>n$, and for a fixed $\psi \in C(\overline{\Omega})$, $\psi(x) > 0$ in $\Omega$ ,we are interested in the solvability of the boundary value problem

$$(\widetilde{P})_t \qquad \begin{cases} Au(x) - k(x) - t\psi(x) \in \widetilde{f}(u(x)) & \text{in } \Omega \\ \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

where we have put $\widetilde{f}(s) = \{f(s)\}$ if $f(s-) \leq f(s+)$ and $\widetilde{f}(s) = [f(s+),f(s-)]$ if $f(s+) < f(s-)$.

In this frame, following an approach due to Kazdan and Warner, [7 ], and a result of Stuart, [10], [11] , we have the following

THEOREM 1. *Assume that*

i) $$\limsup_{s\to-\infty} f(s)/s < \lambda_1 < \liminf_{s\to+\infty} f(s)/s \ ,$$

*where* $\lambda_1$ *is the first eigenvalue of* $Au = \lambda u$ *in* $\Omega$, $Bu = 0$ *on* $\partial\Omega$. *Then there exists* $t_0$ *such that* $(\widetilde{P})_t$ *has no solution for* $t > t_0$ *and at least one solution for* $t < t_0$.

In the case when $f$ is continuously differentiable and satisfies i) and

ii) $$\limsup_{s\to+\infty} f(s)/s < +\infty ,$$

then a multiplicity result holds, in the sense that the boundary value problem $(\widetilde{P})_t$ has at least two solutions for $t < t_o$, and at least one solution for $t = t_0$ (see Dancer, [5], Amann and Hess, [3], and their remarks concerning related researchs).

However such a multiplicity result does not hold in the discontinuous case as we show in Section 2. Therefore, if we wish to give a result of the Amann-Hess type, we have to consider a weaker notion of solution, which is obtained by filling up all the jumps of $f$ rather than only the downward ones. In other words we study the boundary value problem

$$(\hat{P})_t \qquad \begin{cases} Au(x) - k(x) - t\psi(x) \in \hat{f}(u(x)) & \text{in } \Omega \\ \\ Bu = 0 & \text{on } \partial\Omega , \end{cases}$$

where we have denoted by $\hat{f}(s)$ the convex hull of $\{f(s-), f(s+)\}$ for all $s \in \mathbb{R}$.

Then we have the following

THEOREM 2. *Assume that* $f$ *has only finitely many discontinuities in any compact interval and that all the jumps are upward. Moreover assume that* $f$ *satisfies conditions i) , ii) and*

iii) *for some* $M,r>0$ *the mapping* $f(s)+Ms$ *is increasing on* $[-r,r]$.

*Then there exists* $t_0$ *such that the Dirichlet boundary value problem*

$$\begin{cases} Au(x) - k(x) - t\psi(x) \in \hat{f}(u(x)) & \textit{in} \quad \Omega \\ u = 0 & \textit{on} \quad \partial\Omega \quad , \end{cases}$$

*has no solution for* $t > t_0$, *at least one solution for* $t = t_0$ *and at least two solutions for* $t < t_0$.

*The same result holds in the Neumann case if we replace iii) with the stronger assumption*

iii)' *for all* $r > 0$ *there exists* $M_r > 0$ *such that* $f(s)+M_r s$ *is increasing in the interval* $[-r,r]$ .

Section 1. PROOF OF THEOREM 1.

First of all notice that, by replacing (if necessary) $A$ with $A+\lambda I$ and $f$ with $f+\lambda I$, we can always assume that $\lambda_1 > 0$.

For all $k,q$ let us denote by $\|\cdot\|_{k,q}$ the usual norm in the Sobolev space $W^{k,q}(\Omega)$, and by $\|\cdot\|_q$ the usual norm in $L^q(\Omega)$. Moreover $\|\cdot\|$ will denote the norm in $C^1(\bar{\Omega})$. Since in the sequel we shall always assume that $p>n$, we have the compact imbedding of $W^{2,p}(\Omega)$ into $C^1(\bar{\Omega})$.

By a subsolution (supersolution) of $(\tilde{P})_t$ we mean an element of $W^{2,p}(\Omega)$ such that

$$\begin{array}{lll} Au(x)-(k+t\psi)(x) \le \max \tilde{f}(u(x)) & (\text{respectively} \ge \min \ \tilde{f}(u(x))) & \text{in} \quad \Omega \\ Bu \le 0 & (\text{respectively} \ge 0) & \text{on} \quad \partial\Omega . \end{array}$$

Evidently $u \in W^{2,p}(\Omega)$ is a solution of $(\widetilde{P})_t$ if and only if u is a sub- and a supersolution for $(\widetilde{P})_t$.

The assertion of Theorem 1 easily follows from the following three lemmas.

LEMMA 1.1. *Let* $I = \{t \mid (\widetilde{P})_t$ *has a supersolution*$\}$. *Then* I *is a nonempty interval which is unbounded from bellow and is bounded from above.*

LEMMA 1.2. *For all* $K, t \in \mathbb{R}$, *there exists a subsolution* v *of* $(\widetilde{P})_t$ *such that* $v(x) \leq K$ *in* $\bar{\Omega}$.

LEMMA 1.3. *If* v *and* w *are a sub- and a supersolution for* $(\widetilde{P})_t$ *and* $v \leq w$, *then* $(\widetilde{P})_t$ *has a solution* u *with* $v \leq u \leq w$.

Infact put $t_0 = \sup I$. Then for every $t > t_0$ the boundary value problem $(\widetilde{P})_t$ has no supersolution and therefore no solution. On the other hand for $t < t_0$, there exists a supersolution w of $(\widetilde{P})_t$, and by Lemma 1.2 there exists a subsolution v of $(\widetilde{P})_t$ with $v \leq -\|w\| \leq w$. Hence by Lemma 1.3 there exists a solution u of $(\widetilde{P})_t$ with $v \leq u \leq w$.

The proof of Lemma 1.1 and Lemma 1.2 is essentially contained in [7], whereas Lemma 1.3 is a result due to C.Stuart, [10], [11]. (Notice that Stuart considers only the case when B is the Dirichlet boundary operator, but his arguments can be easily extended to the Neumann case).

## Section 2. -

We give now an example of boundary value problems of type $(\widetilde{P})_t$, which shows that the multiplicity result of Amann and Hess, [3], cannot be extended to this kind of problems.

Consider the Neumann boundary value problem

$$(\widetilde{N})_t \qquad -u''(x) - t \in \widetilde{f}(u(x)) \quad \text{for} \quad x \in ]0,1[,\ u'(0) = u'(1) = 0,$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(s) = s+1$ for $s > 0$, $f(s) = -s-1$ for $s<0$ and $f(0) = \varepsilon \in ]-1,1[$.

Evidently $f$ is a mapping of bounded variation on compact intervals such that conditions i) and ii) hold and $f(s)$ belongs to the interval of extremes $f(s-)$, $f(s+)$ for all $s \in \mathbb{R}$. Then by Theorem 1 there exists $t_0$ such that $(\widetilde{N})_t$ is solvable for $t < t_0$ and is not solvable for $t > t_0$. We have $t_0 \geq 1$, since the constant mapping $u(x) = t - 1$ is a solution of $(\widetilde{N})_t$ for all $t \leq 1$. We shall prove that $(\widetilde{N})_t$ has a unique solution for $t = -1$, which proves our statement.

In fact let $u \in W^{2,p}(]0,1[) \subset C^1([0,1])$ be a solution of $(\widetilde{N})_{-1}$, i.e. such that

$$-u''(x) + 1 = \begin{cases} u(x) + 1 & \text{in } \{x \in [0,1] \, u(x) > 0\}, \\ \varepsilon \in ]-1,1[ & \text{in } \{x \in [0,1] \, u(x) = 0\}, \\ -u(x) - 1 & \text{in } \{x \in [0,1] \, u(x) < 0\}. \end{cases}$$

We shall prove that $u(x) = -2$. To do this it is sufficient to prove that $u(x) \leq 0$ in $[0,1]$. In fact one has that meas $\{x \in [0,1] | u(x) = 0\} = 0$, (since $\varepsilon \neq 1$), and therefore $u$ is the unique solution of the boundary value problem $u''(x) - u(x) - 2 = 0$ in $]0,1[$, $u'(0) = u'(1) = 0$, and this is just the constant mapping $u(x) = -2$.

To prove that $u(x) \leq 0$ in $[0,1]$, assume on the contrary that $u(x) > 0$ for some $x \in [0,1]$. Then there exist $0 \leq x_0 < x_1 \leq 1$ such that $u(x) > 0$

in $]x_0, x_1[$, and $u(x_0) = 0$, (respectively $u(x_1) = 0$), if $x_0 > 0$, (respectively $x_1 < 1$). Then u is a solution in $]x_0, x_1[$ of $u'' + u = 0$, i.e. $u(x) = c_1 \cos x + c_2 \operatorname{sen} x$ where the constants $c_1, c_2$ are such that

$$u'(0) = u'(1) = 0 \qquad \text{if } x_0 = 0,\ x_1 = 1,$$

$$u'(0) = 0,\ u(x_1) = 0 \qquad \text{if } x_0 = 0,\ x_1 < 1,$$

$$u(x_0) = 0,\ u(x_1) = 0 \qquad \text{if } 0 < x_0,\ x_1 < 1,$$

$$u(x_0) = 0,\ u'(1) = 0 \qquad \text{if } 0 < x_0,\ x_1 = 1.$$

But since $0 < x_1 - x_0 \leq 1 < \pi/2$, it is easy to see that $c_1 = c_2 = 0$ in all cases, so that $u(x) = 0$ in $]x_0, x_1[$, contraddicting our assumption.

Section 3. - PROOF OF THEOREM 2.

Without lost of generality we can assume that $f(s) = f(s-) = \min \hat{f}(s)$ for all $s \in \mathbb{R}$. Then if we put $t_0 = \sup\{t \mid (\tilde{P})_t$ has a supersolution$\}$, since any solution of $(\tilde{P})_t$ is a solution of $(\hat{P})_t$ and any solution of $(\hat{P})_t$ is a supersolution for $(\tilde{P})_t$, from Theorem 1 it follows that $(\hat{P})_t$ is solvable for $t < t_0$, and not solvable for $t > t_0$. Hence we have only to prove that

(I) $(\hat{P})_t$ is solvable for $t = t_0$,

(II) $(\hat{P})_t$ has at least two solutions for $t < t_0$.

The proof of (I) is essentially contained in [3], and we shall not reproduce it here.

To prove (II) assume on the contrary that for some $t^* < t_0$ the problem $(\hat{P})_{t^*}$ has a unique solution $u^*$. We shall reach a contraddiction through several steps.

First of all notice that by assumptions i) and ii), and by the fact that $f$ is of bounded variation on compact intervals, there exist $\lambda'$, $\lambda''$, $c$ such that $0 < \lambda' < \lambda_1 < \lambda''$ and

$$f(s) > \max\ (\lambda' s + c, \lambda'' s + c) \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.1}$$

LEMMA 3.1. *We can assume that there exists* $K \in \mathbb{R}$ *such that* $f$ *is continuous in* $K$ *and* $f(s) = \text{cost.} = f(K)$ *for all* $s \leq K$.

*Proof.* Let $u_1$ be the solution of $Au(x) - \lambda' u(x) = c+k(x) + (t^*-1)\psi(x)$ in $\Omega$, $Bu = 0$, and let $K < -\| u_1 \|$ such that $f$ is continuous in $K$. Then we consider the boundary value problem

$$(\hat{P}^*)_t \quad \begin{cases} Au(x) - k(x) - t\psi(x) \in \hat{f}^* u(x)) & \text{in } \Omega \\ \\ Bu = 0 & \text{on } \partial\Omega \end{cases},$$

where $f^*$ is defined by $f^*(s) = f(s)$ for $s > K$ and $f^*(s) = f(K)$ for $s \leq K$. By (3.1) and by the maximum principle one easily proves that for $t > t^*-1$ any solution of $(\hat{P})_t$ is a solution of $(\hat{P}^*)_t$ and viceversa, (see [4] section 3, for a similar situation). Since in the sequel we shall be interested in the solvability of $(\hat{P})_t$ for $t^*-1 < t < t_0+1$, we can replace $f$ with $f^*$ and the lemma is proved.

Now, since $f$ is of bounded variation on compact intervals and all the jumps are upward, $f$ has the form $f = g - h$, where $g$ and $h$ are strictly increasing and $h$ is continuous, (see [10], [11]).

LEMMA 3.2. *There exist a subsolution* $v$ *and a supersolution* $w$ *of* $(\tilde{P})_{t^*}$ *such that* $v(x) \leq \min(K, w(x))$ *and* $u^*$ *is interior to the set*

$$X = \left\{ u \in C^1(\bar\Omega) \,\middle|\, Bu = 0,\ v \le u \le w \right\}.$$

*Proof.* For an $\varepsilon$ sufficiently small by Theorem 1 there exists a solution $w$ of $(\tilde P)_{t^*+\varepsilon}$; then $w$ is a supersolution of $(\tilde P)_{t^*}$. Then by Lemma 1.2 there exists a subsolution $v$ of $(\tilde P)_{t^*}$ such that $v \le \min(K, -\|w\|, -\|u^*\|-1)$ and therefore $v \le w$. Since $v \le w$ and $u^*$ is the unique solution of $(\tilde P)_{t^*}$, we have that $v \le u^* \le w$, and therefore $u^* \in X$. Moreover, since $v \le -\|u^*\|-1$, one has $v(x) < u^*(x)$ in $\bar\Omega$; hence for some $r > 0$ we have that

$$z \in C^1(\bar\Omega),\ \| z - (u^* - v) \| < r \Rightarrow z \ge 0. \tag{3.2}$$

Therefore in order to prove that $u^*$ is interior to $X$ it is sufficient to prove that there exists $r > 0$ such that

$$z \in C^1(\bar\Omega),\ Bz = 0,\ \| z - (w - u^*) \| < r \Rightarrow z \ge 0. \tag{3.3}$$

We prove first that $u^*(x) < w(x)$ in $\Omega$. Assume on the contrary that $u^*(x_0) = w(x_0)$ for some $x_0 \in \Omega$. By the continuity of $h, u^*, w$ and $\psi$ there exists $r > 0$ such that $B(x_0,r) \subset \Omega$ and $h(u^*(x)) - h(w(x)) + \varepsilon\psi(x) > 0$ for all $x \in B(x_0,r)$. Now, since $u^*$ and $w$ are solutions of $(\tilde P)_{t^*}$ and $(\tilde P)_{t^*+\varepsilon}$ respectively and $f$ has no downward jump, then we have $(u^*-w)(x) \lneq 0$ on $\partial B(x_0,r)$ and $A(u^*-w)(x) = f(u^*(x)) - f(w(x)) - \varepsilon\psi(x) = g(u^*(x)) - g(w(x)) - h(u^*(x)) + h(w(x)) - \varepsilon\psi(x) < 0$ in $B(x_0,r)$. From this, by the maximum principle, it follows $u^*(x) < w(x)$ in $B(x_0,r)$ contrarily to our assumption.

We distinguish now the case that $B$ is the Dirichlet operator from the case when $B$ is the Neumann one. In the first case we have $u^*(x) =$

$w(x) = 0$ on $\partial\Omega$. Then with the same argument of Lemma 1 of [4] one proves that $\partial(u^* - w)/\partial\nu > 0$ on $\partial\Omega$, and by Lemma 2 of [4] one has that (3.3) holds for some $r > 0$.

In the case that $Bu = \partial u/\partial\nu$, then we prove that $u^*(x) < w(x)$ in $\Omega$, and from this (3.3) easily follows. In fact let $r > 0$ such that $|u^*(x)| \leq r$ and $|w(x)| \leq r$ for all $x \in \bar{\Omega}$. Then we have by iii)' that $f(u^*(x)) - f(w(x)) \leq -M_r(u^*(x) - w(x))$ and therefore $A(u^*-w)(x)+M_r(u^*-w)(x) \leq A(u^*-w)(x)-f(u^*(x)) + f(w(x)) = -\varepsilon\,\psi(x) < 0$ for all $x \in \Omega$. From this, since we have $\partial(u-w)/\partial\nu = 0$ on $\partial\Omega$, by the strong maximum principle, (see [2], Lemma 1), it follows that $u^*(x) < w(x)$ in $\bar{\Omega}$, as desired.

LEMMA 3.3. *There exists a sequence* $(f_n)_n$ *of continuous mappings such that:*

(3.4) $\quad f_n(s) \leq f(s-) = f(s),$

(3.5) $\quad f_n(s) \geq \max\,(\lambda's+c), \lambda''s+c),$

(3.6) $\quad g_n = f_n + h$ *is increasing,*

(3.7) $\quad s_n \longrightarrow s \Rightarrow \left[\liminf_{n\to\infty} f_n(s_n), \limsup_{n\to\infty} f_n(s_n)\right] \subset \hat{f}(s).$

*Proof.* Let $z_1 < z_2 < \dots z_i < \dots$ be the sequence of the points where $f$ is discontinuous; (obviously $K < z_1$). Let us fix $i$ and put for all $n \in N$ and for all $s \in [z_i, z_{i+1}]$, (with the convention that $z_{i+1} = z_i + 1$ if $f$ is continuous on $[z_i, +\infty[$):

$$\gamma_{i,n}(s)=(2n/(z_{i+1}-z_i))(f(z_i+(z_{i+1}-z_i)/2n)-f(z_i))(s-z_i)+f(z_i)$$

$$\delta_{i,n}=\sup\{z \in [z_i, z_i+(z_{i+1}-z_i)/2n] \mid \gamma_{i,n}(s) \leq f(s) \quad \text{for} \quad s \in [z_i, z]\}$$

$$f_{i,n}(s) = \begin{cases} \gamma_{i,n}(s) & \text{for } s \in \left[z_i, \delta_{i,n}\right] \\ \\ f(s) & \text{for } s \in \left[\delta_{i,n}, z_{i+1}\right]. \end{cases}$$

It is easily seen that $f_{i,n}$ is continuous in $[z_i, z_{i+1}]$. On the other hand since $f$ is continuous on the left, one has $f_{i,n}(z_{i+1}) = f(z_{i+1}) = f_{i+1,n}(z_{i+1})$.

Hence the mapping $f_n$ defined by $f_n(s) = f(s)$ if $s < z_1$ and $f_n(s) = f_{i,n}(s)$ if $s \in [z_i, z_{i+1}]$, is continuous. Moreover one easily proves that $f_n$ satisfies conditions (3.4), (3.5) and (3.6). Finally assume that $s_n \to s$. If $s \neq z_i$ for all i, we have $f_n(s_n) = f(s_n)$ for n great and therefore $f_n(s_n) \to f(s) \in \hat{f}(s)$. Let now $s = z_i$ for some i and let $(f_{n_k}(s_{n_k}))_k$ be a converging subsequence. If the set $\{k \mid s_{n_k} \leq z_i\}$ is infinite, then one has $\lim_{k\to\infty} f_{n_k}(s_{n_k}) = f(z_i) \in \hat{f}(s)$. If the set $\{k \mid s_{n_k} \leq z_i$ is finite, then for k great one has $z_i < s_{n_k}$, and therefore by (3.4), (3.6) one has

$f(z_i) = f_{n_k}(z_i) = g_{n_k}(z_i) - h(z_i) < g_{n_k}(s_{n_k}) - h(z_i) = f_{n_k}(s_{n_k}) + h(s_{n_k}) - h(z_i) \leq f(s_{n_k}) + h(s_{n_k}) - h(z_i)$. From this and from the continuity of h it follows that $f(s-) \leq \lim f_{n_k}(s_{n_k}) \leq f(s+)$, which proves the assertion.

LEMMA 3.4. *Put for convenience* $f_0 = f$ *and consider for all* $n \in N \cup \{0\}, t \in [t^*-1, t_0+1]$ *the boundary value problem*

$$(\hat{P}_n)_t \qquad Au(x) - k(x) - t\psi(x) \in \hat{f}_n(x)) \quad \textit{in } \Omega\ , \ Bu = 0 \quad \textit{on } \partial\Omega .$$

*Then there exists* $\rho_0 > 0$ *such that for any solution* $u$ *of* $(\hat{P}_n)_t$ *one has* $\| u \|_{2,p} \leq \rho_0$.

*Proof.* Assume on the contrary that for all $j \in \mathbb{N}$ there exist $t_j \in [t^*-1, t_0+1]$, $n_j \in \mathbb{N} \cup \{0\}$ such that $(\hat{P}_{n_j})_{t_j}$ has a solution $u_j$ with $\|u_j\|_{2,p} > j$. Since $f_{n_j}(s) \leq \lambda' s + c$ for all $s \in \mathbb{R}$, by the maximum principle one has again $u_j \geq K$, for all $j \in \mathbb{N}$. On the other hand, by ii) and the fact that $f$ is of bounded variation on compact intervals, there exist $\lambda > 0, c' > 0$ such that $f(s) < \lambda s + c'$ for all $s > K$. From this and from (3.4) it follows that $f_n(s+) \leq \lambda s + c'$ for all $s > K$, and therefore

$$\lambda'' u_j(x) + c \leq f_{n_j}(u_j(x)) \leq f_{n_j}(u_j(x)+) \leq \lambda u_j(x) + c' \tag{3.8}$$

for all $j \in \mathbb{N}$. Hence, if we put $\hat{u}_j = u_j / \| u_j \|_{2,p}$, $w_j = A\hat{u}_j + (t_j\psi + k)/\|u_j\|_{2,p}$, we have

$$\lambda''(\hat{u}_j(x) + c/ \|u_j\|_{2,p} \leq w_j(x) \leq \lambda \hat{u}_j(x) + c'/\| u_j\|_{2,p}. \tag{3.9}$$

From this it follows that $(Au_j)_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega)$. Since the inverse $A^{-1}$ is compact from $L^p(\Omega)$ to $\{u \in C^1(\overline{\Omega})\ Bu = 0\}$, we can assume that $(\hat{u}_j)_{j\in\mathbb{N}}$ converges to $\hat{u}$ in $C^1(\overline{\Omega})$. Moreover by the maximum principle $A^{-1}$ is increasing. Hence by (3.9) one has

$$A^{-1}(\lambda''\hat{u}_j + c/ \| u_j \|_{2,p}) \leq A^{-1}(w_j) = \hat{u}_j - A^{-1}(t_j\psi + k)/\| u_j \|_{2,p}.$$

By passing to the limit one has $A^{-1}(\lambda''\hat{u}) \leq \hat{u}$ i.e. $(I - \lambda'' A^{-1})\hat{u} \geq 0$, where $\hat{u} \neq 0$, since $\| \hat{u} \|_{2,p} = \lim_{j\to\infty} \| \hat{u}_j \|_{2,p} = 1$, and $\hat{u}(x) = \lim_{j\to\infty} u_j(x)/\|u_j\|_{2,p} \geq 0$, for all $x \in \Omega$ . This is impossible since $A^{-1}$ is a strongly positive compact endomorphism on the space $\{u \in C^1(\bar{\Omega}) \mid Bu = 0\}$, and $\lambda'' > \lambda_1$, (see [1], Theorem 3.2).

LEMMA 3.5. *Let* $t \in [t^*-1, t_0+1]$, *and let* $Y$ *be a closed subset of the space* $\{u \in C^1(\bar{\Omega}) \mid Bu=0\}$. *For all* $j$ *let* $u_j \in Y$ *be a solution of* $(\hat{P}_{n_j})_t$ *where* $(n_j)_{j\in\mathbb{N}}$ *is a strictly increasing sequence of natural numbers.*

*Then* $(\hat{P})_t$ *has a solution in* $Y$.

*Proof*. By Lemma 3.4 one has $\| u_j \|_{2,p} \le \rho_0$ for all $j \in \mathbb{N}$. Since $W^{2,p}(\Omega)$ is reflexive and is compactly imbedded into $C^1(\bar{\Omega})$, we can assume that $(u_j)_j$ converges weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\bar{\Omega})$ to a mapping $u \in W^{2,p}(\Omega)$. From this it follows that $u \in Y$ and $Bu = 0$. On the other hand, by (3.8) we have that $(f_{n_j}(u_j))_j$ is bounded in $L^p(\Omega)$ and therefore we can assume that $(f_{n_j}(u_j))_j$ is weakly convergent in $L^p(\Omega)$ to a mapping $\theta$ . From this and from Fatou's lemma it follows that $\liminf_{j\to\infty} f_{n_j}(u_j(x)) \le \theta(x) \le \limsup_{j\to\infty} f_{n_j}(u_j(x))$, and therefore by (3.7) one has $\theta(x) \in \hat{f}(u(x))$.

The proof will be reached if we prove that $Au(x) = t\psi(x)+k(x)+\theta(x)$.

In fact all $\Phi \in \mathring{W}^{1,p}(\Omega)$ or $\Phi \in W^{1,p}(\Omega)$ according to whether B is the Dirichlet or the Neumann boundary operator, one has

$$\sum_{i\ell} \int_\Omega a_{i\ell} D_i u_j D_\ell \Phi + \sum_i \int_\Omega D_i(a_i u_j)\Phi + \int_\Omega a_0 u_j \Phi =$$

$$= \int_\Omega (t\psi + k + f_{n_j}(u_j))\Phi$$

for all $j \in \mathbb{N}$. Passing to the limit as $j \longrightarrow \infty$, one has

$$\sum_{i\ell} a_{i\ell} D_i u D_\ell \Phi + \sum_i \int_\Omega D_i (a_i u) \Phi + \int_\Omega a_0 u \Phi =$$

$$= \int_\Omega (t\psi + k + \theta) \Phi \ .$$

From this, since $u \in W^{2,p}(\Omega)$ and $Bu = 0$, one has the assertion.

REMARK 3.1. *By the imbedding of $W^{2,p}(\Omega)$ into $C^1(\overline{\Omega})$ and by lemma 3.4 there exists a constant, (which will be denoted again by $\rho_0$), such that for any solution $u$ of $(\hat{P}_n)_t$ with $t \in [t^*-1, t_0+1]$ one has $\|u\| \leq \rho_0$.*

In the sequel the balls considered have to be intended into the space $\{u \in C^1(\overline{\Omega}) \mid Bu = 0\}$.

LEMMA 3.6. *Let $\rho_1 > 0$ such that $B(u^*, \rho_1) \subset X$, (see lemma 3.2). Then there exists $n_0$ such that for all $n \geq n_0$ one has:*

I) $(\hat{P}_n)_{t_0+1}$ *has no solution,*

II) *every solution of $(\hat{P}_n)_{t^*}$ belongs to $B(u^*, \rho_1)$.*

*Proof of* I). Assume on the contrary that there exists a strictly increasing sequence $(n_j)_j$ of natural numbers such that $(\hat{P}_{n_j})_{t_0+1}$ has a solution $u_j$ for all $j$. Then by Remark 3.1 we have $\| u_j \| \leq \rho_0$; from this and from Lemma 3.5 it follows that $(\hat{P})_{t_0+1}$ has a solution $u \in \overline{B}(0, \rho_0)$, contraddicting the definition of $t_0$.

*Proof of* II). Assume on the contrary that there exists a strictly increasing sequence $(n_j)_{j \in \mathbb{N}}$ of natural numbers such that $(\hat{P}_{n_j})_{t^*}$ has a solution $u_j$ in $\overline{B}(0, \rho_0) \setminus B(u^*, \rho_1)$ for all $j$. Then by Lemma 3.5 there exists a solution of

$(\hat{P})_{t*}$ in $\bar{B}(0, \rho_0)\setminus B(u^*, \rho_1)$, contraddicting the assumption that $u^*$ is the unique solution of $(\hat{P})_{t*}$.

REMARK 3.2. *By modifying, (if necessary), the mapping* g *and* h *in* $\mathbb{R}\setminus]\rho,\rho[$, *where* $\rho = \max(\rho_0, \|v\|, \|w\|)$, *we can assume from now on that* h *is continuous, strictly increasing and bounded. (Obviously now* g *and* $g_n = f_n + h$ *are increasing only into the interval* $]-\rho,\rho[$*). Then we have that the map* $A+H: \{u \in W^{2,p}(\Omega) \mid Bu=0\} \longrightarrow L^p(\Omega)$, *(where* H *is the Nemytskii operator associated to* h*), is onto by a degree argument and is one-to-one by the maximum principle. Moreover its inverse* $(A+H)^{-1}$ *is continuous from* $L^p(\Omega)$ *to* $W^{2,p}(\Omega)$ *and therefore completely continuous from* $L^p(\Omega)$ *to the space* $E = \{u\in C^1(\Omega) \mid Bu = 0\}$.

In fact assume on the contrary that $(A+H)^{-1}$ is not continuous in some $z\in L^p(\Omega)$. Then there exist $\varepsilon > 0$ and a sequence $z_n \in L^p(\Omega)$ such that $z_n\to z$ as $n\longrightarrow\infty$ and $\|(A+H)^{-1}(z_n) - (A+H)^{-1}z\|_{2,p} \geq \varepsilon$ for all n. If we put $u_n = (A+H)^{-1}(z_n)$, then we have that the sequence $Au_n=H(u_n)+z_n$ is bounded in $L^p(\Omega)$. Therefore by the compactness of $A^{-1}:L^p(\Omega)\longrightarrow E$, there exists a subsequence $u_{n_k}$ which converges to some $u\in E$.

Hence by the continuity of H, we have that $Au_{n_k}\longrightarrow z-H(u)$. From this by the continuity of $A^{-1}$ from $L^p(\Omega)$ to $W^{2,p}(\Omega)$, it follow that $u\in W^{2,p}(\Omega)$ and $u_{n_k}\longrightarrow u=A^{-1}(z-Hu)$ in $W^{2,p}(\Omega)$. Hence one has $u = (A+H)^{-1}(z)$ and $u_{n_k}=(A+H)^{-1}(z_{n_k})$ converges to u, contraddicting the assumption $(A+H)^{-1}(z_n)-u\|_{2,p} \geq \varepsilon$.

Finally notice that by using the maximum principle one easily has that $(A+H)^{-1}$ is an increasing mapping.

LEMMA 3.7. *For all* $n\in\mathbb{N}$ *and for all* $t\in[t^*-1,t_0+1]$ *put*

$$G_n(t,u)(x)=g_n(u(x))+t\psi(x)+k(x) \quad \textit{for all} \quad x\in\Omega .$$

*Then the mapping* $(A+H)^{-1}G_n(t,.)$ *is completely continuous in* E.

*Moreover for* n *great one has*

$$\deg(I-(A+H)^{-1}G_n(t,.)\ B(u,\rho_1),0) = 1.$$

*Proof.* The first part of the assertion follows from Remark 3.2 and from the continuity of $g_n$.

Now put $T_n = (A+H)^{-1}G_n(t^*,.)$ for all n. By (3.4) and the fact that $f_n(s) = f(s)$ for $s \leq K$ and $v(x) \leq K$ in $\bar{\Omega}$, we have that w and v are respectively a supersolution and a subsolution for $(\tilde{P}_n)_{t^*}$ for all n, and therefore we have $v \leq T_n(v)$ and $T_n(w) \leq w$. On the other hand, since $g_n$ is increasing in $]-\rho,\rho[$, we have that $T_n$ is increasing in X. Therefore one has that $T_n(X) \subset X$. Finally since X is bounded in $C(\bar{\Omega})$, by the continuity of $g_n$ and the complete continuity of $(A+H)^{-1}G_n(t,.)$, we have that $T_n(X)$ is relatively compact. Let $R_n>\rho_1$ such that $T_n(X)\subset B(u^*,R_n)$ and let r be a retraction of E on X, (see [6], Theorem 4.1). Then we have

$$u \neq \mu T_n(r(u))+(1-\mu)u \quad \text{for all} \quad u\in\partial B(u,R_n),\ \mu\in[0,1] ,$$

and therefore by the homotopy invariance property of the Leray-Schauder degree one has for all $n\in\mathbb{N}$

$$\deg(I-T_n r,B(u^*,R_n),0) = \deg(I-u^*,B(u^*,R_n),0) = 1.$$

Now let $n\in\mathbb{N}$ and $u \in B(u^*,R_n)$ such that $u = T_n(r(u))$. Then $u\in X$ and therefore $u = T_n(u)$ since $r(u)=u$. Hence by Lemma 3.6 one has $u\in B(u^*,\rho_1)$ for $n > n_0$. From this, from the properties of the degree and from the fact that $r(u) = u$ for all $u\in B(u^*,\rho_1)\subset X$, one has for $n>n_0$

$$\deg(I-T_n,B(u^*,\rho_1),0) = \deg(I-T_n r,B(u^*,\rho_1),0) =$$

$$\deg(I-T_n r,\ B(u^*,R_n),0) = 1.$$

We are now in a position of concluding the proof of Theorem 2. In fact by part (I) of Lemma 3.6 one has for all $n>n_0$

$$\deg(I-(A+H)^{-1}G_n(t_0+1,.),B(0,\rho_0),0) = 0,$$

and therefore

$$\deg(I-(A+H)^{-1}G_n(t^*,.),\ B(0,\rho_0),0) = 0$$

by the homotopy invariance property. From this and from part (II) of Lemma 3.6 it follows that

$$\deg(I-T_n,B(u^*,\rho_1),0 = \deg(I-T_n,B(0,\rho_0),0) = 0,$$

contraddicting the assertion of Lemma 3.7. Hence Theorem 2 is completely proved.

REFERENCES

[1] AMANN H. *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review, 18, (1976),620-709.

[2] AMANN H. - CRANDALL M.G. *On some existence theorems for semilinear elliptic equations*, Indiana Univ. Math.J., 27,(1978),779-790.

[3] AMANN H.- HESS P. *A multiplicity result for a class of elliptic boundary value problems*, Proc. Royal Soc. of Edinburgh, 84A,(1979),145-151.

[4] BASILE N.- MININNI M. *A multiplicity result for a class of elliptic B.V.P.s of the Ambrosetti-Prodi type*, Ann.Univ.di Ferrara, Sez.VII,26, (1980),91-101.

[5] DANCER E.N.*On the ranges of certain weakly nonlinear elliptic partial differential equations*, J.Math.Pures Appl., 57,(1978),351-366.

[6] DUGUNDJI J.*An extension of Tietze's theorem*, Pac.J.Math.,1,(1951),353--367.

[7] KAZDAN J.L.-WARNER F.W. *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math., 28,(1975),567-597.

[8] NECAS J. *Les Methodes Directes en Theorie des Equations Elliptiques*, Masson et C.ie, Paris, (1977).

[9] SCHWARTZ J.T. *Nonlinear Functional Analysis*, Gordon & Breach, New York (1968).

[10] STUART C.A. *Maximal and minimal solutions of elliptic differential equations with discontinuous nonlinearities*, Math. Z., 163, (1978), 239-249.

[11] STUART C.A. *Elliptic differential equations with discontinuous nonlinearities*, (preprint).

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