Note di Matematica Vol.I , 203-212 (1981)

COMPLETENESS FOR NON NORMAL INTUITIONISTIC MODAL LOGICS

Gisèle FISCHER SERVI

Sunto. *Facendo seguito a un precedente lavoro sulla completezza di certi calcoli modali intuizionisti "normali", si introducono dei modelli di tipo Kripke per una classe di calcoli modali intuizionisti non normali, pervenendo a un risultato di validità e completezza. Le tecniche usate per la completezza seguono lo schema algebrico di Lemmon,* [4] .

1. In this paper we consider a class of intuitionistic modal logics called ∗-IC's, where ∗ stands for one of the following non normal modal calculi: C2, D2, E3, ET, E2S, EB, E5, E, L (see [4] for a description of these systems). In [2] (1) we obtained model-theoretic characterizations of ∗-IC's where ∗ was normal. The non normal case will require some modifications but the general ideas are the same as in [2]. Although we will rely upon knowledge of the terminology and some of the proofs in [1,2], in order to make this paper more readable, we shall recall the following

(1) For complete proofs of theorems stated in [2], see also [1].

facts. Each $*$-IC is obtained by a traslation map T from $*$-IC formulas to formulas of (S4,$*$)-C, a classical bimodal calculus containing two primitive modal operators M_1 and M_2. Roughly speaking, the deductive structure of (S4,$*$)-C is determined by the axioms and rules of S4 (for M_1), the axioms and rules of the $*$-system (for M_2) and the following "connecting axiom schemes:

(1) $$M_2 M_1 a \longrightarrow M_1 M_2 a$$

(2) $$M_2 L_1 a \longrightarrow L_1 M_2 a .$$

Given a bimodal calculus (S4,$*$)-C, the logic $*$-IC is defined as the set of those formulas whose T-transforms are theorems of (S4,$*$)-C.

As in [2], our main concern will be to describe Kripke-type bimodal semantics. For, once a completeness theorem for (S4,$*$)-C is established, it is easy - following the guideline of thanslation T - to derive completeness for $*$-IC.

2. First we must extend all bimodal semantical notions to the non normal case. Thus, let us define a *$*$-generalized double model structure* ($*$-GDMS) as a quadruple $M = (S,Q,R_1,R_2)$ where S is a non-empty set $Q \subseteq S$ and R_1 and R_2 are relations on S such that R_1 is reflexive and transitive, R_2 has the properties required for the $*$-calculus (2) and

(3) *For all $m,n \in S$, if $m \in Q$ and mR_1n, then $n \in Q$;*

(4) *For all $m,n,p \in S$, if mR_1n and mR_2p and $n \notin Q$, then there is $u \in S$ such that nR_2u and pR_1u;*

(5) *For all $m,n,p \in S$, if mR_2nR_1p, then there exists $u \in S$ such that mR_1u and either $u \in Q$ or uR_2p.*

(2) e.g., if the Kripke model structures for $*$ are transitive, then R_2 is transitive.

As in [2], a *bimodal model* is a map $v : V \times S \longrightarrow \{0,1\}$, where V is the set of propositional variables. Given a bimodal model v, let v' be the extension of v to formulas, defined as in [2] as far as propositional connectives and M_1 are concerned; on the other hand, $v'(M_2\beta, m)$ is defined according to whether $m \in Q$ or not, namely for $m \notin Q$, $v'(M_2\beta, m)$ is defined as in [2], while if $m \in Q$, then $v'(M_2\beta, m) = 1$. Furthermore the notions of "verifying in m", "bimodal validity" and "(S4,∗)-C validity" are analogous to those given in [2]. Obviously, when $Q = \emptyset$, all these notions are just those given in [2].

THEOREM 1 (validity). *If* $\vdash_{(S4,*)-C} \alpha$, *then* α *is* (S4,∗)-C *valid.*

Proof. By induction on the length of the proofs in (S4,∗)-C. It is straightforward that the axioms and rules of S4 and those of the ∗ calculus are valid in all ∗-GDMS's. So, let $M = (S, C, R_1, R_2)$ be a ∗-GDMS, v be a bimodal model on M and $m \in S$. In order to prove that v verifies (1) in m, suppose first that $m \in Q$. Then $v'(M_2\alpha, m) = 1$ and since $m R_1 m$, $v'(M_1 M_2\alpha, m) = 1$. On the other hand if $m \notin Q$ and $v'(M_2 M_1\alpha, m) = 1$, then there are $n, p \in S$ such that $mR_2 nR_1 p$ and $v'(\alpha, p) = 1$. Now by (5) let $u \in S$ be such that $mR_1 u$ and either $u \in Q$ or $uR_2 p$. In both cases, $v'(M_2\alpha, u) = 1$ and $v'(M_1 M_2\alpha, m) = 1$.

As for (2), assume first that $m \in Q$. Consider any $n \in S$ such that $mR_1 n$; according to (3), $n \in Q$ and so $v'(M_2\alpha, n) = 1$; but n is arbitrary, hence $v'(L_1 M_2\alpha, m) = 1$. Assume now that $m \notin Q$ and

(6) $$v'(M_2 L_1\alpha, m) = 1.$$

We show that supposing

(7) $$v'(L_1 M_2\alpha, m) = 0$$

leads to a contradiction. By (6), there is $p \in S$ such that mR_2p and

(8) $$pR_1t \quad \textit{implies} \quad v'(\mathbf{a},t) = 1 \qquad (t \in S).$$

Now by (7), there exists $n \in S$ such that mR_1n and $v'(M_2\mathbf{a},n)=0$. This shows both that $n \notin Q$ and

(9) $$nR_2t \quad \textit{implies} \quad v'(\mathbf{a},t) = 0 \qquad (t \in S).$$

At this point we have both mR_2p and mR_1n with $n \notin Q$, so by (4) there is $t \in S$ such that pR_1t and nR_2t. Now, via (8) and (9), this leads to a contradiction.

Actually it is possible to show a stronger result. Consider a relational structure $M = (S,Q,R_1,R_2)$ such that $S \neq \emptyset$, $Q \subseteq S$, R_1 is a reflexive and transitive relation on S and R_2 is a $*$-relation on S. Note that it is easy to extend, with respect to this relational structure, the concepts of "model" and "bimodal validity" introduced formerly for $*$-GDMS's. Then we can prove the following

THEOREM 2. *Let* $M = (S,Q,R_1,R_2)$ *be as above. Then* M *is a* $*$-GDMS *if and only if* (1) *and* (2) *are bimodally valid in* M.

Proof. If M is a $*$-GDMS, then (1) and (2) are bimodally valid in M, by theorem 1. Conversely, assuming that either one of (3),(4),(5) does not hold for M, we shall prove that either (1) or (2) is not bimodally valid in M. If (3) does not hold in M, then there are $m,n \in S$ such that $m \in Q$, mR_1n and $n \notin Q$. Now let $a \in V$ and $v(a,p) = 0$ for all $p \in S$ such that nR_2p. Since $m \in Q$, $v'(M_2L_1a,m) = 1$ while $v'(M_2a,n) = 0$, which yields $v'(L_1M_2a,m) = 0$. Hence (2) is not bimodally valid in M. Similarly, suppose that condition (4) is not met in M , i.e. there are $m,n,p \in S$

such that $n \notin O$, $mR_1 n$, $mR_2 p$, but there is no $u \in S$ such that $nR_2 u$ and $pR_1 u$. For some $a \in V$, stipulate that $v(a,t) = 0$ for all $t \in S$ such that $nR_2 t$ and that $v(a, r) = 1$ for all $r \in S$ such that $pR_1 r$. Then clearly $v'(M_2 L_1 a,m) = 1$, while $v'(L_1 M_2 a,m) = 0$ so that (2) is not bimodally valid in M. Finally if (5) does hold in M, then $mR_2 nR_1 p$ for some $n,m,p \in S$ and

(10) $\quad$ *$u \notin Q$ and not $uR_2 p$, for each u such that $mR_1 u$.*

Now, for some $a \in V$, let

(11) $$v(a,p) = 1$$

and if $X = \{u : mR_1 u\}$, put

(12) $\quad$ *$v(a,t) = 0$ for all $t \in S$ such that $uR_2 t$, for some $u \in X$.*

Note that because of (10), conditions (11) and (12) are compatible. Using (12) and (10), we have $v'(M_2 a,u) = 0$ for every $u \in X$, so that $v'(M_1 M_2 a,m) = 0$. On the other hand from (11) we infer $v'(M_2 M_1 a,m) = 1$ and hence (1) is not bimodally valid in M.

Now recall that the Lindenbaum algebra of (S4,✱)-C is a ✱-bimodal algebra (see [4], [2]). Hence in order to prove completeness following Lemmon's techniques, we show first that (as in the normal case) the dual notion of a ✱-bimodal algebra is a ✱-GDMS. Before we proceed, let us modify some of the definitions introduced in [1]. So, let B be a Boolean algebra, K_1, K_2 operators on B with K_1 a hemimorphism, and let $R_i \subseteq S^2$, be the dual of K_i, (i = (1,2), where S is the Stone space of B. Let $Q \subseteq S$ be the following set

(13) $$Q = \left\{m : K_2 0 \in m\right\}$$

Furthermore let $\Phi : R \longrightarrow \mathcal{B}(S)$ be the Stone embedding, with $\mathcal{B}(S)$ and $\bar{K}_1$ defined as in [1], while

(14) $$\bar{K}_2 A = \left\{ m : m \in Q \quad \text{or there is } n \in A,\ mR_2 n \right\}.$$

As shown in [4], ϕ preserves K_1 and K_2. Last we recall a result due to R. J. BLATTER (see [3], p.165) to the effect that the direct image of a closed set under a Boolean relation is again closed. Actually Blattner does not use the full strenghth of his hypothesis, since his proof holds in fact for a relation which is dual to any function between Boolean algebras. Hence,

REMARK. *The direct image of a closed set under* R_2 *is closed.*

THEOREM 3. *If* $K_2 K_1 x \leq K_1 K_2 x$ *for all* $x \in B$, *then* R_1 *and* R_2 *satisfy condition* (5).

Proof. Let $mR_2\ nR_1\ p$ for all $m,n,p \in S$. As in [1], lemma 2, this yields that for all $x \in B$ if $p \in \Phi(x)$ then $m \in \phi(K_1\ K_2 x) = \bar{K}_1 \bar{K}_2 \phi(x)$. By definition of $\bar{K}_1$, we have that if $p \in \Phi(x)$, then there is $u \in \bar{K}_2\ \Phi(x)$ such that $mR_1\ u$. Hence by (14), for all $x \in B$ if $p \in \Phi(x)$, then for some u, $mR_1 u$ and either $u \in Q$ or there is $v \in \Phi(x)$ such that $uR_2\ v$. Now by predicate logic this yields that either one of the following holds

(15) *there is* $u \in Q$ *such that* $mR_1\ u$,

(15') *for all* $x \in B$ *such that* $p \in \Phi(x)$, *there are* $u, v \in S$ *with* $v \in \Phi(x)$ *and* $mR_1\, uR_2\, v$.

Now if (15), then (5) clearly follows. So, suppose that (15') and put $Y = \left\{ v : \text{there is } u \in S, mR_1\, uR_2\, v \right\}$. Then

(16) *for all* $x \in B$, *if* $p \in \Phi(x)$ *then* $\Phi(x) \cap Y \neq \emptyset$,

which implies that $p \in \bar{Y}$ the closure of Y. It remains to be shown that $p \in Y$. Note first thet since R_1 is a Boolean relation, the set $\{u : mR_1 u\}$ is closed in S. Then by Remark, Y is also a closed subset of S and thus $p \in Y$.

THEOREM 4. *If* $K_2 I_1 x \leq I_1 K_2 x$ *for all* $x \in B$, *then* R_1 *and* R_2 *satisfy conditions* (3) *and* (4).

Proof. Suppose that (3) does not hold in S. Then

(17) *there are* $m \in Q$ *and* $n \in S$ *such that* $mR_1 n$ *and* $n \notin Q$.

By (13) we have that $K_2 I_1 0 = K_2 0 \in m$ and $-K_2 0 \in n$. Since $mR_1 n$, $-I_1 K_2 0 = K_1 - K_2 0 \in m$ and bec use m is a proper filter, $K_2 I_1 0 \cap -I_1 K_2 0 \neq 0$; this is turn means that $K_2 I_1 0 \not\leq I_1 K_2 0$. Suppose instead that

there are $m, n, p \in S$ *with* $n \notin Q$ *such that* $mR_1 n$ *and* $mR_1 p$ *but for no* $u \in S$, $nR_2 u$ *and* $pR_1 u$.

Then the sets $X = \{u : nR_2 u\}$ and $Y = \{v : pR_1 v\}$ are disjoint. Note that $X = R_2(n)$ and $Y = R_1(p)$, hence (see Remark) they are both closed in S. Just as in [1], Lemma 3, we can find a clopen set A such that $Y \subseteq A$ and $X \subseteq -A$ and

(18) *for every* u, *if* $nR_2 u$ *then* $u \in -A$,

(19) *for every* v, *if* $pR_1 v$ *then* $v \in A$.

From (18) and the fact that $n \notin Q$, we infer $n \in \bar{I}_2 - A$. But since $mR_1 n$, $m \in \bar{K}_1 \bar{I}_2 - A = -\bar{I}_1 \bar{K}_2 A$. On the other hand (19) implies $m \in K_2 \bar{I}_1 A$. The proof then proceeds as in [1], hence there is $x \in B$ such that $K_2 I_1 x \not\leq$

$\nvdash I_1 K_2 x$.

COROLLARY 1. *The dual notion of a* $*$*-bimodal algebra a* $*$-GDMS.

THEOREM 5 (Completeness). $\vdash_{(S4,*)-C} \alpha$ *iff* α *is* $(S4,*)$*-valid.*

Proof . Validity is given by Theorem 1. On the other hand if $\nvdash_{(S4,*)-C} \alpha$ then α is not true in the Lindenbaum algebra $(B,\bar{K}_1 \bar{K}_2)$ of $(S4,*)$-C. Since $\varphi: (B,K_1,K_2) \longrightarrow (\mathcal{B}(S),\bar{K}_1,\bar{K}_2)$ is an embedding, α is not true in $(\mathcal{B}(S),\bar{K}_1,\bar{K}_2)$. Now it is easy to check that a wff is true in the algebra $(\mathcal{B}(S),\bar{K}_1,\bar{K}_2)$ iff it is bimodally valid in (S,Q,R_1,R_2) (extend proof in [4]); hence α is not bimodally valid in (S,Q,R_1,R_2). But by Corollary 1, (S,Q,R_1,R_2) is a $*$-GDMS and the theorem is proved.

3. - The methods used to prove completeness for non-normal intuitionistic modal logics are as in [2]. In the last part of this paper we shall examine only those features of the proof which depend upon the fact that Q can be non void. The reader will then be able to draw the semantic conclusions following what is sketched out in [2].

First, we define modal intuitionistic semantic concepts for the non normal case. If M = (S,Q,R_1,R_2) is a $*$-GDMS, then an *intuitionistic model* on M is a function w defined on all pairs $(a,m) \in V \times S$, whose range is $\{0,1\}$ and such that if $w(a,m) = 1$ then $w(a,n) = 1$ for all $n \in S$ such that $mR_1 n$. For each intuitionistic model w, let $\bar{w}$ be the extension of w to all modal wff's satisfying

$\bar{w}(\alpha,m)$ *is defined as in an ordinary intuitionistic model on* (S,R_1) *for non-modal connectives;*

$\bar{w}(M\beta,m)$ *and* $\bar{w}(L\beta,m)$ *are defined as in* [2], *if* $m \notin Q$;

$\bar{w}(M\beta,m) = 1$ *and* $\bar{w}(L\beta, m) = 0$, *if* $m \in Q$.

The definition of "w verifies α in m", "α is intuitionistically valid in M" and "α is $*$-IC valid" are as in [2].

We conclude with the following

LEMMA 2. *Let* M *be a* $*$-GDMS. *Then*

(i) *for every intuitionistic model* w *on* M, *there is a bimodal model* v *on* M *such that*

(20) $\bar{w}(\alpha,m) = v'(T\alpha,m)$ $(m \in S,$ α a wff),

where v' *is the extension of* v *given in n.2;*

(ii) *for every bimodal model* v *on* M, *there is an intuitionistic model* w *on* M *satisfying* (20).

Proof. Given an intuitionistic model w on M, we define a bimodal model v by putting $v(a,m) = w(a,m)$. Vice versa, given v, we define w by $w(a,m) = v'(L_1 a,m)$; note that (as in the normal case) w is in fact an intuitionistic model on M. Now the proof of (20) follows by induction on the height of α, and is identical to that of [2], but for the case when $m \in Q$ and α is either $L\beta$ or $M\beta$. But in this case, (20) holds for any intuitionistic model w and any bimodal v (regardless of their mutual connection) $\bar{w}(M\beta,m) = 1 = v'(M_2\ T\beta,m) = v'(TM\beta,m)$; and $\bar{w}(L\beta,m) = 0 = v'(L_1\ L_2 T\beta,m)$, since $v'(L_2 T\beta,m) = 0$ and $mR_1 m$.

BIBLIOGRAFIA

[1] FISCHER SERVI G., *Remarks on Halmos' duality Theory*, Boll. U.M.I. 18-A (1981), 457-460.

[2] FISCHER SERVI G., *Semantics for a class of intuitionistic modal calculi* in *Italian Studies in the Philosophy of Science*, M. L. Dalla Chiara Ed., Reidel 1981, 59-72.

[3] HALMOS P. R., *Algebraic Logic* New York 1962.

[4] LEMMON E. J., *Algebraic semantics for modal Logics*, 1,2, J. Symbolic Logic, 31 (1966), 46-65, 191-218.

Lavoro pervenuto alla Redazione l'8 Maggio 1981
ed accettato per la pubblicazione il 20 Luglio 1981
su parere favorevole di G. Lolli e R. Magari