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New generalizations of lifting modules

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Abstract. In this paper, we call a module M almost $\mathcal{I}$-lifting if, for any element $\phi \in S = End_R(M)$, there exists a decomposition $r_M\ell_S(\phi) = A \oplus B$ such that $A \subseteq \phi M$ and $\phi M \cap B \ll M$. This definition generalizes the lifting modules and left generalized semiregular rings. Some properties of these modules are investigated. We show that if $f_1 + \cdots + f_n = 1$ in S, where f_i 's are orthogonal central idempotents, then M is an almost $\mathcal{I}$-lifting module if and only if each f_iM is almost $\mathcal{I}$-lifting. In addition, we call a module M π-$\mathcal{I}$-lifting if, for any $\phi \in S$, there exists a decomposition $\phi^n M = eM \oplus N$ for some positive integer n such that $e^2 = e \in S$ and $N \ll M$. We characterize semi-π-regular rings in terms of π-$\mathcal{I}$-lifting modules. Moreover, we show that if M_1 and M_2 are abelian π-$\mathcal{I}$-lifting modules with $Hom_R(M_i, M_j) = 0$ for $i \neq j$, then $M = M_1 \oplus M_2$ is a π-$\mathcal{I}$-lifting module.

Keywords: Lifting module; $\mathcal{I}$-Lifting module; Semiregular ring; Semi-π-regular ring.

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1 Introduction

Throughout this paper, R will denote an arbitrary associative ring with identity, M a unitary right R-module and $S = End_R(M)$ the ring of all R-endomorphisms of M. We will use the notation $N \ll M$ to indicate that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. $N \trianglelefteq M$ means that N is a fully invariant submodule of M (i.e., $\forall \phi \in End_R(M),\ \phi(N) \subseteq N$). For all $I \subseteq S$, the left and right annihilators of I in S are denoted by $\ell_S(I)$ and $r_S(I)$, respectively. We also denote $r_M(I) = \{x \in M \mid Ix = 0\}$, for $I \subseteq S$; $\ell_S(N) = \{\phi \in S \mid \phi(N) = 0\}$, for $N \subseteq M$. A ring R is called a *semiregular* ring if for each $a \in R$, there exists $e^2 = e \in aR$ such that $(1 - e)a \in J(R)$ [7].

A module M is called *lifting* if for every $A \leq M$, there exists a direct summand B of M such that $B \subseteq A$ and $A/B \ll M/B$ [6].

In [1], we introduced $\mathcal{I}$-lifting modules as a generalization of lifting modules. Following [1], a module M is called *$\mathcal{I}$-lifting* if for every $\phi \in S$ there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq \mathrm{Im}\phi$ and $M_2 \cap \mathrm{Im}\phi \ll M_2$. It is obvious that every lifting module is $\mathcal{I}$-lifting while the converse in not true (the $\mathbb{Z}$-module $\mathbb{Q}$ is $\mathcal{I}$-lifting but it is not lifting). It is easily checked that R_R is an $\mathcal{I}$-lifting module if and only if R is a semiregular ring.

In this paper, we call a module M *almost $\mathcal{I}$-lifting* if, for any element $\phi \in S = End_R(M)$, there exists a decomposition $r_M\ell_S(\phi) = A \oplus B$ such that $A \subseteq \phi M$ and $\phi M \cap B \ll M$. In [11], a ring R is called *left almost semiregular* if R_R is almost $\mathcal{I}$-lifting. Such rings are studied in [11] and named as left generalized semiregular rings. In this note our aim is to generalize the results of [11] from the ring case to the module case.

In Section 2, first, we give a new characterization of $\mathcal{I}$-lifting modules by modifying the definition of almost $\mathcal{I}$-lifting modules (Theorem 2.4). Next, we give conditions under which an almost $\mathcal{I}$-lifting module is $\mathcal{I}$-lifting (Proposition 2.6 and Corollary 2.7). We also prove the following which generalizes [11, Theorem 1.14]:

Let $f_1 + \cdots + f_n = 1$ in S, where f_i 's are orthogonal central idempotents. Then M is an almost $\mathcal{I}$-lifting module if and only if each f_iM is almost $\mathcal{I}$-lifting (see Corollary 2.12).

In Section 3, we call a module M *π-$\mathcal{I}$-lifting* if, for any $\phi \in S$, there exists a decomposition $\phi^n M = eM \oplus N$ for some positive integer n such that $e^2 = e \in S$ and $N \ll M$. A ring R is called *semi-π-regular* if R_R is a π-$\mathcal{I}$-lifting module. Semi-π-regular rings are investigated in [11]. A π-$\mathcal{I}$-lifting module generalizes the notion of lifting module as well as that of a semi-π-regular ring. We investigate some properties of π-$\mathcal{I}$-lifting modules. We give conditions under which a π-$\mathcal{I}$-lifting module is $\mathcal{I}$-lifting (Corollary 3.11). We characterize semi-π-regular rings in terms of π-$\mathcal{I}$-lifting modules (Theorem 3.14). It is shown that the class of some abelian π-$\mathcal{I}$-lifting modules is closed under direct sums (Proposition 3.17).

2 Almost $\mathcal{I}$-lifting modules

In this section, we study the module-theoretic version of left generalized semiregular rings defined by Xiao and Tong [11].

Definition 2.1. Let M be a right R-module. M is called *almost $\mathcal{I}$-lifting* if, for every $\phi \in S$, there exists a decomposition $r_M\ell_S(\phi) = A \oplus B$ such that $A \subseteq \phi M$ and $B \cap \phi M \ll M$. A ring R is called a *left almost semiregular* if R_R is almost $\mathcal{I}$-lifting. Such rings are named as left generalized semiregular rings in [11].

Proposition 2.2. Let M be a right R-module. If M is $\mathcal{I}$-lifting, then M is almost $\mathcal{I}$-lifting.

Proof. Let $\phi \in S$. Then there exists a decomposition $M = A \oplus B$ such that $A \subseteq \phi M$ and $B \cap \phi M \ll M$. By modular law, we have $r_M \ell_S(\phi) = A \oplus (B \cap r_M \ell_S(\phi))$ and $(r_M \ell_S(\phi) \cap B) \cap \phi M = B \cap \phi M \ll M$. Hence M is almost $\mathcal{I}$-lifting. QED

The following example shows that almost $\mathcal{I}$-lifting modules need not be $\mathcal{I}$-lifting.

Example 2.3. Let ${}_RV_R$ be a bimodule over a ring R. The trivial extension of R by V is the direct sum $T(R, V) = R \oplus V$ with multiplication $(r+v)(r'+v') = rr' + (rv' + vr')$. It is shown in [7] that the trivial extension $R = T(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is a commutative principally injective ring, but it is not semiregular since $R/J(R) \cong \mathbb{Z}$. Hence R_R is almost $\mathcal{I}$-lifting which is not $\mathcal{I}$-lifting.

Theorem 2.4. Let M be a right R-module. Then the following are equivalent:

(1) M is $\mathcal{I}$-lifting;

(2) For any $\phi \in S$, there exists a decomposition $r_M \ell_S(\phi) = A \oplus B$, where $A \subseteq \phi M$, A is a summand of M and $B \cap \phi M \ll M$.

Proof. (1) $\Rightarrow$ (2) By Proposition 2.2.

(2) $\Rightarrow$ (1) Let $\phi \in S$ and $r_M \ell_S(\phi) = A \oplus B$, where $A \subseteq \phi M$, A is a summand M and $B \cap \phi M \ll M$. Then $\phi M = A \oplus (B \cap \phi M)$, where A is a summand of M and $B \cap \phi M \ll M$. Hence M is $\mathcal{I}$-lifting. QED

By Theorem 2.4, we obtain the following characterization of semiregular rings.

Corollary 2.5. The following are equivalent for a ring R:

(1) R is semiregular;

(2) For any $a \in R$, there exists a decomposition $\ell_R r_R(a) = P \oplus Q$, where $P = Re \subseteq Ra$ for some $e^2 = e \in R$ and $Q \cap Ra \ll R$;

(3) For any $a \in R$, there exists a decomposition $r_R \ell_R(a) = P \oplus Q$, where $P = eR \subseteq aR$ for some $e^2 = e \in R$ and $Q \cap aR \ll R$.

A module M is called *semi-projective* if for any epimorphism $f : M \to N$, where N is a submodule of M, and for any homomorphism $g : M \to N$, there exists $h : M \to M$ such that $fh = g$. As easily seen, M is semi-projective if and only if, for every cyclic right ideal $I \subseteq End_R(M)$, $I = Hom(M, IM)$.

Proposition 2.6. Let M be a semi-projective almost $\mathcal{I}$-lifting module with $RadM \ll M$. If there exists $e^2 = e \in S$ such that $\ell_S(\phi) = \ell_S(e)$ for any $\phi \in S$, then M is $\mathcal{I}$-lifting.

Proof. Let $\phi \in S$. Then there exists a decomposition $r_M \ell_S(\phi) = A \oplus B$ such that $A \subseteq \phi M$ and $B \cap \phi M \ll M$. Since $\ell_S(\phi) = \ell_S(e)$, we have $r_M \ell_S(\phi) = r_M \ell_S(e) = eM$ and so $eM = A \oplus B$. As A and B are direct summands of M, we can write $eM = fM \oplus gM$ for some $f^2 = f \in S$, $g^2 = g \in S$. By [10, 18.4], we get $Hom_R(M, eM) = Hom_R(M, fM) + Hom_R(M, gM)$. Since M is semi-projective, $eS = fS + gS$. As $A \cap B = 0$, $fS \cap gS = 0$. Thus $eS = fS \oplus gS$. Since $fM \subseteq \phi M$ and $gM \cap \phi M \ll M$, $fS \subseteq \phi S$ and $gS \cap \phi S \subseteq Hom_R(M, RadM)$. Since $\ell_S(\phi) = \ell_S(e)$, $r_S\ell_S(\phi) = r_S\ell_S(e) = eS$ and so $\phi = e\phi$. Let $e = \alpha + \beta$, where $\alpha = \phi h \in fS$ and $\beta \in gS$. Then $\phi = e\phi = \phi h\phi + \beta\phi$ and $\phi h = \phi h\phi h + \beta\phi h$. As $\phi h - \phi h\phi h = \beta\phi h \in gS \cap fS = 0$, ϕh is an idempotent. Moreover, we have $(1 - \phi h)\phi = \phi - \phi h\phi = \beta\phi \in gS \cap \phi S \subseteq Hom_R(M, RadM)$, hence $(1 - \phi h)\phi M \subseteq RadM \ll M$. Therefore M is $\mathcal{I}$-lifting. QED

Corollary 2.7. Let $r_M\ell_S(\phi)$ is a direct summand of a semi-projective module M for any $\phi \in S$. If M is an almost $\mathcal{I}$-lifting module with $RadM \ll M$, then M is $\mathcal{I}$-lifting.

Proof. Let $\phi \in S$. By assumption, $r_M\ell_S(\phi) = eM$ for some $e^2 = e \in S$. Then $\ell_S(\phi) = \ell_S(e)$ and so M is $\mathcal{I}$-lifting by Proposition 2.6. QED

Corollary 2.8. (See [11, Corollary 1.6]) If $r_R\ell_R(a)$ is a direct summand of R for any $a \in R$ and R is a left almost semiregular ring, then R is semiregular.

A ring R is called *left Rickart* if for every $a \in R$ there exists an idempotent $e \in R$ such that $\ell_R(a) = Re$ [3].

Corollary 2.9. Let S be a left Rickart ring. If M is a finitely generated semi-projective almost $\mathcal{I}$-lifting module, then M is $\mathcal{I}$-lifting.

An idempotent element $e^2 = e \in R$ is called *left* (resp. *right*) *semicentral* in R if $Re = eRe$ (resp. $eR = eRe$) [4]. It is well known that $e^2 = e \in S$ is a left semicentral idempotent iff eM is a fully invariant submodule of M.

Proposition 2.10. Let M be an almost $\mathcal{I}$-lifting module. Then every fully invariant direct summand of M is almost $\mathcal{I}$-lifting.

Proof. Let M be an almost $\mathcal{I}$-lifting module and K a fully invariant direct summand of M. Then there exists a left semicentral idempotent $e^2 = e \in S$ such that $K = eM$. Let $\phi \in End_R(eM)$. Then there exists a decomposition $r_M\ell_S(\phi e) = P \oplus L$ where $P \subseteq \phi eM$ and $L \cap \phi eM \ll M$. Note that the endomorphism ring of $K = eM$ is eSe. We claim that $r_{eM}\ell_{eSe}(\phi) = eP \oplus eL$. Since $\phi \in eSe$, $1 - e \in \ell_S(\phi) \subseteq \ell_S(\phi e)$. Thus for every $t \in L$, we have $(1 - e)t = 0$, which implies that $eL = L$. Similarly, $eP = P$. Take any $y \in eP \subseteq e\phi eM$, where $y = ey_1$, $y_1 \in P \subseteq r_M\ell_S(\phi e)$. Then for every $\psi \in \ell_{eSe}(\phi) \subseteq \ell_S(\phi) \subseteq \ell_S(\phi e)$, $\psi y_1 = 0$. As $y_1 \in P \subseteq \phi eM$, $y_1 = \phi em_1$ for some $m_1 \in M$. Thus we have $\psi y = \psi ey_1 = \psi e\phi em_1 = \psi\phi em_1 = \psi y_1 = 0$. Hence $y \in r_{eM}\ell_{eSe}(\phi)$ and $eP \subseteq r_{eM}\ell_{eSe}(\phi)$. Similarly, $eL \subseteq r_{eM}\ell_{eSe}(\phi)$. On the other hand, let $x \in r_{eM}\ell_{eSe}(\phi)$. Then for every $f \in \ell_S(\phi)$, we have $efe\phi eM = fe\phi eM = f\phi eM = 0$. Thus

$efe \in \ell_{eSe}(\phi)$ and so $efex = 0$ which gives $fx = fex = efex = 0$ since $x \in eM$ and e is left semicentral. Hence $r_{eM}\ell_{eSe}(\phi) \subseteq r_M\ell_S(\phi)$. Take $x = x_1 + x_2$, where $x_1 \in P$ and $x_2 \in L$. Then $x = ex = ex_1 + ex_2 \in eP + eL$. This shows that $r_{eM}\ell_{eSe}(\phi) = eP \oplus eL$. Since $eP \subseteq e\phi eM = \phi eM$, it is enough to show that $eL \cap \phi eM \ll eM$. Note that $eL \cap \phi eM = L \cap \phi eM \ll M$. Thus $eL \cap \phi eM = eL \cap e\phi eM \ll eM$ since $eM \leq^{\oplus} M$. Therefore $K = eM$ is almost $\mathcal{I}$-lifting. QED

Theorem 2.11. Let e and f be orthogonal central idempotents of S. If eM and fM are almost $\mathcal{I}$-lifting modules, then $gM = eM \oplus fM$ is almost $\mathcal{I}$-lifting.

Proof. First, note that $g = e+f$ is central idempotent. Next, let $\phi \in End_R(gM) \cong gSg = gS$. Then $e\phi \in eS$ and $f\phi \in fS$. Take $x \in r_{gM}\ell_{gS}(\phi)$. Then for any $\psi \in \ell_{eS}(e\phi)$, we have $\psi e\phi = 0$ and so $\psi\phi = e\psi\phi = \psi e\phi = 0$ this implies that $g\psi\phi = 0$ and $g\psi \in \ell_{gS}(\phi)$. Hence $\psi(x) = g\psi(x) = 0$ and so $\psi ex = e\psi(x) = 0$, hence $ex \in r_{eM}\ell_{eS}(e\phi)$. By hypothesis, $ex \in r_{eM}\ell_{eS}(e\phi) = P_e \oplus L_e$ where $P_e \subseteq e\phi eM = \phi eM$ and $L_e \cap \phi eM \ll eM$. Similarly, $fx \in r_{fM}\ell_{fS}(f\phi) = P_f \oplus L_f$ where $P_f \subseteq \phi fM$ and $L_f \cap \phi fM \ll fM$. Then $x = gx = ex + fx \in P_e \oplus P_f \oplus L_e \oplus L_f$ since e and f are orthogonal. Hence $r_{gM}\ell_{gS}(\phi) \subseteq P_e \oplus L_e \oplus P_f \oplus L_f$. On the other hand, let $x \in L_e$ and $\psi \in \ell_{gS}(\phi)$, then $\psi\phi = 0$, and so $e\psi e\phi = e\psi\phi = 0$. Thus $e\psi \in \ell_{eS}(e\phi)$. As $L_e \subseteq r_{eM}\ell_{eS}(e\phi)$, $e\psi x = 0$. Hence $\psi x = \psi ex = 0$ and so $L_e \subseteq r_{gM}\ell_{gS}(\phi)$. Similarly, $L_f, P_e, P_f \subseteq r_{gM}\ell_{gS}(\phi)$. Note that $P_e \oplus P_f \subseteq \phi eM \oplus \phi fM = \phi gM$. This shows that $r_{gM}\ell_{gS}(\phi) = P_e \oplus P_f \oplus L_e \oplus L_f$. It is easily checked that $(L_e \oplus L_f) \cap (\phi eM + \phi fM) \subseteq (L_e \cap \phi eM) \oplus (L_f \cap \phi fM) \ll eM \oplus fM = gM$. Hence $(L_e \oplus L_f) \cap \phi gM \ll gM$. Therefore gM is almost $\mathcal{I}$-lifting. QED

Corollary 2.12. Let $f_1 + \cdots + f_n = 1$ in S, where f_i 's are orthogonal central idempotents. Then M is an almost $\mathcal{I}$-lifting module if and only if each f_iM is almost $\mathcal{I}$-lifting.

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $a, e^2 = e \in R$.

Corollary 2.13. If S is an abelian ring, then any finite direct sum of almost $\mathcal{I}$-lifting modules is almost $\mathcal{I}$-lifting.

Theorem 2.14. Let e be an idempotent of S such that $SeS = S$. If M is an almost $\mathcal{I}$-lifting module, then eM is almost $\mathcal{I}$-lifting.

Proof. Let $\phi \in End_R(eM) \cong eSe$. Then there exists a decomposition $r_M\ell_S(\phi e) = P \oplus L$, where $P \subseteq \phi eM$ and $L \cap \phi eM \ll M$. We claim that $r_{eM}\ell_{eSe}(\phi) = eP \oplus eL$. Since $1 - e \in \ell_S(\phi e)$, we have $(1 - e)t = 0$ for all $t \in L$, thus $eL = L$. Similarly, $P = eP$. Thus $eP \cap eL = 0$. Clearly, $eP = P \subseteq r_{eM}\ell_{eSe}(\phi)$ and

$eL = L \subseteq r_{eM}\ell_{eSe}(\phi)$. On the other hand, take $x \in r_{eM}\ell_{eSe}(\phi)$ and write $1 = \sum_{i=1}^{n} f_i e g_i$ for some f_i and g_i in S. Then for any $\psi \in \ell_S(\phi)$, we get $eg_i\psi e\phi = eg_i\psi\phi = 0$ for each i. This gives $eg_i\psi ex = 0$ for each i, which implies that $\psi x = \psi ex = (\sum_{i=1}^{n} f_i eg_i)\psi ex = 0$ since $x \in eM$. Hence $r_{eM}\ell_{eSe}(\phi) \subseteq r_M\ell_S(\phi)$. Thus $x = s+t$ for some $s \in P$ and $t \in L$. So $x = ex = es+et \in eP+eL$. This follows that $r_{eM}\ell_{eSe}(\phi) = eP \oplus eL$. This completes the claim. It remains to show that $eL \cap \phi eM \ll eM$. Since $eP \subseteq e\phi eM = \phi eM$. Note that $eL \cap \phi eM = L \cap \phi eM \ll M$. Thus $eL \cap \phi eM = eL \cap e\phi eM \ll eM$ since $eM \leq^{\oplus} M$. QED

Proposition 2.15. If M is an almost $\mathcal{I}$-lifting semi-projective module, then $Z({}_SS) \subseteq J(S)$.

Proof. Let $0 \neq s \in Z({}_SS)$. Then for each element $t \in S$, $st \in Z({}_SS)$. Let $u = 1 - st$, then $u \neq 0$. Since $\ell_S(st)$ is essential in S and $\ell_S(u) \cap \ell_S(st) = 0$, $\ell_S(u) = 0$. Thus $M = r_M\ell_S(u) = P \oplus L$, where $P \subseteq uM$ and $uM \cap L \ll M$. Hence $P = eM$ for some $e^2 = e \in S$. It is sufficient to prove that $e = 1$. If not, there exists $0 \neq \psi(1-e) \in S(1-e) \cap \ell_S(st)$ because $\ell_S(st)$ is essential in S. This implies that $\psi(1-e)u = \psi(1-e)$. Let $m \in M$, then $um = em' + a$ for some $m' \in M$ and $a \in L$. Then $\psi(1-e)um = \psi(1-e)a$. Thus $\psi(1-e)m = \psi(1-e)a$ and so $\psi(1-e)(m-a) = 0$ for all $m \in M$ and some $a \in L$. Note that $a = um - em' \in uM \cap L \subseteq Rad(M)$ and so $aR \ll M$. It is easy to see that $M = e(m-a)R + aR + (1-e)(m-a)R$. Since $aR \ll M$, we have $M = e(m-a)R + (1-e)(m-a)R$. Thus for all $m \in M$, $m = e(m-a)r_1 + (1-e)(m-a)r_2$ for some $r_1, r_2 \in R$. Hence $(1-e)m = (1-e)(m-a)r_2$. Therefore $0 = \psi(1-e)(m-a)r_2 = \psi(1-e)m$ for all $m \in M$. Hence $\psi(1-e) = 0$, a contradiction. So $e = 1$ and $M = eM = uM$. Since M is semi-projective, $S = uS$. Thus $s \in J(S)$. QED

Recall that a ring R is *left principally injective* (*P-injective*) if every principal right ideal is a right annihilator. Following [8], the ring R is *left almost principally injective* (*AP-injective*) if, for any $a \in R$, aR is a direct summand of $r_R\ell_R(a)$.

Lemma 2.16. Let M be a right finitely generated projective R-module with $S = End_R(M)$. Then:

(1) If, for any $\phi \in S$, there exists a decomposition $r_M\ell_S(\phi) = \phi M \oplus X$ for some $X \leq M$, then S is a left almost principally injective ring.

(2) If S is a left almost principally injective ring and M is a self-generator, then, for all $\phi \in S$, there exists a decomposition $r_M\ell_S(\phi) = \phi M \oplus X$ for some $X \leq M$.

Proof. (1) Let $\phi \in S$ and $r_M\ell_S(\phi) = \phi M \oplus X$ for some $X \leq M$. Note that $r_S\ell_S(\phi)M \subseteq r_M\ell_S(\phi)$. Thus $r_S\ell_S(\phi)M = \phi M \oplus (X \cap r_S\ell_S(\phi)M)$. Then we have $Hom_R(M, r_S\ell_S(\phi)M) = Hom_R(M, \phi M) + Hom_R(M, X \cap r_S\ell_S(\phi)M)$. Since M is finitely generated projective, by [5, 4.19], $r_S\ell_S(\phi) = \phi S + Hom_R(M, X \cap r_S\ell_S(\phi)M)$. As $\phi M \cap X = 0$, $\phi S \cap Hom_R(M, X \cap r_S\ell_S(\phi)M) = Hom_R(M, \phi M) \cap Hom_R(M, X \cap r_S\ell_S(\phi)M) = Hom_R(M, \phi M \cap X) = 0$. Thus $r_S\ell_S(\phi) = \phi S \oplus Hom_R(M, X \cap r_S\ell_S(\phi)M)$. Therefore S is a left almost principally injective ring.

(2) Let $\phi \in S$ and $r_S\ell_S(\phi) = \phi S \oplus J$ for some $J \leq S_S$. Since M is a self-generator, $r_S\ell_S(\phi)M = r_M\ell_S(\phi)$. Thus $r_M\ell_S(\phi) = \phi M + JM$. As $\phi S \cap J = 0$, by [5, 4.19], we have $Hom_R(M, \phi M) \cap Hom_R(M, JM) = 0$. Therefore $Hom_R(M, \phi M \cap JM) = 0$. Since M is a self-generator we get $\phi M \cap JM = 0$. It follows that $r_M\ell_S(\phi) = \phi M \oplus JM$. QED

Corollary 2.17. Let M be a finitely generated projective module with $Rad(M) = 0$. If M is almost $\mathcal{I}$-lifting, then S is a left almost principally injective ring. The converse is true whenever M is a self-generator.

Proof. For every $\phi \in S$, there exists a decomposition $r_M\ell_S(\phi) = P \oplus L$, where $P \subseteq \phi M$ and $L \cap \phi M \ll M$. By hypothesis, $L \cap \phi M = 0$. Clearly, $r_M\ell_S(\phi) = \phi M + L$, so $r_M\ell_S(\phi) = \phi M \oplus L$. Therefore S is left almost principally injective by Lemma 2.16. The converse is clear by Lemma 2.16. QED

3 π-$\mathcal{I}$-lifting modules

Definition 3.1. A module M is called *π-$\mathcal{I}$-lifting* if, for any $\phi \in S$, there exists a decomposition $\phi^n M = eM \oplus N$ for some positive integer n such that $e^2 = e \in S$ and $N \ll M$.

A ring R is called *semi-π-regular* if R_R is a π-$\mathcal{I}$-lifting module. Such rings are studied in [11].

It is clear that every $\mathcal{I}$-lifting module is π-$\mathcal{I}$-lifting. The following example shows that there exists a π-$\mathcal{I}$-lifting module which is not $\mathcal{I}$-lifting.

Example 3.2. Let $R = M = \{(x_1, x_2, \ldots, x_n, x, x, \ldots\} \mid x_1, x_2, \ldots, x_n \in M_2(\mathbb{Z}_2), x \in \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. Then M_R is a π-$\mathcal{I}$-lifting R-module but not $\mathcal{I}$-lifting (see [11, Example 4.5]).

Lemma 3.3. Let M be a semi-projective module and F be a fully invariant submodule of M. Then the following are equivalent for an element $\phi \in S$ and any positive integer n:

(1) There exists $e^2 = e \in \phi^n S$ with $(\phi^n - e\phi^n)M \subseteq F$.

(2) There exists $e^2 = e \in \phi^n S$ with $\phi^n M \cap (1 - e)M \subseteq F$.

(3) $\phi^n M = eM \oplus N$ where $e^2 = e \in S$ and $N \subseteq F$.

Proof. (1) $\Rightarrow$ (2) If $x \in \phi^n M \cap (1-e)M$, then $x = \phi^n m = (1-e)m = (1-e)m'$ for some $m, m' \in M$. Thus $x = (1-e)\phi^n m \in F$.

(2) $\Rightarrow$ (3) It is clear that $\phi^n M = eM \oplus [\phi^n M \cap (1-e)M]$. Set $N = \phi^n M \cap (1-e)M$.

(3) $\Rightarrow$ (1) First we show that $e^2 = e \in \phi^n S$. Consider the epimorphisms $\phi^n : M \to \phi^n M$ and $e : M \to eM$. Since M is semi-projective, there exists a homomorphism $g \in S$ such that $\phi^n g = ie = e$, where $i : eM \to \phi^n M$ is the inclusion map. Hence $e \in \phi^n S$. Since $\phi^n M = eM \oplus N$, for every $m \in M$, we have $\phi^n m = em' + n$ for some $m' \in M$ and $n \in N$. Then $\phi^n m - e\phi^n m = n - en \in F$ because $N \subseteq F$. Hence $(\phi^n - e\phi^n)M \subseteq F$. QED

Corollary 3.4. Let M be a finitely generated semi-projective module. Then the following are equivalent for an element $\phi \in S$ and any positive integer n:
(1) There exists $e^2 = e \in \phi^n S$ with $(\phi^n - e\phi^n)M \ll M$.
(2) There exists $e^2 = e \in \phi^n S$ with $\phi^n M \cap (1-e)M \ll M$.
(3) $\phi^n M = eM \oplus N$ where $e^2 = e \in S$ and $N \ll M$.

Proposition 3.5. Let M be a projective module, and let $S = End_R(M)$. Then $J(S) = \nabla(M)$, where $\nabla(M) = \{\phi \in S \mid \mathrm{Im}\phi \ll M\}$. Moreover, S is a semi-π-regular ring if and only if M is a π-$\mathcal{I}$-lifting module.

Proof. By [11, Theorem 4.12]. QED

A module M is said to have the *exchange property* if for any module X and decomposition $X = M' \oplus Y = \bigoplus_{i\in I} N_i$, where $M' \cong M$, there exist submodules $N_i' \subseteq N_i$ for each i such that $X = M' \oplus (\bigoplus_{i\in I} N_i')$. If this condition holds for finite sets I, the module M is said to have the *finite exchange property.*

Corollary 3.6. Let M be a projective π-$\mathcal{I}$-lifting module. Then M is a module with the finite exchange property.

Proof. By [11, Corollary 4.11] and Proposition 3.5. QED

Corollary 3.7. The following are equivalent for a ring R.
(1) $M_n(R)$ is semi-π-regular for every positive integer n.
(2) Every finitely generated projective R-module is π-$\mathcal{I}$-lifting.

Proof. (1) $\Rightarrow$ (2) Let M be a finitely generated projective R-module. Then $M \cong eR^n$ for some positive integer n and $e^2 = e \in M_n(R)$. Hence S is isomorphic to $eM_n(R)e$. By (1), and [11, Corollary 4.2], S is semi-π-regular. Thus M is π-$\mathcal{I}$-lifting by Proposition 3.5.

(2) $\Rightarrow$ (1) Note that $M_n(R)$ can be viewed as the endomorphism ring of a projective R-module R^n for any positive integer n. By (2), R^n is π-$\mathcal{I}$-lifting. Therefore $M_n(R)$ is semi-π-regular by Proposition 3.5. $\boxed{QED}$

Corollary 3.8. Let M be a projective module and N a fully invariant submodule of M. If M is π-$\mathcal{I}$-lifting, then so is M/N.

Proof. Let $f \in S$ and $g : M \to M/N$ denote the natural epimorphism. Since N is fully invariant, we have $\mathrm{Ker}g \subseteq \mathrm{Ker}gf$. By the Factor Theorem, there exists a unique homomorphism f^* such that $f^*g = gf$. Hence we define a homomorphism $\phi : S \to End_R(M/N)$ with $\phi(f) = f^*$ for any $f \in S$. As M is projective, ϕ is an epimorphism. Thus $End_R(M/N) \cong S/\mathrm{Ker}\phi$. By Proposition 3.5, S is semi-π-regular, and so is $S/\mathrm{Ker}\phi$ by [11, Corollary 4.2]. Therefore M/N is π-$\mathcal{I}$-lifting duo to Proposition 3.5 again. $\boxed{QED}$

Recall that an R-module M is called *duo* if every submodule of M is fully invariant.

Corollary 3.9. Let M be a projective duo module. If M is π-$\mathcal{I}$-lifting, then M/N is also π-$\mathcal{I}$-lifting for every submodule N of M.

Proposition 3.10. Let M be a projective π-$\mathcal{I}$-lifting module. Then $Rad(M)$ is small in M.

Proof. Let $N \subseteq M$ be any submodule with $N + Rad(M) = M$. If $g : M \to M/N$ is the natural map, then there exists $f : M \to Rad(M)$ with $gf = g$. Then $g = gf = \cdots = gf^n$ for any positive integer n. Since M is π-$\mathcal{I}$-lifting, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq \mathrm{Im}f^n$ and $M_2 \cap \mathrm{Im}f^n \ll M_2$. Note that $M_1 \subseteq \mathrm{Im}f^n \subseteq \mathrm{Im}f \subseteq Rad(M)$. By [10, 22.3], $M_1 = 0$ and so $\mathrm{Im}f^n \ll M$. Hence $f^n \in \nabla = Jac(S)$, thus $g = 0$ and so $N = M$. This shows that $Rad(M) \ll M$. $\boxed{QED}$

Recall that a projective module is *semiperfect* if every homomorphic image has a projective cover [10].

Corollary 3.11. If R is a semiperfect ring, then the following are equivalent for a projective R-module M:

(1) M is semiperfect;
(2) $End_R(M)$ is semiregular;
(3) $End_R(M)$ is semi-π-regular;
(4) M is $\mathcal{I}$-lifting;
(5) M is π-$\mathcal{I}$-lifting;
(6) $Rad(M) \ll M$.

Proof. (1) ⇒ (2) ⇒ (6) ⇒ (1) By [19, Corollary 3.7].

(3) ⇔ (5) By Proposition 3.5.

(2) ⇒ (4) Since M is projective, it is well known that, $J(S) = \nabla(M)$ where $\nabla(M) = \{\alpha \in S \mid \mathrm{Im}\alpha \ll M\}$. Assume that $f \in S$, then there exists an idempotent $e \in S$ such that $eS \subseteq fS$ and $(1-e)fS \subseteq J(S) = \nabla(M)$. Therefore $M = eM \oplus (1-e)M$, $eM \subseteq fM$ and $(1-e)fM \ll M$. Hence M is $\mathcal{I}$-lifting.

(4) ⇒ (5) is clear.

(5) ⇒ (6) By Proposition 3.10. QED

Proposition 3.12. Let M be a π-$\mathcal{I}$-lifting module. Then every direct summand of M is also π-$\mathcal{I}$-lifting module.

Proof. Let $M = N \oplus P$ and $S_N = End_R(N)$. Define $g = f \oplus 0|_P$, for any $f \in S_N$, and so $g \in S$. By hypothesis, there exist a positive integer n and a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq \mathrm{Im}g^n$ and $\mathrm{Im}g^n \cap M_2 \ll M_2$. Hence $M_1 \subseteq \mathrm{Im}g^n = f^n N \leq N$. Thus $N = M_1 \oplus (M_2 \cap N)$, $M_1 \subseteq f^n N$ and $M_2 \cap N \cap f^n N \ll M_2$, this means that N is π-$\mathcal{I}$-lifting. QED

Corollary 3.13. Let R be a semi-π-regular ring. Then, for any $e^2 = e \in R$, $M = eR$ is a π-$\mathcal{I}$-lifting module.

We now characterize semi-π-regular rings in terms of π-$\mathcal{I}$-lifting modules.

Theorem 3.14. Let R be a ring. Then R is a semi-π-regular ring if and only if every cyclic projective R-module is π-$\mathcal{I}$-lifting.

Proof. The sufficiency is clear. For the necessity, let $M = mR$ be a cyclic projective module. Then $R = r_R(m) \oplus I$ for some right ideal I of R. Let $\phi : I \to M$ denote the isomorphism and $f \in S$. By Corollary 3.13, there exist a positive integer n and a decomposition $I = K_1 \oplus K_2$ such that $K_1 \subseteq (\phi^{-1} f \phi)^n I = (\phi^{-1} f^n \phi) I$ and $K_2 \cap (\phi^{-1} f^n \phi) I \ll K_2$. Hence $\phi I = \phi K_1 \oplus \phi K_2$. So $M = \phi K_1 \oplus \phi K_2$, $\phi K_1 \subseteq f^n \phi I = f^n M$ and $\phi K_2 \cap f^n M \ll \phi K_2$. This shows that M is π-$\mathcal{I}$-lifting. QED

Theorem 3.15. Let R be a ring and consider the following conditions:

(1) Every free R-module is π-$\mathcal{I}$-lifting;

(2) Every projective R-module is π-$\mathcal{I}$-lifting;

(3) Every flat R-module is π-$\mathcal{I}$-lifting.

Then (3) ⇒ (2) ⇔ (1). Moreover, (2) ⇒ (3) holds for finitely presented modules.

Proof. (3) ⇒ (2) ⇒ (1) Clear. (1) ⇒ (2) Let M be a projective R-module. Then M is a direct summand of a free R-module F. By (1), F is π-$\mathcal{I}$-lifting and so

M is π-$\mathcal{I}$-lifting by Proposition 3.12. (2) $\Rightarrow$ (3) is obvious from the fact that finitely presented flat modules are projective. QED

Lemma 3.16. Let M be a module and $f \in S$. If $eM \subseteq \mathrm{Im} f^n$ for some central idempotent $e \in S$ and a positive integer n, then $eM \subseteq \mathrm{Im} f^{n+1}$.

Proof. Let $f \in S$ and $eM \subseteq \mathrm{Im} f^n$ for some central idempotent $e \in S$ and a positive integer n. Let $em \in eM \subseteq \mathrm{Im} f^n$, then $em = f^n(x)$ for some $x \in M$. Since $e(x) \in \mathrm{Im} f^n$, $e(x) = f^n(y)$ for some $y \in M$. Hence $em = ef^n(x) = f^n e(x) = f^n(f^n(y)) = f^{n+1}(f^{n-1}(y)) \in \mathrm{Im} f^{n+1}$. Thus $eM \subseteq f^{n+1}(M)$. QED

A module M is called *abelian* if $fem = efm$ for any $f \in S$, $e^2 = e \in S, m \in M$ [9]. Note that M is an abelian module if and only if S is an abelian ring.

Proposition 3.17. Let M_1 and M_2 be abelian R-modules. If M_1 and M_2 are π-$\mathcal{I}$-lifting with $Hom_R(M_i, M_j) = 0$ for $i \neq j$, then $M = M_1 \oplus M_2$ is a π-$\mathcal{I}$-lifting module.

Proof. Let $f \in S$, then $\mathrm{Im} f = \mathrm{Im} f_1 \oplus \mathrm{Im} f_2$ where $f_1 \in End_R(M_1)$, $f_2 \in End_R(M_2)$. As M_i is π-$\mathcal{I}$-lifting, there exist positive integers m, n, and a direct summand X_i of M_i and a small submodule Y_i of M_i such that $\mathrm{Im} f_1^n = X_1 \oplus Y_1$ and $\mathrm{Im} f_2^m = X_2 \oplus Y_2$. Set $X = X_1 \oplus X_2$, then X is a direct summand of M. Consider the following cases:

(i) Let $n = m$. Clearly, $\mathrm{Im} f^n = \mathrm{Im} f_1^n \oplus \mathrm{Im} f_2^n = X + (Y_1 \oplus Y_2)$ and $Y_1 \oplus Y_2 \ll M_1 \oplus M_2 = M$.

(ii) Let $n < m$. By Lemma 3.16, $X_1 \subseteq \mathrm{Im} f_1^m$ so $X = X_1 \oplus X_2 \subseteq \mathrm{Im} f_1^m \oplus \mathrm{Im} f_2^m = \mathrm{Im} f^m$ and $\mathrm{Im} f_1^m = X_1 \oplus (Y_1 \cap \mathrm{Im} f_1^m)$ such that $Y_1 \cap \mathrm{Im} f_1^m \ll M_1$. Hence $\mathrm{Im} f^m = \mathrm{Im} f_1^m \oplus \mathrm{Im} f_2^m = X + ((Y_1 \cap \mathrm{Im} f_1^m) \oplus Y_2)$ and $(Y_1 \cap \mathrm{Im} f_1^m) \oplus Y_2 \ll M$. Therefore M is π-$\mathcal{I}$-lifting.

(iii) Let $m < n$. Since M_2 is abelian, the proof is similar to case (ii). QED

References

[1] T. Amouzegar, *A generalization of lifting modules*, Ukrainian Math. J., DOI: 10.1007/s11253-015-1042-z, **66**, n. 11, (2015).

[2] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York, 1992.

[3] E. P. Armendariz, *A note on extensions of Baer and p.p.-rings*, J. Austral. Math. Soc., **18** (1974), 470-473.

[4] G. F. Birkenmeier, *Idempotents and completely semiprime ideals*, Comm. Algebra, **11** (1983), 567-580.

[5] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting modules -supplements and projectivity in module theory, Frontiers in Mathematics, Birkhäuser, 2006.

[6] S. H. Mohamed and B. J. Müller, Continuous and Discrete Modules, London Math. Soc. Lecture Notes Series 147, Cambridge, University Press, 1990.

[7] W. K. Nicholson, *Semiregular modules and rings*, Canad. J. Math., **28** (1976), 1105-1120.

[8] S. Page and Y. Zhou, *Generalizations of principally injective rings*, J. Algebra, **206** (1998), 706–721

[9] J. E. Roos, *Sur les categories spectrales localement distributives*, C. R. Acad. Sci. Paris Ser. A-B, **265** (1967), 14-17.

[10] R. Wisbauer, Foundations of module and ring theory, Gordon and Breach, Reading, 1991.

[11] G. Xiao, W. Tong, *Generalizations of semiregular rings*, Comm. Algebra, **33** (2005), 3447-3465.