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On the relationships between the factors of upper and lower central series in groups and other algebraic structures

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Abstract. We discuss some new recent development of the well known classical R. Baer and B. Neumann theorems. More precisely, we want to show the thematic which evolved from the mentioned classical results describing the relations between the central factor-group and the derived subgroup in an infinite group. We track these topics not only in groups, but also in some other algebraic structures.

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The concepts of the center, upper central series, and lower central series are very important notions not only in group theory but also in some other algebraic structures. If G is a group, then the central factor-group $G/\zeta(G)$ and the derived subgroup $[G, G]$ indicate how far the group G is from being abelian: Thus if $G/\zeta(G)$ or $[G, G]$ is trivial, then G is abelian. The question about relations between these two groups arises naturally. The following classical result on infinite groups is the first answer on this question.

Theorem BN. Let G be a group, C a subgroup of the center $\zeta(G)$ such that G/C is finite. Then the derived subgroup $[G, G]$ is finite.

In many papers, this theorem is called the Schur's theorem. I. Schur was a very famous algebraist; he proved many important and interesting results. But he did not work in infinite groups and did not discover **Theorem BN**. We try to trace the history of this result, the history of its occurrence, and in particular, to find out how it has been attributed to I. Schur. Usually, the authors who call this result the Schur's theorem makes a reference on the article [17] (we must admit that we have done it also). However, we one cannot find **Theorem BN** there. In this paper, I. Schur introduced a concept of the group, which now called the ***Schur multiplicator*** or ***Schur multiplier*** (only for finite groups!), and obtained some properties of this group. In the modern terminology, the Schur multiplier $\mathbf{M}(G)$ of a group G is exactly the second cohomology group $\mathbf{H}^2(G, \mathbf{U}(\mathbf{C}))$. Initially **Theorem BN** appears in the paper of B.H. Neumann [16]. But at the end of this paper B.H. Neumann recognized that he received a letter from R. Baer, in which R. Baer mentioned that this result is a corollary of a more general result, which has been proved in his paper [1]. In fact, Theorem 3 of this paper proved that if a normal subgroup H of a group G has finite index, then the factor $([G, G] \cap H)/ [H, G]$ is also finite. Nevertheless, R. Baer placed **Theorem BN** in its usual form and gives another proof of it in his paper [2].

In connection to **Theorem BN** the following natural question appears: Are the order $| G/\boldsymbol{\zeta} G) |= \mathbf{t}$ and the order of the derived subgroup $[G, G]$ connected? If it is happened, what kind of a function defines this relation? B.H. Neumann posed this question in his paper [16]. He obtained the first bound for the order of $| [G, G] |$ making a reduction to the case of finite groups, and then using some results of I. Schur. R. Baer in the paper [2] has obtained the bound for the exponent of $[G, G]$. It looks like the following.

Theorem 1.1. *Let G be a group.*
If $| G/\zeta(G) |= t$, then $| [G, G] |\leq t^m$ where $m = t^2 + 1$.
If $| G/\zeta(G) |= t$, then $[G, G]^m = \langle 1 \rangle$ where $m = t^2$.
The best yet bound has been obtained by J. Wiegold [22], [23].

Theorem 1.2. *Let G be a group. Suppose that central factor-group $G/\zeta(G)$ is finite and has order* $\mathbf{t}$.

(i) *Then $| [G, G] |\leq \mathbf{w}(\mathbf{t})$ where $\mathbf{w}(\mathbf{t}) = \mathbf{t}^m$ and $\mathbf{m} = 1/2 \log_2(t-1)$.*

(ii) *If $\mathbf{t} = p^n$, where p is a prime, then $[G, G]$ is a p-group of order at most $\mathbf{p}^{1/2n(n-1)}$.*

(iii) *For each prime p and each integer $n > 1$ there exists a p-group G with $| G/\zeta(G) |= p^n$ and $[G, G] = \mathbf{p}^{1/2n(n-1)}$.*

When $| G/\zeta(G) |$ has more than one prime divisor, one can do better, though here the picture is less clear. For the proof of these results J. Wiegold also used the results about Schur multiplier, but talking on **Theorem BN** he refers on [16].

P.Hall was the first researcher who named **Theorem BN** the Schur's theorem. He did it ЭТО in his famous lectures on nilpotent groups [6], [8]. He gives this theorem with his own proof, and simply wrote the name I. Schur at this result without making any reference. Thus the name ***Schur Theorem*** was introduced by P. Hall. The reputation of P. Hall was very influential, so inheriting its many algebraists start using the name ***Schur's Theorem*** for **Theorem BN**.

We recall some definitions.

The ***upper central series*** of a group G is the ascending central series

$$\langle 1\rangle = \zeta_0(G) \leq \zeta(G) = \zeta_1(G) \leq \dots \zeta_\alpha(G) \leq \zeta_{\alpha+1}(G) \leq \dots \zeta_\gamma(G)$$

of characteristic subgroups such that $\zeta_{\alpha+1}(G)/\zeta_\alpha(G) = \boldsymbol{\zeta}(G/\zeta_\alpha(G))$ for all ordinals $\alpha < \gamma$, $\zeta_\lambda(G) = \cup_{\beta<\alpha} \zeta_\beta(G)$ for all limit ordinals $\lambda < \gamma$ and $\boldsymbol{\zeta}(G/\zeta_\gamma(G)) = \langle 1\rangle$.

The last term $\boldsymbol{\zeta}_\gamma(G)$ of the upper central series is called the ***upper hypercenter*** of G and will be denoted by $\boldsymbol{\zeta}_\infty(G)$. An ordinal γ is called the ***hypercentral length*** of G and will be denoted by $\mathbf{zl}(G)$. We note that J.C. Lennox and J.E. Roseblade [14] used for zl(G) the term the ***upper central height***.

The ***lower central series*** of a group G is the descending series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots \gamma_\alpha(G) \geq \gamma_{\alpha+1}(G) \geq \dots \gamma_\delta(G)$$

of characteristic subgroups defined by the rule $G = \gamma_1(G)$, $\gamma_2(G) = [G, G]$, and recursively $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$ for all ordinals α, $\gamma_\lambda(G) = \cap_{\mu<\lambda} \gamma_\mu(G)$ for the limit ordinals λ, and $\gamma_\delta(G) = [\gamma_\delta(G), G]$. The last term $\boldsymbol{\gamma}_\delta(G)$ of this series is called the ***lower hypocenter of* G**.

Initial relations between the upper and lower central series are well known: If $G = \boldsymbol{\zeta}_n(G)$ for some positive integer n, then $\gamma_{n+1}(G) = \langle 1\rangle$, and conversely. It is not true for infinite ordinals: An infinite dihedral 2-group D is a hypercentral group of the hypercentral length $\omega + 1$, but $\gamma_2(D) = \gamma_3(G)$ is a non-trivial infinite subgroup. On the other hand, by Magnus's theorem, every non-abelian free group has a lower central series of length and has a trivial center.

We can remark that if G is a hypercentral group with $\mathbf{zl}(G) = \omega$, then $\boldsymbol{\gamma}_{\omega+1}(G) = \langle 1\rangle$. This result has been proved by D.M. Smirnov in his paper [18].

In this connection, the following natural question arises: What can we say about the groups having a finite upper (respectively lower) central series which terminates not necessary on G (respectively on $\langle 1\rangle$)? In particular, what we can say about a group G such that $G/\boldsymbol{\zeta}_{\mathbf{K}}(G)$ is finite.

Theorem B. *Let G be a group. Suppose that there exists a positive integer k such that $\zeta_k(G)$ has finite index. Then $\gamma_{k+1}(G)$ is finite.*

This theorem is called the **Baer theorem**. But the paradox is that R. Baer did not give a direct proof of this result. In the aforementioned article [2], R. Baer notes that this result can be obtained from **Zusatz zum Endlichkeitssatz** of this paper. Title "the Baer theorem" for the first time also occurred in P. Hall lectures. P. Hall proves some generalization of **Theorem BN**, and as a corollary of it, derived **Theorem B**. He again named this Corollary as R. Baer theorem without any references.

We note that in this case, there also exists a function β_1 such that $|\gamma_{k+1}(G)| \leq \boldsymbol{\beta}_1(\mathbf{t}, \mathbf{k})$ where $\mathbf{t} = |G/\zeta_k(G)|$.

This function is defined recursively:

$$\boldsymbol{\beta}_1(\mathbf{t}, 1) = \boldsymbol{w}(\mathbf{t}), \boldsymbol{\beta}_1(\mathbf{t}, 2) = \boldsymbol{w}(\boldsymbol{w}(\mathbf{t})) + \mathbf{t}\boldsymbol{w}(\mathbf{t}), \dots,$$

$\boldsymbol{\beta}_1(\mathbf{t}, \mathbf{k}) = \boldsymbol{w}(\boldsymbol{\beta}_1(\mathbf{t}, \mathbf{k}-1)) + \mathbf{t} \cdot \boldsymbol{\beta}_1(\mathbf{t}, \mathbf{k}-1)$ where $\boldsymbol{w}(\mathbf{t}) = \mathbf{t}^{\mathbf{m}}$ and $\mathbf{m} = \frac{1}{2}\mathbf{log}_2(t+1)$.

The fact that $\gamma_{k+1}(G)$ is finite, implies that G includes the least normal subgroup L such that G/L is nilpotent. It leads us to the concept of a nilpotent residual. There will be appropriate to remind the concept of a residual in its most general form.

Let G be a group, $\boldsymbol{\mathcal{X}}$ be a class of groups, and

$\mathbf{Res}_{\boldsymbol{\mathcal{X}}}(G) = \{H \mid H$ is normal subgroup such that $G/H \in \boldsymbol{\mathcal{X}}\}$.

Then the intersection $G^{\mathcal{X}}$ of all normal subgroups from a family $\mathbf{Res}_{\boldsymbol{\mathcal{X}}}(G)$ is called the ***$\boldsymbol{\mathcal{X}}$-residual*** of a group G. If $\mathbf{Res}_{\boldsymbol{\mathcal{X}}}(G)$ has the least element L, then $L = G^{\mathcal{X}}$ and $G/G^{\mathcal{X}} \in \boldsymbol{\mathcal{X}}$. But in general $G/G^{\mathcal{X}} \notin \boldsymbol{\mathcal{X}}$.

If $\boldsymbol{\mathcal{X}} = \mathbf{a}$ is the class of all abelian groups, then $\mathbf{a}$-residual $G^{\mathbf{a}}$ is exactly the derived subgroup $[G, G]$ of a group G. In particular, $G/G^{\mathbf{a}} \in \mathbf{a}$.

If $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{N}}_{\mathbf{c}}$ is the class of all nilpotent groups having nilpotency class at most $\mathbf{c}$, then $\boldsymbol{\mathcal{N}}_{\mathbf{c}}$-residual $G^{\boldsymbol{\mathcal{N}}_{\mathbf{c}}}$ is exactly the subgroup $\boldsymbol{\gamma}_{\mathbf{c}+1}(G)$. In particular, $G/G^{\mathbf{N_c}} \in \boldsymbol{\mathcal{N}}_{\mathbf{c}}$.

But if $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{N}}$ is the class of all nilpotent groups, then, in general, $G/G^{\boldsymbol{\mathcal{N}}} \notin \boldsymbol{\mathcal{N}}$. Moreover, this factor-group can be non locally nilpotent. For example, if G is a free group, then by Magnus theorem $\boldsymbol{\gamma}_{\boldsymbol{\omega}}(G) = \langle 1 \rangle$. It follows that $G^{\boldsymbol{\mathcal{N}}} = \langle 1 \rangle$, so that $G/G^{\mathcal{N}}$ is a free group.

Therefore the following question appears: For what group G the factor-group $G/G^{\boldsymbol{\mathcal{N}}}$ is locally nilpotent?

The finiteness of $\gamma_{k+1}(G)$ implies that the nilpotent residual of G is finite. After **Theorem B** we show some bound for order of $\boldsymbol{\gamma}_{k+1}(G)$, and hence for the nilpotent residual. This bound was significantly improved in the paper [13].

Theorem 1.3. *Let G be a group, Z be the upper hypercenter of G and L a nilpotent residual of G. Suppose that $\mathbf{zl}(G)$ is finite and G/Z has finite order t. Then the following assertions hold:*

(i) *L is finite and there exists a function β_2 such that $|L| \leq \beta_2(t)$;*

(ii) *G/L is nilpotent and there exists a function β_3 such that* $\mathbf{ncl}(G/L) \le \beta_3(t)$.

It is interesting to observe that the order of the nilpotent residual depends only of the order of the factor-group by the upper hypercenter, but not depends of $\mathbf{zl}(G)$.

M. de Falco, F. de Giovanni, C. Musella and Ya. P. Sysak [7] have obtained a generalization of **Theorem B** in the following form.

Theorem 1.4. *Let G be a group, Z be the upper hypercenter of G. If G/Z is finite, then G includes a finite normal subgroup L such that G/L is hypercentral.*

Theorem 1.4 generates the following natural question: Are there orders of G/Z and L connected in some way? In particular, is the order of L bounded?

L.A. Kurdachenko, J. Otal and I.Ya. Subbotin answered on this question in the paper [10].

Theorem 1.5. *Let G be a group and Z be the upper hypercenter of G. Suppose that G/Z is finite, $G/Z = t$. Then G includes a finite normal subgroup L such that G/L is hypercentral and $\mid L \mid \le t^d$ where* $\mathbf{d} = \frac{1}{2}(\mathbf{log}_2 t + 1)$.

The technique used in the last paper enables us to obtain a stronger result.

Let G be a group and L a locally nilpotent subgroup of G. We say that L is ***locally hypercentrally embedded in* G** if for every finitely generated subgroup F of G the intersection $F \cap L$ lies in the upper hypercenter of F.

Theorem 1.6. *Let G be a group and L be a locally nilpotent normal subgroup of G. Suppose that G/L is finite, $\mid G/L \mid = t$. If L is locally hypercentrally embedded in G, then G includes a finite normal subgroup K such that G/K is locally nilpotent. Moreover, $\mid K \mid \le t^d$ where $\mathbf{d} = \frac{1}{2}(\mathbf{log}_2 t + 1)$. In particular, the locally nilpotent residual R of G is finite and G/R is locally nilpotent.*

Consider now some analogs of the above concepts and results in some other algebraic structures. Naturally we can not consider all the algebraic structures in which some analogs of the above group-theoretic results are considered. Our choice is essentially determined by our needs and tastes. One of the first analogs of **Theorem BN** was proved in topological groups. The case of topological groups has its own specific nature. The compactness is a natural analog of finiteness here. But the situation with the terms of upper and lower central series is more complicated. The terms of the upper central series having finite numbers are closed subgroups, but the rest in general not. The terms of the lower central series, in general, are not closed. An analog of **Theorem BN** has been obtained by V.I. Ushakov in the paper [21] and looks as the following.

Theorem 2.1. *Let G be a locally compact groups. If the factor-group $G/\boldsymbol{\zeta}(G)$ is compact, then the closure of $[G, G]$ is also compact.*

Using this theorem and the fact that the centralizers of any sets of a topological group are closed by adapting one of the proofs of **Theorem B**, it is possible to prove the following result

Theorem 2.2. *Let G be a locally compact groups. Suppose that there exists a positive integer k such that $G/\zeta_k(G)$ is compact. Then the closure of $\boldsymbol{\gamma}_{\mathbf{k+1}}(\mathbf{G})$ is also compact.*

Another major algebraic structure where the concept of the upper and lower central series emerged and have become familiar for a long time are Lie algebras. ??????? ????????, ??? there is a certain analogy between groups and Lie algebras: In Lie algebras it is possible to obtain analogs of many group theoretical results, and vice versa. Moreover, in some cases, in Lie algebras some group theoretical results can get a deeper promotion (of course, in the case, when one considers a Lie algebra over a field). More details on it one can find in a survey article of I.N. Stewart [19]. So, it is natural to look for analogs of the discussed results in the Lie algebras.

Let L be a Lie algebra over a field F and

$$\boldsymbol{\zeta}(L) = \{x \in L \mid [x, y] = 0 \text{ for each element } y \in L\}$$

be the center of L. Clearly $\boldsymbol{\zeta}(L)$ is a subalgebra, moreover, $\boldsymbol{\zeta}(L)$ is an ideal of L. As for groups we define here the upper central series

$$\langle 0\rangle = \zeta_0(L) \leq \zeta_1(L) \leq \zeta_2(L) \leq \ldots \leq \zeta_\alpha(L) \leq \zeta_{\alpha+1}(L) \leq \ldots \zeta_\gamma(L) = \zeta_\infty(L)$$

of an Lie algebra L by the following rule $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α and $\zeta_\gamma(L) = \cup_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. The last term $\boldsymbol{\zeta}_\infty(L)$ of this series is called the ***upper hypercenter*** of L.

If $L = \boldsymbol{\zeta}_\infty(L)$, then L is said to be ***hypercentral*** Lie algebra.

Define the lower central series of L by the following rule: $L_1 = \gamma_1(L) = L$, $\gamma_2(L) = [L, L]$ and recursively $\gamma_{\alpha+1}(L) = [\gamma_\alpha(L), L]$ for all ordinals α and $\gamma_\lambda(L) = \cap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ.

For Lie algebras the following analog of **Theorem BN** is well-known.

Theorem 3.1. *Let L be a Lie algebra over a field F. If the factor-algebra $L/\boldsymbol{\zeta}(G)$ has finite dimension $\mathbf{d}$, then the derived ideal $[L, L]$ also has finite dimension. Moreover, $\mathbf{dim}_F([L, L]) \leq \frac{\mathbf{1}}{\mathbf{2}}\mathbf{d}(\mathbf{d}+1)$.*

Some extensions of this result have been obtained by I.N. Stewart in the paper [20]. In particular, the results of this paper implies the following analog of **Theorem B**.

Theorem 3.2. *Let L be a Lie algebra over a field F. Suppose that there exists a positive integer k such that $\boldsymbol{\zeta}_k(L)$ has finite codimension. Then $\boldsymbol{\gamma}_{k+1}(G)$ has finite dimension.*

As for groups, for Lie algebras one can define the concept of the residual. More concretely, let L be a Lie algebra and $\boldsymbol{\mathcal{X}}$ be a class of Lie algebras, then put

$$\mathbf{Res}_{\boldsymbol{\mathcal{X}}}(L) = \{H \mid H \text{ is an ideal of } L \text{ such that } L/H \in \boldsymbol{\mathcal{X}}\}.$$

Then the intersection $L_{\mathcal{X}}$ of all ideal from the family $\mathbf{Res}_{\boldsymbol{\mathcal{X}}}(G)$ is called the ***$\mathcal{X}$-residual*** of algebra L. Thus, a natural question regarding an analog of **Theorem 1.3** arises. Such analog was obtained by L.A. Kurdachenko, A.A. Pypka and I.Ya. Subbotin in the paper [12].

Theorem 3.3. *Let L be a Lie algebra over a field F and R be a nilpotent residual of L. Suppose that there exists a positive integer k such that $\boldsymbol{\zeta}_k(L)$ has finite codimension $\mathbf{d}$. Then L/R is nilpotent and R has finite dimension at most $\frac{1}{2}\mathbf{d}(\mathbf{d}+3)$.*

This result is an intermediate step toward the following analog of **Theorems 1.4**, **1.5**, which was also obtained in [12].

Theorem 3.4. *Let L be a Lie algebra over a field F. Suppose that the upper hypercenter of L has finite codimension $\mathbf{d}$. Then L includes an ideal R such that L/R is hypercentral and R has finite dimension at most $\frac{1}{2}\mathbf{d}(\mathbf{d}+3)$.*

Leibniz algebras are interesting generalizations of Lie algebras. The concept of Leibniz algebra was introduced by J.L. Loday [15]. Since then it became very popular mainly because of its applications in physics.

Let L be an algebra over a field F, then L is called a ***Leibniz algebra*** (more precisely a ***left Leibniz algebra***), if its multiplication satisfies the Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]] \text{ for all } a,b,c \in A.$$

If L is a Leibniz algebra such that $[a,a] = 0$ for every element $a \in L$, then it is not hard to check that L is a Lie algebra. In other words we can consider Leibniz algebras as a non-asymmetrical analogy of Lie algebras. For some important results of the theory of Lie algebras there are analogs in Leibniz algebras. Very often these analogs are direct, but certain analogs have some specificity.

Let L be a Leibniz algebra over a field F. If A, B are subspaces of L, then $[A,\, B]$ will denote a subspace, generated by all elements $[a,\, b]$ where $a \in A$, $b \in B$.

As usual, a subspace A of L is called a ***subalgebra*** of L, if $[x,\, y] \in H$ for every $x,\, y \in H$. It follows that $[A,\, A] \le A$.

A subalgebra A is called a ***left*** (respectively ***right***) ideal of L, if $[y,\, x] \in A$ (respectively $[x,\, y] \in A$) for every $x \in A$, $y \in L$. In other words, if A is a left (respectively right) ideal, then $[L,\, A] \le A$ (respectively $[A,\, L] \le A$).

A subalgebra A of L is called an ***ideal*** of L (more precisely, a ***two-sided ideal***) if it is both a left ideal and a right ideal, that is $[x, y]$, $[y, x] \in A$ for every $x \in A$, $y \in L$.

If A is an ideal of L, we can consider a ***factor-algebra*** L/A. It is not hard to see that this factor-algebra also is Leibniz algebra.

Let L be a Leibniz algebra over a field F, M be non-empty subset of L, then $\langle M\rangle$ denotes the subalgebra of L generated by M.

Denote by $\mathbf{Leib}(L)$ the subspace generated by the elements $[a, a]$, $a \in L$.

It is possible to show that $\mathbf{Leib}(L)$ is an ideal of L, and if H is an ideal of L such that L/H is a Lie algebra, then $\mathbf{Leib}(L) \le H$.

The ideal $\mathbf{Leib}(L)$ is called the ***Leibniz kernel*** of an algebra L.

We note also the following important property of the Leibniz kernel:

$$[[a, a], x] = 0 \text{ for arbitrary elements } a, x \in L.$$

Let L be a Leibniz algebra over a field F, M be a non-empty subset of L and H be a subalgebra of L. Put

$$\mathbf{Ann}_{H^{\mathbf{left}}}(M) = \{a \in H \mid [a, M] = 0\}, \mathbf{Ann}_{H^{\mathbf{right}}}(M) = \{a \in H \mid [M, a] = 0\}.$$

The subset $\mathbf{Ann}_{H^{\mathbf{left}}}(M)$ is called the ***left annihilator*** or ***left centralizer*** of M in the subalgebra H; the subset $\mathbf{Ann}_{H^{\mathbf{right}}}(M)$ is called the ***right annihilator*** or ***right centralizer*** of M in the subalgebra H. The intersection

$$\mathbf{Ann}_H(M) = \mathbf{Ann}_{H^{\mathbf{left}}}(M) \cap \mathbf{Ann}_{H^{\mathbf{right}}}(M) = \{a \in H \mid [a, M] = \langle 0\rangle = [M, a]\}$$

is called the ***annihilator*** or ***centralizer*** of M in the subalgebra H.

It is not hard to see that all these subsets are subalgebras of L. Moreover, if M is a left ideal of L, then $\mathbf{Ann}_{L^{\mathbf{left}}}(M)$ is an ideal of L. If M is an ideal of L, then $\mathbf{Ann}_L(M)$ is an ideal of L.

The ***left*** (respectively ***right***) ***center*** $\boldsymbol{\zeta}^{\mathbf{left}}(L)$ (respectively $\boldsymbol{\zeta}^{\mathbf{right}}(L)$) of L is defined by the rule

$$\boldsymbol{\zeta}^{\mathbf{left}}(L) = \{x \in L \mid [x, y] = 0 \text{ for each element } y \in L\}.$$

(respectively

$$\boldsymbol{\zeta}^{\mathbf{right}}(L) = \{x \in L \mid [y, x] = 0 \text{ for each element } y \in L\}).$$

By above noted, the left center of L is an ideal, moreover $\mathbf{Leib}(L) \le \boldsymbol{\zeta}^{\mathbf{left}}(L)$, so that $L/\boldsymbol{\zeta}^{\mathbf{left}}(L)$ is a Lie algebra.

The ***center*** $\boldsymbol{\zeta}(L)$ of L is the intersection of the annihilators of all the elements of L. In other words,

$$\boldsymbol{\zeta}(L) = \{x \in L \mid [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

Thus the center is an annihilator of whole algebra L, it is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$.

The right center is an subalgebra of L, and, in general, the left and right centers are different; they even may have different dimensions. It is possible to construct an example to illustrate this statement.

Define now the lower central series of L

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \ldots \geq \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \ldots \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \cap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\delta(L)$ is called the ***lower hypocenter*** of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \ldots, L] \ldots L]$ is a left normed product of k copies of L.

A Leibniz algebra L is called ***nilpotent***, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be ***nilpotent of nilpotency class*** $\mathbf{c}$ if $\gamma_{\mathbf{c}+1}(L) = \langle 0 \rangle$, but $\gamma_{\mathbf{c}}(L) \neq \langle 0 \rangle$. We denote by $\mathbf{ncl}(L)$ the nilpotency class of L.

It is useful to show some properties of Leibniz algebras.

Proposition 4.1. *Let L be an Leibniz algebra over a field F. Then*

(i) If H is an ideal of L, then $[H, H]$ is an ideal of L, in particular the derived subalgebra $[L, L]$ is a ideal.

(ii) If H is an ideals of L, then $[L, H]$ is a subalgebra of L.

(iii) If H is an ideals of L, then $[H, L]$ is a subalgebra of L.

(iv) If H is an ideals of L, then $[L, H] + [H, L]$ is an ideal of L.

(v) If H is an ideal of L, then $[\gamma_j(H), \gamma_k(H)] \leq \gamma_{j+k}(H)$ for every positive integers j, k.

(vi) If H is an ideal of L, then $\gamma_j(H)$ is an ideal of L for each positive integer j. In particular, $\gamma_j(L)$ is an ideal of L for each positive integer j.

(vii) If H is an ideal of L, then $\gamma_j(\gamma_k(H)) \leq \gamma_{j\,k}(H)$ for every positive integers j, k.

We remark that if A, B are ideals of a Leibniz algebra L, then, in general, $[A, B]$ need not be an ideal. A correspondent examples has been constructed by D. Barnes in his paper [3].

Consider a factor $\gamma_k(L)/\gamma_{k+1}(L)$, $k \in \mathbb{N}$. By definition, $[L, \gamma_k(L)] = \gamma_{k+1}(L)$. By above property (iii), $[\gamma_k(L), L] = [\gamma_k(L),\, \gamma_1(L)] \leq \gamma_{k+1}(L)$. Let

$$\langle 0 \rangle = C_0 \leq C_1 \leq \ldots \leq C_\alpha \leq C_{\alpha+1} \leq \ldots C_\gamma = L$$

be an ascending series of ideals of the Leibniz algebra L. This series is called ***central*** if $C_{\alpha+1}/C_\alpha \leq \zeta(L/\zeta_\alpha(L))$ for each ordinal $\alpha < \gamma$. In other words, $[C_{\alpha+1}, L]$, $[L,\, C_{\alpha+1}] \leq C_\alpha$ for each ordinal $\alpha < \gamma$.

Define now the upper central series

$$\langle 0 \rangle = \zeta_0(L) \leq \zeta_1(L) \leq \zeta_2(L) \leq \ldots \leq \zeta_\alpha(L) \leq \zeta_{\alpha+1}(L) \leq \ldots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals $\alpha, \zeta_\lambda(L) = \cup_{\mu<\lambda}\, \zeta_\mu(L)$ for a limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\boldsymbol{\zeta}_\infty(L)$ of this series is called the ***upper hypercenter*** of L. Denote by $\mathbf{zl}(L)$ the length of upper central series of L.

Suppose now that a Leibniz algebra L has a finite central series

$$\langle 0 \rangle = C_0 \leq C_1 \leq \ldots \leq C_n = L.$$

Then

(i) $\gamma_j(L) \leq C_{n-j+1}$, *so that* $\gamma_{n+1}(L) = \langle 0 \rangle$.

(ii) $C_j \leq \zeta_j(L)$, *so that* $\zeta_n(L) = L$.

(iii) L *is nilpotent and* $\mathbf{ncl}(L) \leq n$. *Furthermore, the upper central series of* L *is finite,* $\boldsymbol{\zeta_\infty}(L) = L$, $\mathbf{zl}(L) \leq n$, *moreover* $\mathbf{ncl}(L) = \mathbf{zl}(L)$.

The following example shows that the finiteness of codimension of the left center does not imply the finiteness of dimension of the derived ideal.

Let F be a field. Put $L = Fe_1 \oplus Fe_2 \oplus Z$ where a subspace Z has a countable basis $\{z_n, \mid n \in \mathbb{N}\}$. Put $[z_n,\, x] = 0$ for every $x \in L$ and

$$[e_1, e_1] = [e_2, e_2] = [e_1, e_2] = [e_2, e_1] = z_1, [e_1, z_1] = [e_2, z_1] = 0.$$

It is possible to check that L is a Leibniz algebra. By it construction, **Z is a left center of L**, the **right center coincides with the center and coincides with Fz**$_1$, so that the left center has finite codimension (and therefore, infinite dimension) and the right center and the center have finite dimension. By its construction, $[\mathbf{L}, \mathbf{L}] = \mathbf{Z}$. Furthermore $\mathbf{Leib}(L) = Z$.

An analog of **Theorem BN** was proved by L.A. Kurdachenko. A.A, Pypka, J. Otal [11] and looks as following.

Theorem 4.2. *Let L be a Leibniz algebra over a field F. Suppose that* $\mathbf{codim}_F(\boldsymbol{\zeta}^{\mathbf{left}}(L)) = \mathbf{d}$ *and* $\mathbf{codim}_F(\boldsymbol{\zeta}^{\mathbf{right}}(L)) = \mathbf{r}$ *are finite. Then $[L, L]$ has finite dimension, moreover* $\mathbf{dim}_F([L, L]) \leq \mathbf{d}(\mathbf{d} + \mathbf{r})$.

In particular, if $\mathbf{codim}_F(\boldsymbol{\zeta}(L)) = \mathbf{d}$ *is finite. Then $[L, L]$ has finite dimension, moreover* $\mathbf{dim}_F([L, L]) \leq \mathbf{d^2}$. *And the Leibniz kernel of L has finite dimension at most* $\frac{1}{2}\mathbf{d}(\mathbf{d} - \mathbf{1})$.

An analog of **Theorem B** for Leibniz algebras is also proved in the paper [11].

Theorem 4.3. *Let L be a Leibniz algebra over a field F. Suppose that* $\mathbf{codim}_F(\boldsymbol{\zeta}_{\mathbf{k}}(L)) = \mathbf{d}$ *is finite. Then $\boldsymbol{\gamma}_{k+1}(L)$ has finite dimension, moreover* $\mathbf{dim}_F(\boldsymbol{\gamma}_{k+1}(L)) \leq \mathbf{2^{k-1}d^{k+1}}$, $k \geq 1$.

As a corollary we obtained the bound for a dimension of $\gamma_{\mathbf{k}+1}(L)$ in Lie algebra L.

Corollary 4.4. *Let L be a Leibniz algebra over a field F. Suppose that* $\mathbf{codim}_F(\boldsymbol{\zeta}_k(L)) = \mathbf{d}$ *is finite. Then $\boldsymbol{\gamma}_{k+1}(L)$ has finite dimension. Moreover,* $\mathbf{dim}_F(\boldsymbol{\gamma}_{k+1}(L)) \leq \frac{\mathbf{1}}{\mathbf{2}}\mathbf{d^{k-1}}(\mathbf{d} - \mathbf{1})$.

The next algebraic structure we considered is a vector space A over a field F and acting on it a group G of non-singular linear transformations. In this case we can consider A as a FG-module. There fore we give more general definition.

Let G be a group, R a ring and A an RG-module. Put

$$\zeta_{RG}(A) = \{a \in A \mid a(g-1) = 0 \text{ for each element } g \in G\} = \mathbb{C}_A(G).$$

Clearly $\zeta_{RG}(A)$ is an RG-submodule of A. This submodule is called the ***RG-center of* A**.

An analog of the derived subgroup is the following. Denote by ϖ_{RG} the augmentation ideal of a group ring RG, that is the two-sided ideal, generated by all elements $g - 1$, $g \in G$. The submodule $[A, G] = A(\varpi RG)$ generated by the elements $a(g - 1)$, $a \in A$, $g \in G$, is called the ***derived submodule*** of A.

The first case here is the case when R is a field. In this case, A becomes a vector space over R, and the natural substitution of the restriction of finiteness becomes the finiteness of dimensionality over R. The group G (more precisely $G/\mathbb{C}_G(A)$) could be considered as a subgroup of a group $\mathbf{GL}(R, A)$ of all non-singular linear transformations of the R-vector space A.

We observe that in reality we cannot have full analogy here. If A is an RG-module, then we have a natural semidirect product $K = A\lambda G$. Instead of the center $\boldsymbol{\zeta}(K)$ we will consider $\boldsymbol{\zeta}(K) \cap A = \mathbb{C}_A(G)$. This observation leads us to the conclusion that we cannot obtain for modules a complete analog of **Theorem BN**. The following example justifies this.

Let A be a vector space over the field F of infinite countable dimension and let $\{a_n \mid n \in \mathbb{N}\}$ be a basis of A. Define the following F-automorphism g_k of A by the rule:

$$a_n g_k = \begin{cases} a_1 + a_k, & \text{if } n = 1 \\ a_n, & \text{if } n > 1. \end{cases}$$

Let $G = \langle g_n \mid n \in \mathbb{N} \rangle$. Clearly, $G = \mathbf{Dr}_{n \in \mathbb{N}} \langle g_n \rangle$. In particular, if $\mathbf{char}(F) = p > 0$, then G is an elementary abelian p-group; if $\mathbf{char}(F) = 0$, then G is a free abelian group,. Then $\boldsymbol{\zeta}_{FG}(A)$ is a subspace $\oplus_{n>1} Fa_n$, so that $\mathbf{codim}_F(\zeta_{FG}(A)) = 1$. In the same time, $[A, G]$ s also a subspace $\oplus_{n>1} Fa_n$, so that $\mathbf{dim}_F(A(\varpi FG))$ is infinite.

This shows that the question about an appropriate module analog for **Theorem BN** should be expressed in the following form.

Let F be a field, A be a vector space over F and G be a subgroup of $GL(F, A)$. Suppose that $\zeta_{FG}(A)$ has finite codimension. For what group G a derived submodule $[A, G]$ has finite dimension?

In the mentioned above example, the group G has an infinite elementary abelian section. This is a hint for considering the following possibility.

Let p be a prime. We say that a group G has ***finite section* p*-rank*** $\mathbf{sr}_p(G) = \mathbf{r}$ if every elementary abelian p-section of G is finite of order at most $p^{\mathbf{r}}$ and there is an elementary abelian p-section A/B of G such that $| A/B |= p^{\mathbf{r}}$.

And similarly, we say that a group G has **finite section 0-rank** $\mathbf{sr_0(G)} = \mathbf{r}$ if for every torsion-free abelian section U/V of G the inequality $\mathbf{r}_{\mathbb{Z}}(U/V) \leq \mathbf{r}$ holds and there is an abelian torsion-free section A/B such that $\mathbf{r}_{\mathbb{Z}}(U/V) = \mathbf{r}$.

Here $\mathbf{r}_{\mathbb{Z}}(A)$ is the $\mathbb{Z}$-rank of an abelian group A (that is a rank A as a $\mathbb{Z}$-module).

We note that if a group G has finite section p-rank for some prime p, then G has finite section 0-rank, moreover $\mathbf{sr_0(G)} \leq \mathbf{sr_p(G)}$.

Recall that a group G has ***finite special rank*** $\mathbf{r(G)} = \mathbf{r}$ if every finitely generated subgroup of G can be generated by $\mathbf{r}$ elements and $\mathbf{r}$ is the least positive integer with this property.

We can see that the concept of section p-rank generalizes the concept of special rank. Indeed if a group G has a finite special rank $\mathbf{r}$, then G has finite section p-rank for every prime p, moreover $\mathbf{sr}_p(G) \leq \mathbf{r}$.

The following result was obtained by M.R. Dixon, L.A. Kurdachenko and J. Otal in the paper [4].

Theorem 5.1. *Let F be a field, A be an F-vector space and G be a subgroup of $\mathbf{GL}(F, A)$. Suppose that $\mathbf{codim}_F(\boldsymbol{\zeta}_{FG}(A)) = \mathbf{c}$ is finite. Then the following assertions hold.*

(A) *If* $\mathbf{char}(F) = 0$ *and* $\mathbf{sr}_0(G) = \mathbf{r}$ *is finite, then* $[A,\, G]$ *has finite dimension.*

(B) *If* $\mathbf{char}(F) = p > 0$ *and* $\mathbf{sr}_p(G) = \mathbf{r}$ *is finite, then* $[A,\, G]$ *has finite dimension.*

Moreover there exists a function $\boldsymbol{\kappa}$ *such that* $\mathbf{dim}_F([A, G]) \leq \boldsymbol{\kappa}(\mathbf{c}, \mathbf{r})$.

Starting from the RG-center, we can construct the upper RG-central series of A:

$$\langle 0 \rangle = \zeta_{RG\,0}(A) \leq \zeta_{RG\,1}(A) \leq \ldots \leq \zeta_{RG\,\alpha}(A) \leq \zeta_{RG\,\alpha+1}(A) \leq \ldots \zeta_{RG\,\gamma}(A),$$

defined the rule $\zeta_{RG\,1}(A) = \zeta_{RG}(A)$ is the center of G, and recursively $\zeta_{RG\,\alpha+1}(A)/\zeta_{RG\,\alpha}(A) = \zeta_{RG}(A/\zeta_{RG\,\alpha}(A))$ for all ordinals $\alpha, \zeta_{RG\,\lambda}(A) = \cup_{\mu<\lambda}\, \zeta_{RG\,\mu}(A)$ for the limit ordinals λ and $\zeta_{RG}(A/\zeta_{RG\,\gamma}(A)) = \langle 0 \rangle$. The last term $\zeta_{RG\,\gamma}(A) = \zeta_{RG\,\infty}(A)$ of this series is called the **upper *RG-hypercenter of*** **A** and the ordinal γ is called the ***RG-hypercentral length of a module*** **A** and will denoted by $\mathbf{zl}_{RG}(A)$. We observe that $|\ \zeta_{RG\,\alpha+1}(A), G] \leq \zeta_{RG\,\alpha}(A)$ for all $\alpha < \gamma$.

If the upper RG-hypercenter of A coincides with A, then A is called ***RG-hypercentral***.

If A is an RG-hypercentral module and $\mathbf{zl}_{RG}(A)$ is finite, then we will say that A is ***RG-nilpotent***.

We defined also the lower RG-central series of A. It is a series

$$A = \gamma_{RG\,1}(A) \geq \gamma_{RG\,2}(A) \geq \ldots \geq \gamma_{RG\,\alpha}(A) \geq \gamma_{RG\,\alpha+1}(A) \geq \ldots$$

defined by the rule $\gamma_{RG\,2}(G) = [A, G]$ and recursively $\gamma_{RG\,\alpha+1}(A) = [\gamma_{RG\,\alpha}(A), G]$ for all ordinals α and $\gamma_{RG\,\gamma}(G) = \cap_{\mu<\lambda}\, \gamma_{RG\,\mu}(G)$ for the limit ordinals λ.

The following linear analog of **Theorem B** has also been obtained in paper [4].

Theorem 5.2. *Let* F *be a field,* A *be an* F*-vector space and* G *be a subgroup of* $\mathbf{GL}(F, A)$*. Suppose that there exists a positive integer* k *such that* $\mathbf{codim}_F(\boldsymbol{\zeta}_{FG\,k}(A)) = \mathbf{c}$ *is finite. Then the following assertions hold.*

(A) *If* $\mathbf{char}(F) = 0$ *and* $\mathbf{sr}_0(G) = \mathbf{r}$ *is finite, then* $\boldsymbol{\gamma}_{FG\,k+1}(A)$ *has finite dimension.*

(B) *If* $\mathbf{char}(F) = p > 0$ *and* $\mathbf{sr}_p(G) = r$ *is finite, then* $\boldsymbol{\gamma}_{FG\,k+1}(A)$ *has finite dimension.*

Moreover there exists a function $\boldsymbol{\kappa_1}$ *such that* $\mathbf{dim}_F(\boldsymbol{\gamma}_{FG\,k+1}(A)) \leq \boldsymbol{\kappa_1(c, r, k)}$.

Some linear analogs of **Theorem GB, GB$_1$** have been obtained by M.R. Dixon, L.A. Kurdachenko and J. Otal in the paper [5].

Theorem 5.3. *Let* F *be a field,* A *be an* F*-vector space and* G *be a subgroup of* $\mathbf{GL}(F, A)$*. Suppose that there exists a positive integer* k *such that* $\mathbf{codim}_F(\boldsymbol{\zeta}_{FG\,\infty}(A)) = \mathbf{c}$ *is finite. Then the following assertions hold.*

(A) If $\mathbf{char}(F) = 0$ *and* $\mathbf{sr}_0(G) = \mathbf{r}_0$ *is finite, then* A *includes an* FG*-submodule* C*, having finite dimension, such that* A/D *is* FG*-hypercentral.*

(B) If $\mathbf{char}(F) = p > 0$ *and* $\mathbf{sr}_p(G) = r$ *is finite, then* A *includes an* FG*-submodule* D*, having finite dimension, such that* A/D *is* FG*-hypercentral.*

Moreover there exist the functions $\boldsymbol{\kappa_2}, \boldsymbol{\kappa_3}$ *such that* $\mathbf{dim}_F(C) \leq \boldsymbol{\kappa_2}(\boldsymbol{c}, \boldsymbol{r_0})$*,* $\mathbf{dim}_F(D) \leq \boldsymbol{\kappa_3}(\boldsymbol{c}, \boldsymbol{r_p})$*.*

The study of this topics for the case of RG-modules where R is not a field is at the beginning stage yet. There are quite many generalizations of the notion of finite dimensionality for rings. The very first one is based on the following property of a finite dimensional vector space: every finite dimensional vector space has a finite F-composition series. Recently we obtained the following analog of **Theorem BN**. But first we need some definitions.

Let R be an integral domain and A an R-module. Suppose that A has a finite composition series

$$\langle 0 \rangle = C_0 \leq C_1 \leq \ldots \leq C_n = A$$

of submodules. Then $C_j/C_{j-1} = R(c_j + C_{j-1}) \cong_R R/\mathbf{Ann}_R(c_j + C_{j-1})$. Since C_j/C_{j-1} is a simple R-module, $\mathbf{Ann}_R(c_j + C_{j-1}) = \mathbf{Ann}_R(C_j/C_{j-1})$ is a maximal ideal of R. Then a factor-ring $R/\mathbf{Ann}_R(c_j + C_{j-1})$ is a field. We recall that every two composition series of A are isomorphic. It follows that the length $\mathbf{n}$ of composition series and the sets

$$\mathbf{Spec}(A) = \{\mathbf{char}(F_j) \mid F_j = R/\mathbf{Ann}_R(C_j/C_{j-1})), 1 \leq j \leq n\},$$

are the invariants of module A. The length of composition series of A is called the composition length of A and will denotes by $\mathbf{c}_R(A)$.

As we mentioned above, an analog of **Theorem BN** for the case when R is a field takes place only under restrictions imposed on the abelian p-sections of G where $p = \mathbf{char}(F)$. Therefore it is natural to keep here the same restrictions. Next result has been obtained by L.A. Kurdachenko, I.Ya. Subbotin, V.A. Chupordya in the paper [9].

Theorem 5.4. *Let* R *be an integral domain,* G *be a group and* A *be an* RG*-module. Suppose that* $A/\zeta_{RG}(A)$ *has finite composition series as an* R*-module. If a group* G *has finite section* p*-rank* $\mathbf{r}_p$ *for every* $p \in \mathbf{Spec}(A/\zeta_{RG}(A))$*, then* $[A,G]$ *has finite* R*-composition series and* $\mathbf{Spec}([A,G]) \subseteq \mathbf{Spec}(A/\zeta RG(A))$*. Moreover, there exists a function* $\boldsymbol{\kappa_4}$ *such that* $\mathbf{c}_R([A,G]) \leq \boldsymbol{\kappa_4}(\boldsymbol{r_p}, \mathbf{c}_R(A/\zeta_{RG}(A))/p \in \mathbf{Spec}_R(A/\zeta_{RG}(A))$*.*

For the case $R = \mathbb{Z}$ we have

Theorem 5.5. *Let G be a group and A be a $\mathbb{Z}G$-module. Suppose that $A/\zeta_{ZG}(A)$ is finite. If a group G has finite section p-rank $\mathbf{r}_p$ for every $p \in \Pi(A/\zeta_{RG}(A))$, then $[A,G]$ is finite and $\Pi([A,G]) \subseteq \Pi(A/\zeta_{RG}(A))$. Moreover, there exists a function $\boldsymbol{\kappa_5}$ such that*

$$|\,[A,G]| \leq \boldsymbol{\kappa_5}(\boldsymbol{r_p}, |\,A/\zeta_{RG}(A)\,|\,/p \in \mathbf{\Pi}(A/\zeta_{RG}(A))).$$

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